Available online at www.eccomasproceedia.org
Eccomas Proceedia COMPDYN (2025) 5486-549

ECCOMAS
Proceedia

COMPDYN 2025
10th ECCOMAS Thematic Conference on
Computational Methods in Structural Dynamics and Earthquake Engineering
M. Papadrakakis, M. Fragiadakis (eds.)
Rhodes Island, Greece, 15-18 June 2025

IMPACT OF A TWO-PARAMETER ELASTIC FOUNDATION ON EDGE WAVE PROPAGATION IN AN ORTHOTROPIC LAYER

Saad Althobaiti

Department of Sciences and Technology, Ranyah University College, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia
e-mail: snthobaiti@tu.edu.sa

Abstract. *This paper examines the elastodynamics of a propagating edge wave on a semi-infinite orthotropic layer under the influence of a two-parameter foundation. The dispersion relations of the stated edge problem amidst the influence of the Pasternak's and Hetenyi's elastic foundations have been analyzed; the two famous members of the two-parameter foundation family. Moreover, the dynamic response of the medium to the variation of these involving parameters of both elastic foundations has been captured, including the full examination of the respective cutoff and static expressions. Lastly, the study ends with the analysis of the phase and group velocities in the beseeched elastic foundations.*

Keywords: Orthotropic medium; edge wave; two-parameter foundation; elastodynamics; dispersion analysis.

ISSN:2623-3347 © 2025 The Authors. Published by Eccomas Proceedia.
Peer-review under responsibility of the organizing committee of COMPDYN 2025.
doi: 10.7712/120125.12827.25952

1 INTRODUCTION

Propagation of edge waves in elastic plates has been studied in both the past and recent literature owing to its vast applications in elastodynamics [1, 2]. Various studies have been carried out concerning isotropic, anisotropic, and orthotropic materials, among others [3]. In addition, some of the recent studies examined the relevance of edge wave propagations in plates resting over elastic foundations, including the impact of a two-parameter elastic beneath an isotropic plate [4], and that of an orthotropic bar supported by the classical Winkler's elastic foundation [5] to mention a few.

However, due to the availability of vast literature concerning the elastic foundations [6, 7, 8], mostly in engineering fields, in addition to their untapped relevance, the present study examines the dispersion of edge waves, propagating on a semi-infinite orthotropic plate while rested over two-parameter foundations, the so-called Pasternak's and Hetenyi's elastic foundations [9]. More precisely, these foundations feature the Laplacian operator, which further compounds the wave behavior in the media. An analytical approach will be utilized to obtain the consequential characteristic equations in each case, in addition to the acquisition and analysis of the resulting phase and group velocities. Lastly, the study intends to analyze the impact of all the foundational parameters on the dispersion of waves in the medium.

sectionProblem formulation

The equation of motion for a homogeneous elastic orthotropic layer over the region $-\infty < x < \infty$, $0 \leq y < \infty$, and $-h \leq z \leq h$, in the presence of contact pressure or forcing term $p(x, y, t)$ is expressed as [1, 2, 3]

$$D_x u_{xxxx} + 2H u_{xxyy} + D_y u_{yyyy} + 2\rho h u_{tt} + p(x, y, t) = 0, \tag{1}$$

where $u = (x, y, t)$ is the flexural displacement, D_x and D_y are the respective bending stiffnesses along x, y, directions, sequentially, $H = 2D_{xy} + D_1$, h is the half-thickness of the layer, while ρ is the density of the material. Notably, upon considering $p(x, y, t) = 0$, one recovered the unforced Kirchhoff equation of motion. In addition, on the edge of the structure, see Figure 1 for the geometry of the plate that is adopted from Althobaiti et al. [5]; at $y = 0$, the following boundary conditions are imposed when no modified shear force and moment are presumed

$$D_1 u_{xx} + D_y u_{yy} = 0, \quad \text{and} \qquad (4D_{xy} + D_1)u_{xxy} + D_y u_{yyy} = 0, \quad \text{at} \qquad y = 0. \tag{2}$$

Moreover, in connection to the forcing term $p(x, y, t)$, this study examines the case of two

