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INTEGRATED APPROACH FOR ESTIMATING FAILURE PROBABILITIES USING METHOD OF MOMENTS AND CONTROL VARIATES WITH SPLITTING

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Abstract. *Calculating failure probabilities in engineering systems is crucial to ensure acceptable reliability levels and avoid catastrophic consequences. However, traditional methods, such as Monte Carlo Simulation (MCS), often require an excessive number of samples to estimate low failure probabilities, frequently encountered in engineering structures. This paper proposes a novel procedure that combines the strengths of two complementary methods to improve reliability estimation while reducing computational costs. The first method is the Method of Moments (MoM), which estimates failure probabilities using explicit formulas based on statistical moments of the performance function. A key task for MoM is moment estimation, which is usually conducted using a dimension reduction method. The efficiency of this method is influenced by the number of random variables, which may limit its applicability in high-dimensional problems. The second method, Control Variates with Splitting, provides unbiased estimators of statistical moments using fewer samples, improving computational efficiency. This method achieves this by leveraging information from both high- and low-fidelity models. However, Control Variates with Splitting alone is not suited for accurately calculating failure probabilities. The proposed approach exploits the advantages of both methods, focusing on moments up to the fourth order to balance efficiency and accuracy. Its performance is validated by comparing it with the traditional Method of Moments. Results demonstrate that the proposed method achieves accurate failure probability estimates with fewer samples than standard approaches, offering a practical and efficient solution to the limitations of conventional reliability analysis techniques.*

Keywords: Reliability analysis, Failure probability, Control Variates, Method of Moments.

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1 INTRODUCTION

Estimating failure probabilities is vital in engineering for evaluating system reliability under uncertain conditions. This probability, denoted as P_F, is typically computed as an integral over the failure domain, which depends on a performance function $G(\boldsymbol{\theta})$ that defines safe and failure states:

$$P_F = \int_{G(\boldsymbol{\theta})\leq 0} f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\, d\boldsymbol{\theta}, \tag{1}$$

where $\boldsymbol{\theta}$ is the vector of uncertain parameters, $f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ is the joint probability density function (PDF) of $\boldsymbol{\theta}$, and the failure domain is defined by $G(\boldsymbol{\theta}) \leq 0$.

Small failure probabilities, such as those associated with structural failures under extreme loads or fatigue in aerospace components, present computational challenges due to their rarity. Monte Carlo Simulation (MCS) provides unbiased results but requires a large number of samples, making it prohibitive for large-scale or strongly nonlinear models. Variance reduction techniques like Control Variates (CV) have been employed to improve efficiency, though their effectiveness diminishes for extreme events where correlations between models weaken [1, 2].

To address these challenges, alternative approaches have been developed to improve computational efficiency in failure probability estimation. One such approach is the use of surrogate models to approximate expensive performance function evaluations. These models include Kriging [3, 4, 5], polynomial chaos expansion [6, 7], and response surface methods [8, 9], which enable more efficient evaluations, although their accuracy depends on the quality of the surrogate model.

Additionally, active learning strategies have been proposed to further reduce computational cost by adaptively selecting the most informative samples. Recent advancements include Bayesian frameworks that optimize sample selection to enhance accuracy while reducing computational cost [10, 11]. These approaches have demonstrated significant improvements in estimating low failure probabilities.

Beyond surrogate modeling and active learning, probability density-based methods provide an alternative for reliability estimation in stochastic dynamic systems. These approaches aim to approximate the probability distribution of structural responses directly. Recent formulations based on probability density evolution and integral equations have been developed to improve efficiency in nonlinear and high-dimensional problems [12, 13].

Finally, another class of methods focuses on analytical approximations of failure probabilities using statistical moments of the performance function. The Method of Moments (MoM) [14] utilizes statistical moments of the performance function to derive probability estimates. MoM has demonstrated accuracy across various engineering applications [15, 16], yet efficiently computing higher-order moments without introducing errors remains an open problem.

In this work, we propose a novel reliability analysis approach that combines MoM with CV, introducing new estimators for the third and fourth moments of the performance function. By leveraging the variance reduction properties of CV, the proposed method achieves significant computational savings while maintaining high accuracy in reliability assessments.

