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REDUCED ORDER MODELING FOR QUANTIFYING THE IMPACT OF VARIABILITY ON THE PERFORMANCE OF AN INFINITE LOCALLY RESONANT METAMATERIAL BEAM

Yves Pillot[1,3], Régis Boukadia[1,3], Lucas Van Belle[1,3] and Elke Deckers[2,3]

[1] KU Leuven
Department of Mechanical Engineering, Division LMSD
Celestijnenlaan 300, 3001 Leuven, Belgium
e-mail: yves.pillot@kuleuven.be, regis.boukadia@kuleuven.be, lucas.vanbelle@kuleuven.be

[2] KU Leuven Campus Diepenbeek
Department of Mechanical Engineering
Wetenschapspark 27, 3590 Diepenbeek, Belgium
e-mail: elke.deckers@kuleuven.be

[3] Flanders Make@KU Leuven
Belgium

Abstract. *Nowadays, metamaterials are advancing towards realizable industrial solutions, making it increasingly important to quantify the impact of variability on their performance. Modeling metamaterials typically relies on Bloch-Floquet theory, which makes describing variability from unit-cell to unit-cell a theoretical challenge, as the assumption of perfect periodicity would become invalid. To address this challenge, this paper proposes a strategy to compute the statistics of the forced response of an infinite metamaterial beam possessing varying resonator properties. The proposed strategy is based on Krylov subspace methods which is suitable for rank 1 perturbation to the system and enables efficient Monte-Carlo simulations to derive the statistics of interest for this problem. The method aims to determine the behavior of the metamaterial response close to stopbands, where the variability of the resonators' properties leads to response modifications. Comparison between the responses of periodic and disordered metamaterials is hence made possible regardless of size effects.*

Keywords: Vibro-Acoustic Metamaterials, Random media, Reduced-Order Modeling.

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1 INTRODUCTION

Metamaterials have recently emerged as a more and more credible potential solution to noise and vibration problems in the industry. Their ability to block the propagation of waves in a given frequency range, called stopband, makes them appealing vibration control devices. However, challenges arise with the industrial implementation of such materials. One of the most important challenges is the one of variability. This point is of critical importance in view of industrial applications, in which performance is desired on a large number of samples subject to manufacturing variability. Indeed, some of samples' vibration properties might not meet the target properties because of the latter.

Designing metamaterials often resorts to Unit-Cell (UC) modeling [1], which assumes perfect translation periodicity in space. This former modeling method is therefore unable to describe some response variations that might be caused by manufacturing imperfections, which are inevitably aperiodic and most of the time differ between the different periods of the metamaterial. Still, some authors applied uncertainty quantification methods on UC models [2, 3, 4], allowing to obtain statistical results on important indicators for metamaterials, such as the stopband width and center, but inevitably assuming variability effects that do not break periodicity. Some works suggest the use of a supercell approach to account for the variability between the metamaterial cells [2], allowing to partially overcome the periodicity restriction at the UC scale imposed by this modeling method. Other numerical investigations on metamaterial variability have been conducted on finite size systems [5, 6, 7].In these works the authors developed methods for fast computation of metamaterials response statistics and allowed them observe different effects due to variability on the vibration attenuation zone of metamaterial partitions, like a shift in central frequency, a modification of width, or an increased variance [5, 8].

To study metamaterials that are infinite and aperiodic, several challenges appear on the numerical side because the metamaterial response evaluation is not as straightforward anymore as in a finite metamaterial simulation, because the operators describing the system of interest are infinite and therefore not invertible numerically. This challenge was addressed through the Wave Finite Element Method (WFEM) [9, 10] for the case of a perfectly periodic systems, but this method possesses no ability to describe media without periodicity.

To alleviate this challenge, the objective of this paper is to develop a reduced-order strategy to compute the forced response of an infinite periodic structure possessing variability. The approach relies on adding a correction to the infinite periodic structure response, obtained by considering the perturbation to the system caused by each UC. Assuming that the displacement response decays away from the forcing location, we show numerically that only a finite number of UC perturbations need to be accounted for in order to obtain a convergent response, making the method suitable for computer simulations. This makes fast evaluation of the metamaterial response statistics possible, allowing for quantification of the impact of random uncertainty on vibration attenuation performance of the metamaterial.