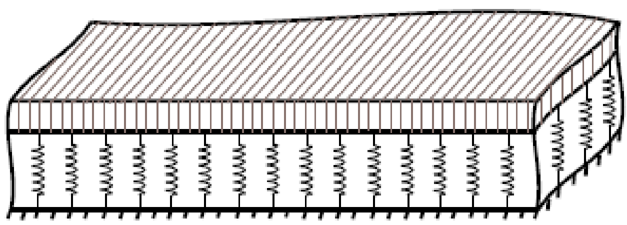


Figure 1: Elastic plate underlying a Winkler's foundation (figure adopted from Althobaiti et al. [5]).

dissimilar two-parameter elastic foundations occupying the region $-\infty < x < \infty$, $0 \leq y < \infty$, and $2h \leq z < \infty$, upon which the governing layer is rested over as follows:

I). The Pasternak elastic foundation [9] when one expresses the function $p(x, y, t)$ as follows

$$p(x, y, t) = \beta u(x, y, t) - \gamma \nabla^2 u(x, y, t), \tag{3}$$

where β is the stiffness of the Winkler's elastic foundation, while γ is the shear modulus parameter for the Pasternak foundation. In addition, ∇^2 is the Laplacian operator expressed as follows $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

II). The Hetenyi elastic foundation [9], where the function $p(x, y, t)$ takes the following expression

$$p(x, y, t) = \beta u(x, y, t) - \gamma \nabla^4 u(x, y, t), \tag{4}$$

where β is the stiffness of the Winkler's elastic foundation, and γ is a flexural parameter of the Hetenyi elastic foundation. Obviously, upon setting the parameter γ to zero in the Eqs. (3)-(4), one obtains the famous Winkler's model [3, 5] that admits the characteristic equation, upon suppressing the effects of Winkler's foundation as follows

$$K^4 = \Omega^2 - 1, \tag{5}$$

where K and Ω are the dimensionless wavenumber and frequency, respectively, which take the following expressions $K = kc\sqrt[4]{\frac{D_x}{\beta}}$ and $\Omega = \omega\sqrt{\frac{2\rho h}{\beta}}$, with c as a dimensionless quality take takes the following expression

$$c^4 = 1 - \frac{\left(\sqrt{D_1^2 + 4D_{xy}^2} - 2D_{xy}\right)^2}{D_x D_y}, \tag{6}$$

the main result by Norries [1] when no elastic foundation was considered. In addition, since we care about the impact of the underlying elastic foundations in Eqs. (3)-(4) on the dispersion of edge waves in the current study, the reported characteristic equation in (5) is now re-expressed to feature the presence of Winkler's elastic parameter as follows

$$K^4 = \Omega^2 - W, \tag{7}$$

with $K = kc\sqrt[4]{h}$ and $\Omega = h\omega\sqrt{\frac{2\rho}{D_x}}$, and $W = \frac{\beta h}{D_x}$; where K and Ω are the dimensionless wavenumber and frequency, respectively, while W is the dimensionless stiffness of the Winkler's elastic foundation. Moreover, setting $W = 1$, in Eq. (7) one recovers Eq. (5).

2 Characteristic equation under Pasternak foundation

Under the consideration of the Pasternak foundation resting beneath the governing plate, the bending edge wave is then analyzed through the acquisition of the consequential characteristic equation. In this case, the following harmonic solution is assumed [5]

$$u(x, y, t) = Ae^{i(kx - \omega t) - k\lambda y}, \tag{8}$$

where A, k, and ω are the waves' amplitude, wavenumber and the frequency, respectively, while λ is the attenuation parameter, with $\Re(\lambda) > 0$; ensuring a bounded solution along the depth of the underlying foundation that is placed at $y = 0$. Accordingly, substituting Eq. (8) into Eq. (1) yields as follows

$$\lambda^4 - \left(\frac{2H}{D_y} + \frac{p}{D_y}\right)\lambda^2 + \frac{\beta + \gamma k^2 - 2h\rho\omega^2 + k^4 D_x}{k^4 D_y} = 0, \tag{9}$$

where $p = \frac{\gamma}{k^2}$ and the roots of the latter equation λ_1 and λ_2 satisfy

$$\lambda_1^2 + \lambda_2^2 = \frac{2H}{D_y} + \frac{\gamma}{k^2 D_y}, \qquad \lambda_1^2\lambda_2^2 = \frac{\beta + k^4 D_x - 2h\rho\omega^2 + \gamma k^2}{k^4 D_y}. \tag{10}$$