The paper is structured as follows: first, we provide a review of MoM fundamentals. Then, we introduce the proposed method, demonstrate its effectiveness through a numerical example, and conclude with a summary of key findings.

2 METHOD OF MOMENTS

The Method of Moments (MoM) approximates failure probability by treating the performance function as a random variable and estimating its cumulative distribution at zero [17]. This approach avoids direct integration over the failure domain, making it more feasible for complex systems. The failure probability is approximated as:

$$P_F \approx \Phi(-\beta_{4M}), \tag{2}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution, and β_{4M} is a reliability index obtained from the statistical moments of the performance function $Z = G(\boldsymbol{\theta})$, given by:

$$\beta_{4M} = \sqrt[3]{\Delta + q} - \sqrt[3]{\Delta - q} + \frac{a_3}{3a_4}, \tag{3}$$

where:

$$\Delta = \sqrt{q^2 + p^3}, \tag{4}$$

$$p = \frac{3a_2a_4 - a_3^2}{9a_4^2}, \quad q = \frac{a_3^3}{27a_4^3} - \frac{a_2a_3}{6a_4^2} - \frac{a_3 - \beta_{2M}/a_1}{2a_4}, \quad \beta_{2M} = \frac{\mu_{Z_1}}{\mu_{Z_2}}. \tag{5}$$

Here, β_{2M} represents the second-moment reliability index, and the parameters a_i are calculated as follows:

$$a_2 = 1 - 3a_4, \quad a_3 = \frac{5 + (35 - \mu_{Z_3}^2)a_4^2}{9a_0 + 30 - 0.8\mu_{Z_3}^2}\mu_{Z_3}, \quad a_4 = \frac{2a_0}{2a_0 + 46(1 - 1/\mu_{Z_4}^2) - \mu_{Z_3}^2}, \tag{6}$$

$$a_0 = \frac{\sqrt{3\mu_{Z_4} - 4\mu_{Z_3}^2 - 5} - 2}{1 - (3\mu_{Z_3}^2 + 1)/\mu_{Z_4}^2}, \quad a_1 = \frac{1}{\sqrt{1 + 2a_3^2 + 6a_4^2}}. \tag{7}$$

The statistical moments μ_{Z_1}, μ_{Z_2}, μ_{Z_3}, and μ_{Z_4} characterize the distribution of the performance function Z. Specifically, μ_{Z_1} represents the mean, μ_{Z_2} the standard deviation, μ_{Z_3} the skewness, and μ_{Z_4} the kurtosis. They are computed as follows:

$$\mu_{Z_1} = E_{Z_1}, \tag{8}$$

$$\mu_{Z_2} = \sqrt{E_{Z_2} - E_{Z_1}^2}, \tag{9}$$

$$\mu_{Z_3} = \frac{1}{\mu_{Z_2}^3}\left(E_{Z_3} - 3E_{Z_1}E_{Z_2} + 2E_{Z_1}^3\right), \tag{10}$$

$$\mu_{Z_4} = \frac{1}{\mu_{Z_2}^4}\left(E_{Z_4} - 4E_{Z_1}E_{Z_3} + 6E_{Z_1}^2E_{Z_2} - 3E_{Z_1}^4\right). \tag{11}$$

The expected values E_{Z_j} correspond to the first four raw moments of Z and are defined as:

$$E_{Z_j} = \mathbb{E}\left[[G(\boldsymbol{\Theta})]^j\right] = \int_{\Omega_{\boldsymbol{\Theta}}} [G(\boldsymbol{\theta})]^j f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta}, \tag{12}$$

where $\mathbb{E}[\cdot]$ denotes the expectation operator, which computes the expected value of a function of the random variable $\boldsymbol{\Theta}$ by integrating over its probability density function. The domain $\Omega_{\boldsymbol{\Theta}}$ represents the entire space of possible values for the random parameter vector $\boldsymbol{\Theta}$.