The paper is organized as follows. Section 2 provides a description of the problem under consideration. Then, Section 3 provides a brief refresher on how the WFEM is used for the forced response computation of infinite structures. Thirdly, section 4 describes the reduced-order modeling strategy used to compute the forced response of a metamaterial in the presence of variability. Finally, section 5 presents a numerical study about the impact of variability on the response of an infinite metamaterial beam. The main findings are summarized in section 6.

2 PROBLEM SETTING

In this paper, a one-dimensional infinite Euler-Bernoulli beam is considered with a periodic attachment of single degree of freedom resonators with the same mass but a stiffness that differs by a small percentage between the different additions. A sketch of this system of interest is given in figure 1. The UCs are labeled with an integer $n \in \mathbb{Z}$. The left-boundary of each cell is labeled similarly. A point force of 1 N is applied on the 0th UC at the basis of the resonator. Generalization to more complicated loading cases can be made by linear combinations and translation of responses to a load on a single UC.

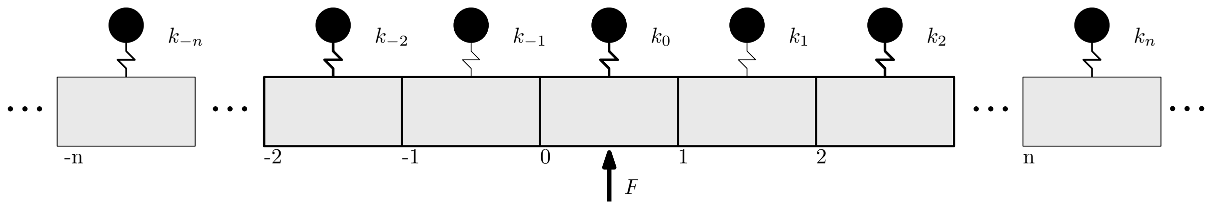

Figure 1: Considered infinite metamaterial beam with spatially periodic addition of resonators possessing a random stiffness.

Computing the response of such a structure is not straightforward because of the detuning of the resonators, breaking the periodicity of the system. However, in a first step, the response of a periodic system with identical values for each resonator stiffness is computed.

3 THE WAVE FINITE ELEMENT METHOD

The WFEM allows to construct a wave basis for periodic structures allowing to compute their forced response. In this section, the WFEM for forced response computation of an infinite periodic waveguide is briefly reviewed.

3.1 Constructing a wave basis to compute periodic structure response

The FE-discretized equations of motion for the n -th UC can be written as

$$D_n(\omega)\mathbf{u}_n^{(0)} = \mathbf{F}_n, \tag{1}$$

where ω is the radial frequency and with D_n, $\mathbf{u}_n^{(0)}$ and $\mathbf{F}_n$ the discretized dynamic stiffness operator, displacement field and force field of the n-th UC for the periodic case. The superscript $^{(0)}$ is used to refer to the displacement vector computed in the periodic case, which will later be used in the reduced-order model to serve as a reference case to construct the perturbation. Thus, assuming that all UCs are identical, and following [11], the infinite structure is assembled by connecting the UCs to their nearest neighbor and the displacement vector is condensed at their right and left boundaries, yielding the following recurrence equation:

$$D_{RL}\tilde{\mathbf{u}}_{n-1}^{(0)} + (D_{RR} + D_{LL})\,\tilde{\mathbf{u}}_n^{(0)} + D_{LR}\tilde{\mathbf{u}}_{n+1}^{(0)} = \tilde{\mathbf{F}}_n, n \in \mathbb{Z}, \tag{2}$$

with D_{RL}, D_{RR}, D_{LL}, D_{LR} the partitions of the stiffness matrix condensed at the left and right boundaries of the n-th UC and $\tilde{\mathbf{u}}_n^{(0)}$, $\tilde{\mathbf{F}}_n$ the displacement and force vector condensed at

n-th UC boundary. Free solutions are sought in the form of Bloch waves $\tilde{\mathbf{u}}_n^{(0)} = \lambda^n \psi$. Inserting such an assumption into the condensed equation of motion (2) and setting $\tilde{\mathbf{F}}_n = 0$ leads to the following palindromic eigenvalue problem:

$$\frac{1}{\lambda} D_{RL}\psi + (D_{RR} + D_{LL})\psi + \lambda D_{LR}\psi = 0, \tag{3}$$