Certainly, assuming the deflection $u(x,y,t)$ to admit the following bounded wave solution

$$u(x,y,t) = A_1 e^{i(kx-\omega t)-k\lambda_1 y} + A_2 e^{i(kx-\omega t)-k\lambda_2 y}, \tag{11}$$

where A_1 and A_2 are constants, k, and ω are the dimensional wavenumber and the frequency, respectively, while λ_1 and λ_2 are attenuation parameters determined from Eq. (1) more explicitly as follows

$$\lambda_j = \sqrt{\frac{H}{D_y} + \frac{p}{D_y} + \frac{1}{2}(-1)^j \kappa}, \qquad j = 1, 2, \qquad \kappa = 2\sqrt{\frac{H^2}{D_y^2} - \frac{D_x}{D_y}\left(1 - c^4\right)}. \tag{12}$$

Furthermore, with the help of the imposed edge boundary conditions at $y = 0$ in Eq. (2), the following characteristic equation is thus obtained

$$\lambda_1^2 \lambda_2^2 D_y^2 + 4\lambda_1 \lambda_2 D_{xy} D_y - D_1\left(\lambda_1^2 + \lambda_2^2\right) D_y + 4 D_1 D_{xy} + D_1^2 = 0, \tag{13}$$

which relations in Eq. (10) triggers the acquisition of the overall characteristic equation as follows

$$\gamma\left(\gamma + 4Hk^2\right) - 4D_y\left(\beta - 2h\rho\omega^2 + \gamma k^2\right) = 4c^4 k^4 D_x D_y, \tag{14}$$

where the positive value of c portrays

$$c^4 = 1 - \frac{\left(\sqrt{D_1^2\left(P_1 + 1\right) + 4D_{xy}^2} - 2D_{xy}\right)^2}{D_x D_y} + P_y\left(\frac{H}{D_x} + \frac{P_x}{4}\right), \tag{15}$$

In addition, P_1, P_x and P_y emerged due to the presence of Pasternak foundation, which are further scaled and expressed as follows $P_1 = \frac{p}{D_1}$, $P_x = \frac{p}{D_x}$, and $P_y = \frac{p}{D_y}$. Moreover, when $P_1 = P_x = P_y = 0$ in Eq. (15), one obtains Eq. (6). However, accounting for the effects of Winkler's elastic foundation as in Eq. (7), alongside featuring the action of the Pasternak foundation, the characteristic equation in Eq. (14) is non-dimensionalized as follows

$$K^8 = K^4\left(\Omega^2 - W\right) + P\left(K^4 \chi_2 + P\chi_1\right), \tag{16}$$

where K, Ω, W, P are the dimensionless wavenumber, frequency, stiffness of the Winkler's foundation, and the elastic moduli of the Pasternak foundation, respectively, as follows

$$K = kc\sqrt[4]{h}, \quad \Omega = h\omega\sqrt{\frac{2\rho}{D_x}}, \quad W = \frac{\beta h}{D_x}, \quad P = \frac{\gamma h k^2}{D_x}, \quad \chi_1 = \frac{c^4 D_x}{4D_y}, \quad \chi_2 = \frac{H}{D_y} - 1. \tag{17}$$

Accordingly, the equation of cut-off frequency and static equation can be obtained by respectively setting $K = 0$ and $\Omega = 0$. In addition, one obtains the resulting phase velocity V^{ph} and the group velocity V^g from Eq. (16) as follows

$$V^{ph} = \frac{\Omega}{K} = \frac{\sqrt[4]{2}\Omega}{\sqrt[4]{\Omega^2 - W + R_1}}, \tag{18}$$

and

$$V^g = \frac{d\Omega}{dK} = \frac{2\, 2^{3/4}\left(P^2\chi_1 + \frac{1}{4}\left(\Omega^2 - W + R_1\right)^2\right)}{R_2\left(\Omega^2 - W + R_1\right)^{3/4}}, \tag{19}$$

where R_1 and R_2 are expressed as follows $R_1 = P\chi_2 + \sqrt{4P^2\chi_1 + \left(\Omega^2 - W + P\chi_2\right)^2}$, and $R_2 = \sqrt{R_3 - P^2\chi_1}$, with R_3 appearing in R_2 expressed as follows
$R_3 = \frac{1}{4}\left(\Omega^2 - W + R_2\right)\left(R_2 + W + \Omega^2 - 2P\chi_2\right)$.