The estimation of the moments given in Equation (12) remains a challenge, as direct computation is often impractical for complex models. To address this, dimensional reduction techniques have been introduced [18, 19, 20] to reduce computational cost. However, these methods do not provide variance estimates for the moments, leaving the uncertainty level of the calculated failure probability unknown. While Monte Carlo Simulation (MCS) offers a way to estimate moment variances, it requires a large number of samples, making it computationally prohibitive. To mitigate this, Control Variates (CV) are introduced as a variance reduction technique, which will be discussed in Section 3.

3 VARIANCE-REDUCED MOMENT ESTIMATION VIA CONTROL VARIATES

3.1 Control Variates

Control Variates (CV) is a technique designed to reduce the variance of an estimator [21]. By introducing a secondary model, referred to as the low-fidelity model, CV allows for efficient variance reduction while retaining the statistical properties of the estimator. The high-fidelity model provides accurate results but at a significant computational cost, as it typically involves detailed numerical simulations, such as finite element methods. In contrast, the low-fidelity model offers approximate results with reduced computational demands, often achieved through coarser discretizations or simplified numerical methods.

The combination of these two models enables the estimator to leverage the computational efficiency of the low-fidelity model while maintaining the accuracy of the high-fidelity model. Variance minimization in CV depends on maximizing the correlation between the two models and minimizing the variance of the low-fidelity estimator. This approach is particularly valuable for reducing computational costs in scenarios where directly increasing sample size is infeasible due to high computational expense. Furthermore, the method proves useful for estimating statistical moments, such as those required in the Method of Moments for failure probability calculations.

To incorporate this variance reduction strategy into the moment estimation process, the CV estimator is applied to the raw moment computation of the performance function $G(\boldsymbol{\Theta})$. This is achieved by adding and subtracting a conveniently chosen term, ensuring that the expected value remains unchanged while reducing variance. The modified estimation is expressed as follows [21]:

$$[G(\boldsymbol{\Theta})]^j = [G(\boldsymbol{\Theta})]^j - \rho_j \left([L(\boldsymbol{\Theta})]^j - [L(\boldsymbol{\Theta})]^j\right), \tag{13}$$

where $L(\boldsymbol{\Theta})$ is a low-fidelity approximation of $G(\boldsymbol{\Theta})$, and ρ_j is the optimal control parameter that minimizes variance. The expression for ρ_j will be introduced later in this section.

Using this formulation, the raw moments of $G(\boldsymbol{\Theta})$ are computed as:

$$E_{Z_j} = \int_{\Omega_{\boldsymbol{\Theta}}} [G(\boldsymbol{\theta})]^j f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta} + \rho_j \int_{\Omega_{\boldsymbol{\Theta}}} \left([L(\boldsymbol{\theta})]^j - [L(\boldsymbol{\theta})]^j\right) f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta} \tag{14}$$

Since direct evaluation of these integrals is impractical for complex systems, the moment estimate is obtained using two sets of samples through a Control Variates estimator, as follows:

$$\widehat{E}_{Z_j}^{\mathrm{CV}} = H\left(G^j, \boldsymbol{\Theta}_n\right) - \rho_j\left[H\left(L^j, \boldsymbol{\Theta}_n\right) - H\left(L^j, \boldsymbol{\Theta}_m\right)\right], \tag{15}$$

where $H(\cdot, \boldsymbol{\Theta})$ denotes the empirical mean computed over the given sample set, with $\boldsymbol{\Theta}_n$ and $\boldsymbol{\Theta}_m$ representing two independent sets of samples. Note that the conveniently added zero in the previous formulation is now utilized, as the estimators associated with the low-fidelity model are computed using different sample groups, ensuring the decorrelation required for variance reduction.

The variance of the control variates estimator for the raw moment is given by:

$$\begin{aligned}\mathbb{V}\left[\widehat{E}_{Z_j}^{\mathrm{CV}}\right] = \mathbb{V}\left[H(G^j, \boldsymbol{\Theta}_n)\right] - 2\rho_j\mathbb{C}\left[H(G^j, \boldsymbol{\Theta}_n), H(L^j, \boldsymbol{\Theta}_n)\right] \\ +\rho_j^2\left\{\mathbb{V}\left[H(L^j, \boldsymbol{\Theta}_n)\right] + \mathbb{V}\left[H(L^j, \boldsymbol{\Theta}_m)\right]\right\}.\end{aligned} \tag{16}$$

In Equation (16), $\mathbb{C}[X, Y]$ denotes the covariance between the random variables X and Y, while $\mathbb{V}[X]$ represents the variance of X.