which allows to obtain a Bloch wave basis. Such eigenvalue problems can be solved numerically according to one of the schemes from [12]. Due to the palindromic structure of the eigenvalue problem, the obtained eigenvalues come in pairs such that they can be ordered as $\left[\lambda_1, \lambda_2, ..., \lambda_M, \lambda_1^{-1}, \lambda_2^{-1}, ...\lambda_M^{-1}\right]$, where M is the number of degrees of freedom at the boundary of the UC and λ_m corresponding to right-going waves and λ_m^{-1} to left-going waves. The associated eigenvectors can then be sorted accordingly as $\left[\psi_1^+, \psi_2^+, ..., \psi_M^+, \psi_1^-, \psi_2^-, ...\psi_M^-\right]$. To compute the forced response of a periodic structure, the displacement field is sought as a superposition of Bloch waves outside the 0-th UC:

$$\tilde{\mathbf{u}}_n^{(0)} = \Psi^+ \Lambda^n \mathbf{q}^+ + \Psi^- \Lambda^{-n} \mathbf{q}^-, n \neq 0, 1, \tag{4}$$

with $\Lambda = [\lambda_1, \lambda_2, ..., \lambda_M]$, $\Psi^+ = \left[\psi_1^+, \psi_2^+, ..., \psi_M^+\right]$, $\Psi^- = \left[\psi_1^-, \psi_2^-, ..., \psi_M^-\right]$, $\mathbf{q}^+$ the left-going wave modal contribution vector and $\mathbf{q}^-$ the right-going wave modal contribution vector. The force is assumed to be applied on the 0-th UC. As the medium is infinite, only waves leaving the loaded UC are considered:

$$\begin{aligned} \tilde{\mathbf{u}}_n^{(0)} &= \Psi^+ \Lambda^{(n+1)} \mathbf{q}^+, n > 0, \\ \tilde{\mathbf{u}}_n^{(0)} &= \Psi^- \Lambda^{-n} \mathbf{q}^-, n \leq 0. \end{aligned} \tag{5}$$

By inserting assumption (5) in equation (2) for cells 0 and 1, a system of equations is obtained, which can be used to determine $\mathbf{q}^+$ and $\mathbf{q}^-$:

$$\begin{bmatrix} D_{RL}\Psi^- \Lambda + (D_{RR} + D_{LL})\Psi^- & D_{LR}\Psi^+ \\ D_{RL}\Psi^- & (D_{RR} + D_{LL})\Psi^+ + D_{LR}\Psi^+ \Lambda \end{bmatrix} \begin{bmatrix} \mathbf{q}^- \\ \mathbf{q}^+ \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{F}}_0 \\ \tilde{\mathbf{F}}_1 \end{bmatrix}. \tag{6}$$

Once the amplitude of the wave modes is known, the response in any UC is straightforwardly evaluated using expression (5).

4 REDUCED-ORDER MODELING OF VARIABILITY

While the conventional WFEM enables one to compute the response of infinite periodic waveguides, a strategy to obtain the forced response of a perturbed waveguide is sought for in this section. To this end, a response correction strategy is developed to compute the response of the waveguide in the presence of a perturbation.

4.1 Problem definition

Variability of the resonators' stiffness is included in the system by introducing a random perturbation operator ΔD_n for each UC in equation (1), leading to:

$$[D_n + \Delta D_n]\,\mathbf{u}_n = \mathbf{F}_n, \tag{7}$$

with $\mathbf{u}_n$ the displacement field for the perturbed case. The method is developed for the case where ΔD_n is of rank 1 which is the case if the stiffness or mass of the resonators is perturbed. Due to the presence of the perturbation, the system is no longer periodic, making direct use of the WFEM impossible. For that reason, a correction approach is needed to compute the displacement field. In a first step, we treat the case of classical perturbation theory. Secondly, an alternative method is proposed to obtain more accurate results.

4.2 Regular perturbation theory

A first approach to solve equation (7) would be to use a first order perturbative scheme and expand the displacement field as:

$$\mathbf{u}_n = \mathbf{u}_n^{(0)} + \sum_{m=-\infty}^{\infty} \mathbf{V}_n^{(m)}, \tag{8}$$

where the $\mathbf{V}_n^{(m)}$ are defined by the solutions to the forced equation

$$D_n(\omega)\mathbf{V}_n^{(m)} = -\Delta D_m \mathbf{u}_n^{(0)}, n \in \mathbb{Z}, \tag{9}$$

and can be computed with the method of Section 3. The main issue with such an approach is that the series might diverge in certain cases of interest. To obtain a well-behaved solution in the regions of interest, an alternative correction approach is described.