3 Characteristic equation under Hetenyi elastic foundation

Accordingly, with the consideration of the Hetenyi elastic foundation earlier expressed in Eq. 4, the equation of motion in Eq. (1) upon deploying Eq. (8) yields

$$\lambda^4 - 2\left(\frac{H-\gamma}{D_y-\gamma}\right)\lambda^2 + \frac{\beta - \gamma k^4 - 2h\rho\omega^2 + k^4 D_x}{k^4(D_y-\gamma)} = 0, \tag{20}$$

with the roots $\lambda_1$and λ_2 satisfying

$$\lambda_1^2 + \lambda_2^2 = 2\left(\frac{H-\gamma}{D_y-\gamma}\right), \qquad \lambda_1^2\lambda_2^2 = \frac{\beta - \gamma k^4 - 2h\rho\omega^2 + k^4 D_x}{k^4(D_y-\gamma)}. \tag{21}$$

Accordingly, deploying the solution in Eq. (11), one further obtains the explicit expressions for the related roots as follows

$$\lambda_j = \sqrt{\frac{H}{D_y}\left(\frac{1+Q_H}{1-Q_y}\right) + \frac{1}{2}(-1)^j\kappa}, \quad j=1,2, \quad \kappa = 2\sqrt{\frac{H^2}{D_y^2}\left(\frac{1-Q_H}{1-Qy}\right)^2 - \frac{D_x}{D_y}\left(\frac{1}{1-Qy}\right)(1-c^4)}, \tag{22}$$

where $Q_y = \frac{\gamma}{D_y}$, and $Q_H = \frac{\gamma}{H}$; recall that $H = D_1 + 2D_{xy}$.

Accordingly, as in the previous subsection, one obtains the resulting characteristic equation as follows

$$2h\rho\omega^2 - \beta + \gamma k^4 = 4c^4k^4D_x, \tag{23}$$

where

$$c^4 = 1 - \frac{\left(\sqrt{2D_1\left(H - D_1Q_1\right) + 4D_{xy}^2 - 4D_1D_{xy} - D_1^2} - 2D_{xy}\sqrt{1-Q_y}\right)^2}{D_xD_y}. \tag{24}$$

In addition, in the absence of Hetenyi's foundation, one sets $Q_1 = Q_y = 0$, where one equally recovers Eq. (6) for both the isotropic and orthotropic mediums. Additionally, the consequential dimensionless characteristic equation is obtained from Eq. (23) as follows

$$K^4(1-Q) = \Omega^2 - W, \tag{25}$$

where K, Ω and W are the dimensionless wavenumber, frequency, and the stiffness of Winkler's foundation, earlier expressed in Eq. (16), while Q is the dimensionless stiffness of Hetenyi's foundation given as

$$Q = \frac{\gamma}{c^4D_x}. \tag{26}$$

Notably, when $Q = 0$ in Eq. (25), one obtains Eq. (6). What is more, the related phase velocity V^{ps} and group velocity V^g are equally obtained as follows

$$V^{ph} = \frac{\Omega\sqrt[4]{1-Q}}{\sqrt[4]{\Omega^2 - W}}, \tag{27}$$

and

$$V^g = \frac{2\sqrt[4]{\left(\Omega^2 - W\right)^3}}{\sqrt[4]{1-Q}\sqrt{\Omega^2 - QW}}. \tag{28}$$

4 Discussion

This section graphically examines the obtained expressions for the respective phase and group velocities with regard to the consideration of Pasternak's and Hetenyi's foundations. Notably, the expressions reported in (18) and (19), and on the other hand, (27) and (28) eventually reduce to those expressions determined by Althobaiti et al. [5] upon disregarding the elastic supports by Pasternak and Hetenyi, respectively. That is, respectively setting $P = 0$ and $Q = 0$ in the said equations, while considering the stiffness of the Winkler's foundation to be unity, that is, $W = 1$; just to coincides with the referenced characteristic equation in Eq. (5) - moreover, Figure 2 portrays the acquired phase and group velocities with two-parameter foundations while setting $W = 1$ and disregarding the second parameter; affirming the already mentioned results. Notably, the two velocity curves coincide when $\Omega = \sqrt{2}$.