Minimizing Equation (16) with respect to ρ_j yields the optimal control parameter [21]:

$$\rho_{j,\mathrm{opt}} = \frac{\mathbb{C}\left[H(G^j, \boldsymbol{\Theta}_n), H(L^j, \boldsymbol{\Theta}_n)\right]}{\mathbb{V}\left[H(L^j, \boldsymbol{\Theta}_n)\right] + \mathbb{V}\left[H(L^j, \boldsymbol{\Theta}_m)\right]}. \tag{17}$$

Substituting this optimal parameter into the variance expression, the minimum achievable variance using the control variates approach is:

$$\mathbb{V}\left[\widehat{E}_{Z_j}^{\mathrm{CV}}\right]_{\min} = \mathbb{V}\left[H(G^j, \boldsymbol{\Theta}_n)\right] - \frac{\left(\mathbb{C}\left[H(G^j, \boldsymbol{\Theta}_n), H(L^j, \boldsymbol{\Theta}_n)\right]\right)^2}{\mathbb{V}\left[H(L^j, \boldsymbol{\Theta}_n)\right] + \mathbb{V}\left[H(L^j, \boldsymbol{\Theta}_m)\right]}. \tag{18}$$

This result quantifies the variance reduction achieved through the control variates method, demonstrating that the variance of the moment estimates is reduced as long as a strong correlation exists between the high- and low-fidelity models.

While Control Variates (CV) estimators are highly effective for variance reduction, they may exhibit bias under certain conditions. This bias is influenced by the dependency of the optimal control parameter on the sample sets used for estimation and its correlation with the low-fidelity model outputs [22, 23]. Specifically, the covariance between the control parameter and the difference in low-fidelity estimators evaluated with shared samples can introduce deviations. To further improve the accuracy of CV estimators, strategies for mitigating bias must be considered, as discussed next.

3.2 Splitting Technique

The splitting technique addresses the bias inherent in traditional Control Variates (CV) estimators by dividing samples into independent groups [24, 25]. This approach ensures that the calculation of the control parameter and the statistics for low-fidelity responses remain uncorrelated. By restructuring samples into three groups, the method achieves unbiased estimates without requiring additional simulations. The splitting-based estimator is constructed as:

$$\widehat{E}_{Z_j}^{\mathrm{CV\text{-}S}} = \frac{1}{3}\sum_{i=1}^{3}\widehat{E}_{Z_j}^{\mathrm{CV},i}, \tag{19}$$

where $\widehat{E}_{Z_j}^{\text{CV-S}}$ is the unbiased estimator using the splitting technique, and $\widehat{E}_{Z_j}^{\text{CV},i}$ represents the Control Variates estimator computed for each independent group i.

Each group-specific estimator is calculated as:

$$\widehat{E}_{Z_j}^{\text{CV},i} = H\left(G^j, \boldsymbol{\Theta}_{i,n^*}\right) - \rho_{j,\text{opt}}^{\tau(i)}\left(H\left(L^j, \boldsymbol{\Theta}_{i,n^*}\right) - H\left(L^j, \boldsymbol{\Theta}_{i,m^*}\right)\right), \tag{20}$$

where $\rho_{j,\text{opt}}^{\tau(i)}$ is the optimal control parameter that minimizes variance and ensures that the control variate correction remains unbiased by leveraging the independent group index $\tau(i)$. This index assigns independent sets of samples, preventing correlation with the low-fidelity statistics and thereby eliminating bias. The expression for $\rho_{j,\text{opt}}^{\tau(i)}$ is:

$$\rho_{j,\text{opt}}^{\tau(i)} = \frac{\mathbb{C}\left[H(G^j, \boldsymbol{\Theta}_{\tau(i),n^*}), H(L^j, \boldsymbol{\Theta}_{\tau(i),n^*})\right]}{\mathbb{V}\left[H(L^j, \boldsymbol{\Theta}_{\tau(i),n^*})\right] + \mathbb{V}\left[H(L^j, \boldsymbol{\Theta}_{\tau(i),m^*})\right]}. \tag{21}$$