4.3 Reduced-order problem definition

Equation (7) is impossible to solve with a WFEM approach. An alternative way of solving it is to rely on the fact that its solution can be sought for as a linear combination of vectors, called the Krylov subspace [13]. To create a reduced-order problem for the system, the approach developed in [7] to compute the response of a perturbed, finite metamaterial beam is extended to the case of an infinite system. The method starts by assuming that the solution can be expanded in a series that matches the moments the perturbation expression (8),

$$\mathbf{u}_n = \mathbf{u}_n^{(0)} + \sum_{m=-\infty}^{\infty} w_m \mathbf{V}_n^{(m)}, \tag{10}$$

with the w_m representing the contribution amplitude of $\mathbf{V}_n^{(m)}$. As the ΔD_n are rank 1, this first order representation of the displacement vector is exact if all $m \in \mathbb{Z}$ are considered. However, in practice, only a finite number of $\mathbf{V}_n^{(m)}$ corresponding to the first few resonators on each side of the forced UC are considered to construct the solution. The assumption behind such a consideration is that resonators far from the forcing location would not contribute significantly to the perturbed response and can be discarded from the model. This claim is later evaluated in the numerical study. The correction to solution $\mathbf{u}_n^{(0)}$ therefore approximately lies in the space Q_n, defined as

$$Q_n = \left[\mathbf{V}_n^{(-N)}, ..., \mathbf{V}_n^{(0)}, ..., \mathbf{V}_n^{(N)}\right], \tag{11}$$

which will be the basis used to project (7). In [7], the system is finite and described by a matrix, making the projection of the operators describing the system straightforward. Since this is not the case here, the notion of projection of the full order system has to be re-thought. The loading and the perturbation being localized on one cell at a time allows to compute the projection at the UC level and to prevent writing the definition of the matrices with infinite operators. Thanks to this, we express the reduced-order model coefficients as a function of the individual UC matrices and displacement vector as follows:

$$\begin{aligned}\hat{D}_{ij} &= \mathbf{V}_j^{(i)*}\Delta D_j \mathbf{u}_j^{(0)},\\ \Delta\hat{D}_{ij} &= \sum_{k=-N}^{N}\mathbf{V}_k^{(i)*}\Delta D_k \mathbf{V}_k^{(j)},\\ \hat{F}_i &= -\sum_{k=-N}^{N}\mathbf{V}_k^{(i)*}\Delta D_k \mathbf{u}_k^{(0)}.\end{aligned} \tag{12}$$

with the superscript $()^*$ denoting the Hermitian transpose. Using these expressions, a reduced-order system can be constructed to compute the components w_n of the perturbation amplitude contribution vector $\mathbf{w}$ of the corrections:

$$\left[\hat{D} + \Delta\hat{D}\right]\mathbf{w} = \hat{\mathbf{F}}, \tag{13}$$

Note that, mathematically, the existence of such a system and its convergence to the solution of interest are not necessarily guaranteed and need to be proved. More derivations on mathematical convergence of the method are required to ensure proper behavior of the projection and will be the subject of a future paper.

5 STUDY OF A METAMATERIAL BEAM WITH VARYING RESONATOR STIFFNESS

To validate the ability of the method developed in section 4 to treat a problem involving metamaterials, the structure defined in section 2 is used as a case study. The properties of the beam used in the study are described in table 1. For the numerical simulation, the waveguide is modeled with 6 elements per UC, with a TVA placed at the UC center. Resonators are spaced with a distance of $L_{UC} = 5cm$. The nominal parameters of the resonators k_{res} and m_{res} are chosen such that m_{res} is 10% of the UC mass and the resonance frequency of the resonators is 400 Hz. Damping is accounted for in the resonators with a structural damping coefficient of $\eta_{tva} = 5\%$. Via a dispersion curve analysis it is predicted that a stopband exists from 400 to 421 Hz, and that the Bragg frequency of the periodic beam is 5502 Hz for this case. The resonators are chosen to deviate from their nominal value by a random factor $\varepsilon p_n k_{res}$ where the p_n are randomly distributed parameters with a continuous, uniform probability distribution defined on the interval $[-1, 1]$ and ε is called the disorder level. The statistics of the studied disordered metamaterial beams are analyzed through the mean and variance of the response over realizations of the beam obtained by sampling the p_n through Monte-Carlo simulations with 1000 samples of each p_n. As a first step, the simulation setup is described. Then a numerical validation of the method is performed. Subsequently, a comparison between a finite structure and the infinite structure is run. Finally, the mean and variance of the response obtained by the simulation are discussed for different levels of disorder.

Table 1: Properties of the beam used in the numerical study

Property	Symbol	Value
Mass density	ρ	$2700\ \mathrm{kg/m^3}$
Young's modulus	E	69 GPa
Height	h	6 mm
Width	b	20 mm
Structural damping coefficient	η	1%

5.1 Numerical validation

To verify that the number of correction vectors accounted for in the reduced-order model is sufficient, an analysis is first performed on a single random realization of the system. The number of UCs in the simulation is gradually increased from 2 to 50 at each side of the forced UC resulting in a simulation of 5 to 101 correction vectors. The results are displayed in figure 2 for 3 different frequencies and number of corrections.

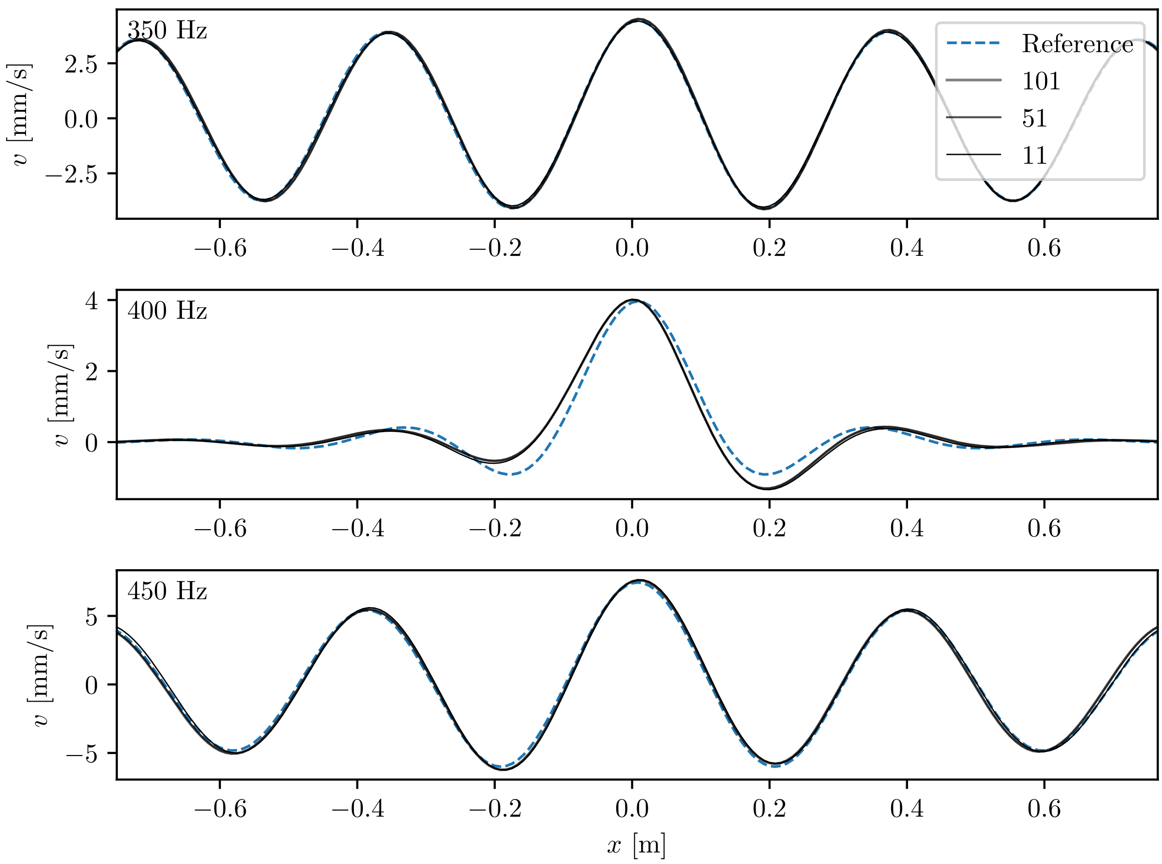

Figure 2: Real part of the velocity field at different frequencies for a single realization of the random metamaterial with different number of corrections accounted for. The number in the legend corresponds to the number of correction vectors accounted for in the reduced-order model.