To implement this approach, the total number of samples is divided into three independent groups. The number of shared samples per group is defined as $n^* = n/3$, while the number of independent samples per group is $m^* = m/3$. The estimators and control parameter in each group are structured as follows:

- $H\left(G^j, \boldsymbol{\Theta}_{i,n^*}\right)$ is the high-fidelity estimator computed using n^* shared samples in group i.
- $H\left(L^j, \boldsymbol{\Theta}_{i,n^*}\right)$ is the low-fidelity estimator computed using the same n^* shared samples in group i.
- $H\left(L^j, \boldsymbol{\Theta}_{i,m^*}\right)$ is the low-fidelity estimator computed using m^* independent samples in group i.

The minimum variance of the splitting-based estimator (19) is given by:

$$\mathbb{V}\left[\widehat{E}_{Z_j}^{\text{CV-S}}\right] = \frac{1}{3^2}\sum_{i=1}^{3}\mathbb{V}\left[\widehat{E}_{Z_j}^{\text{CV},i}\right]. \tag{22}$$

Each variance term inside the summation follows the structure of the variance expression given in Equation (16), but using the corresponding optimal control parameter $\rho_{j,\text{opt}}^{\tau(i)}$, which is obtained independently for each group to maintain unbiased estimation. This ensures that the decorrelation required for variance reduction is effectively applied across the three groups.

This method eliminates bias while improving computational efficiency by leveraging existing samples effectively. It ensures robust variance reduction while maintaining accuracy in statistical moment estimation.

With the moments estimated using Control Variates with Splitting, the Method of Moments (MoM) can then be applied to compute failure probabilities. By integrating these refined moment estimates into MoM, the proposed approach balances efficiency and accuracy, overcoming the computational limitations of traditional moment estimation techniques. This enables a practical and scalable framework for reliability analysis, ensuring precise probability estimates with significantly reduced sample requirements.

4 EXAMPLE

This example investigates a univariate polynomial performance function:

$$G(\theta) \quad = \quad k_1\theta^3 + k_2\theta^2 + k_3\theta + k_4, \tag{23}$$

where θ follows a Gamma distribution with mean $\mu_\theta = 1.0$ and coefficient of variation (COV) of 0.2. The polynomial coefficients k_1, k_2, k_3, k_4 determine the shape of the performance function and influence failure probability estimates. In this example, the values are set as $k_1 = 1.0$, $k_2 = -6.0$, $k_3 = 18$, and $k_4 = -8.0$. The analytical solution provides exact statistical moments $\boldsymbol{\mu}_Z = (4.8832, 1.7852, 0.0188, 2.9354)$ and a failure probability of 2.5271×10^{-3}.

To implement the proposed method, a low-fidelity model is constructed using a second-order Taylor expansion around μ_θ, approximating the original performance function. The method then generates $12,000$ samples for the high-fidelity model and 3×10^6 samples for the low-fidelity model. Applying Control Variates with Splitting, the estimated moments are $(4.8825, 1.7848, 0.0166, 2.9240)$, which are then used in the Method of Moments to compute the failure probability. The result, 2.5632×10^{-3}, shows a relative error of 1.43%, confirming the accuracy of the proposed approach.

To further assess the robustness of the method, its performance is evaluated across different probability distributions. The influence of the input distribution on failure probability estimation is examined for Normal, Lognormal, Gamma, Weibull, and Gumbel distributions, with COV values ranging from 0.1 to 0.5. The failure probability P_F is analyzed as a function of the mean value of the input variable, μ_θ, as shown in Figure 1. The results illustrate how different distributions impact the accuracy of the proposed method.