The velocity field is observed to converge progressively as the number of simulated UCs increases. This means that the contribution of the response due to resonators away from the forcing location becomes less and less important, making it unnecessary to simulate them after a certain point to obtain the desired accuracy. In figure 3, the convergence of the method is assessed by computing the Root-Mean-Square Error (RMSE) between successive correction vector for the real part of the velocity field up to 50 resonators away from the forcing location.

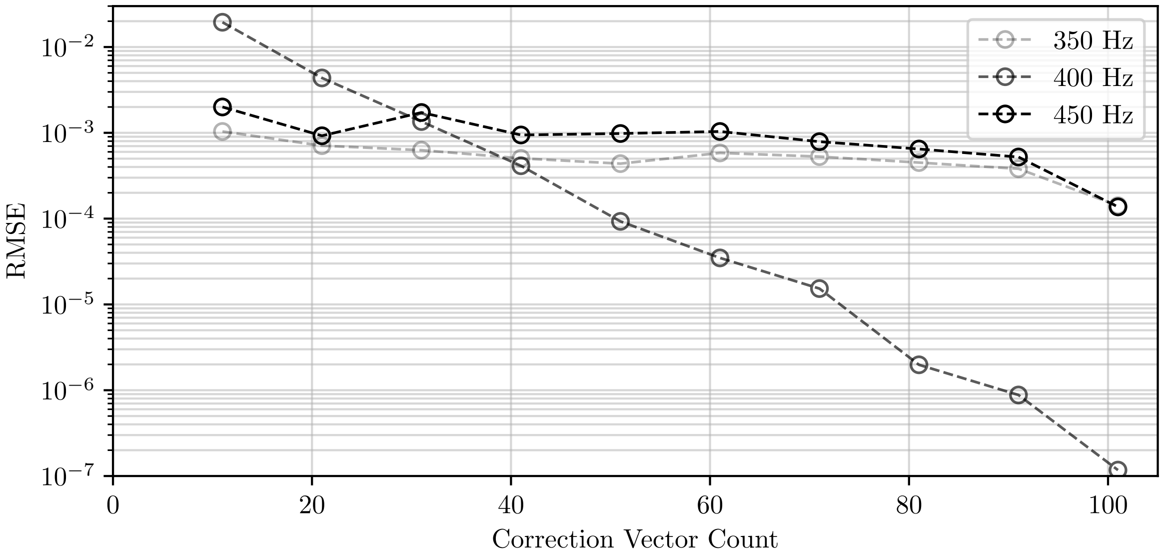

Figure 3: RMSE of the velocity field between successive correction vector counts for different frequencies.

The RMSE of the velocity field is seen to decrease for all frequencies observed, but not at the same speed. Indeed, when the spatial attenuation of the waves becomes smaller, it becomes necessary to use correction vectors corresponding to resonators farther away from the forcing location to represent the system accurately as the spatial extent of the response increases. After this convergence analysis, it is decided that 101 corrections vectors (50 on each side + the forced UC) will be taken into account in the rest of the study to obtain a converged velocity field in the frequency region of attenuation. Outside the attenuation band, a convergence analysis would also be required to verify the behavior of the method when the spatial extent of the response becomes larger. However, such a study is out of the scope of this paper as the vibration attenuation zone of the metamaterial is of interest here.

5.2 Comparison with finite structure statistics

To validate if the infinite structure calculations performed are representative of what would happen with a large finite structure, the response mean is first compared between an infinite beam and its finite counterpart. The finite structure is simulated with clamped boundary conditions at both extremities. It is constituted of 101 UCs and forced on the 51st UC at the basis of the resonator. The results are shown in figure 4 for the average nominal case and for the average of the response.

When comparing the two nominal structures, the finite case response displays modal behavior compared to the infinite case response, which is mostly smooth over the considered frequency range. This is to be expected due to the presence of boundaries creating reflections. However, close to the nominal frequency of the resonators, the finite and infinite system

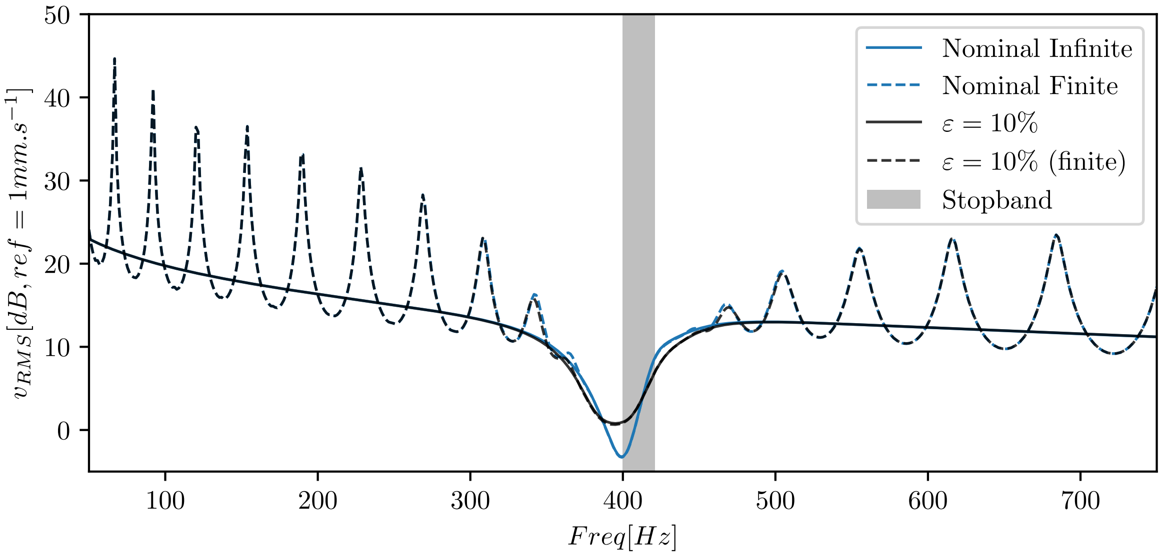

Figure 4: Comparison of the RMS velocity response between a finite and infinite metamaterial beam with and without the presence of variability. The average of the response is represented for a disorder level $\varepsilon = 10\%$ in the disordered case.

responses show a similar trend as the waves generated at the forcing location are attenuated enough so that they do not reach the boundaries of the system anymore. In the case where variability is present in the system, it can be observed that the main deviation to the pure periodic structure case occurs near the resonators' nominal frequency. The attenuation zone is seen to get wider at the cost of reduced attenuation for the mean response in the finite as well as the infinite case. However, the mean response can sometimes not be representative of a single realization, and other statistical indicators are required to provide information on the random behavior of the system. A deeper statistical analysis is then required to understand the statistics of the metamaterial response but is out of the scope of this work. Nevertheless, in the finite as well as the infinite case, the mean response is nearly identical close to 400 Hz, showing the ability of the proposed method to capture the average of a metamaterial response in and around the attenuation region.

5.3 Comparison of different disorder levels

Figure 5 shows the results obtained with the Monte-Carlo simulations of equation (13). The graphs displays the mean of the RMS velocity response for different values of ε. With the amount of variability increasing, a clear trend can be observed for the mean response. The attenuation decreases, and the width of the attenuation zone increases. This can be attributed to the detuning of the resonators that spreads the frequency zone in which they can act but prevents them from acting coherently on the host structure and attenuating its response.

Examining the spread of the responses from figure 5 shows that there is a high variance inside the initial attenuation region. This increase in variance inform us that the performance of the stopband effect might differ for different realizations of the beam, especially when ε is high. These results allow us to conclude that designing of metamaterials that meet the target vibration attenuation characteristics reliably requires precise knowledge about the uncertainty of the resonators applied to the host structure.