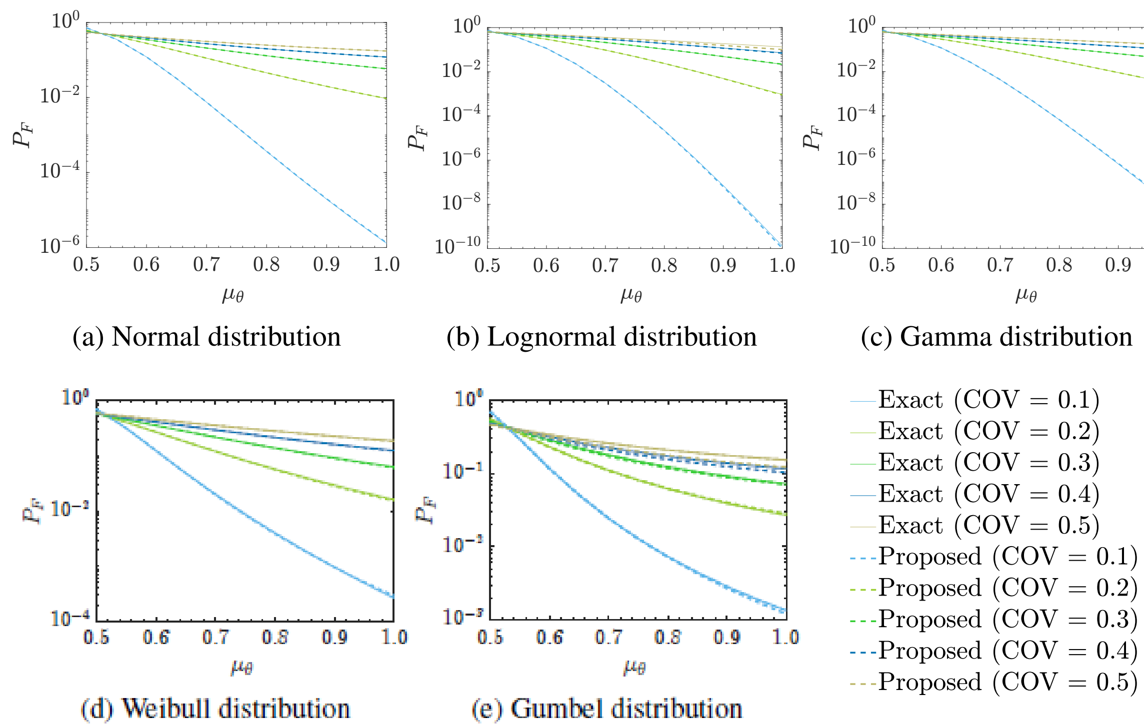

Figure 1: Change of failure probability with mean value of θ.

From Figure 1, the following observations can be made:

- The method accurately estimates failure probabilities for Normal, Gamma, and Weibull distributions across all COV values.
- For strongly non-Gaussian distributions (Lognormal and Gumbel), deviations appear at higher COV levels (> 0.4), as seen in Figure 1b and Figure 1e. This highlights the limitations of MoM under strong non-Gaussianity.

Overall, these results demonstrate that Control Variates with Splitting effectively reduces variance, leading to improved moment estimation and enhancing the reliability of the Method of Moments. The proposed approach shows promise for univariate problems and can be extended to more complex systems.

5 CONCLUSIONS

A reliability analysis method is proposed, combining the Method of Moments (MoM) and Control Variates with Splitting (CV-S). This approach introduces unbiased estimators for the third and fourth raw moments of the model response while reducing variance. The accuracy of the method depends on the non-Gaussianity of the performance function, with reliable estimates achieved for weakly non-Gaussian cases, where skewness and kurtosis remain close to 0 and 3, respectively. In these cases, the method provides accurate failure probability estimations, as demonstrated in the polynomial example. However, for strongly non-Gaussian distributions, discrepancies arise, particularly at higher COV levels, highlighting a key limitation of MoM in capturing complex distributions. The efficiency of the method primarily relies on the accuracy of the low-fidelity model. Nonetheless, its performance may be affected when the first four moments are insufficient to characterize the distribution of the performance function, as observed in cases with strong non-Gaussianity.

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