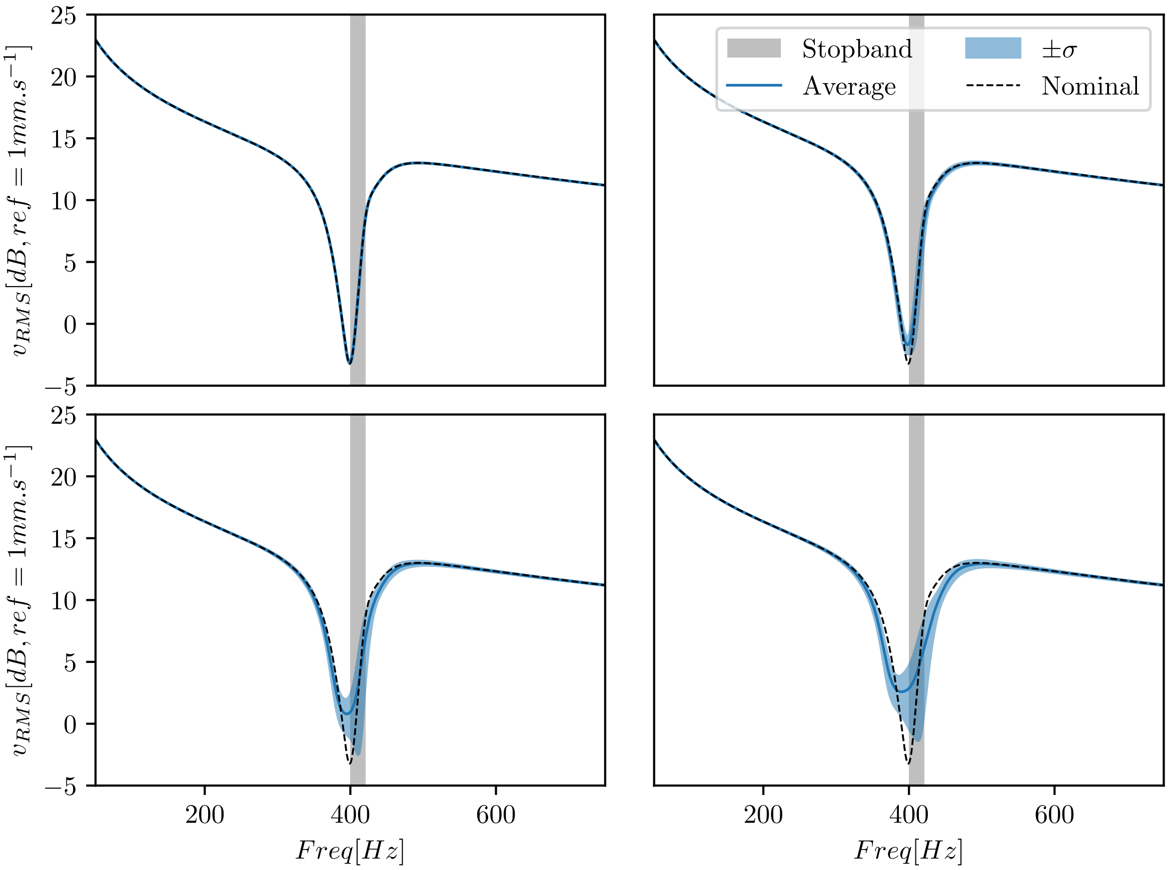

Figure 5: Average of the RMS velocity response of an infinite periodic metamaterial beam for different disorder levels compared to the pure periodic case. The light blue region corresponds to the ± 1 standard deviation. Levels of disorder are $\varepsilon = 1\%$ (top-left), $\varepsilon = 5\%$ (top-right), $\varepsilon = 10\%$ (bottom-left), $\varepsilon = 15\%$ (bottom-right). The gray region highlights the stopband of the periodic metamaterial, obtained by UC modeling.

6 CONCLUSIONS

In this paper, a reduced-order modeling strategy is proposed to analyze the effect of random resonator detuning on an infinite metamaterial structure, providing a method to account for uncertainty effects between the UCs in the design of metamaterials.

This new strategy provides a framework based on UC modeling and Krylov subspaces to construct a reduced-order model. This model can be used to efficiently predict the effect of multiple rank 1 perturbations on the metamaterial response by projection of the UC equations.

A metamaterial beam was used as a case study for the method. The number of correction vectors used in the model was determined by gradually increasing the number of corrections considered in the simulation until convergence. A comparison with a finite structure was also presented to show the ability of the method to represent the behavior of a large structure's mean response. Vibration response statistics were used to show the effect of random stiffness detuning on the attenuation region present due to the metamaterial treatment on the beam. Identifiable effects of randomizing the resonator stiffnesses already reported in the literature were observed in the response computed with the reduced-order model, such as making the vibration attenuation zone wider at the cost of reducing the attenuation provided by the resonator additions.

Even though the method apparently shows numerical convergence, a deeper numerical and mathematical analysis of the method still remains necessary to obtain a rigorous proof that the method indeed converges towards the desired solution.

Extensions of this work will look at more complex variations of the UCs, as the method used here cannot compute an exact correction of the response for perturbation matrices of rank higher than one.

7 ACKNOWLEDGMENTS

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