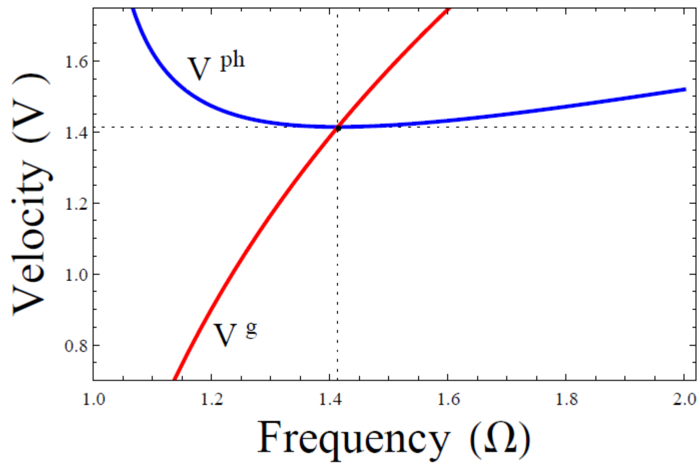
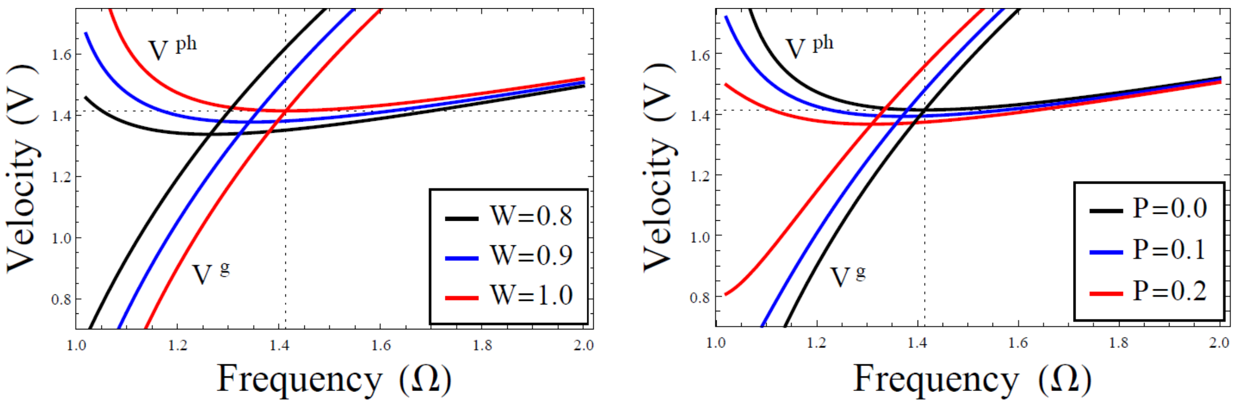


Figure 2: The phase V^{ph} and group V^{g} velocities in both cases against Ω when $W = 1$.

4.1 Impact of Pasternak's elastic foundation

Having accounted for the impact of both the Winkler's and Pasternak's parameters W and P in the acquired phase and group velocities in (18) and (19), it is evident that in the absence of Pasternak parameter P and setting $W = 1$ one acquires the results of Althobaiti et al. [5], where the two velocities are noted to coincide, and further have local a minimum when $\Omega = \sqrt{2}$. Thus, to examine the impact of the foundation parameters, Figure 3 (a) studies the variational impact of dimensionless stiffness of the Winkler's foundation W on the phase and group velocities while disregarding Pasternak's parameter presence, that is, when $P = 0$. Indeed, one notes that both velocities steadily increase with an increase in W. In addition, analyzing the impact of the elastic modulus of Pasternak's foundation, P, Figure 3 (b) captures the scenario after setting $W = 1$, $\chi_1 = 0.35$ and $\chi_2 = 0.55$. In particular, one notes an opposite trend, where an increase in the Pasternak parameter decreases the two velocities.



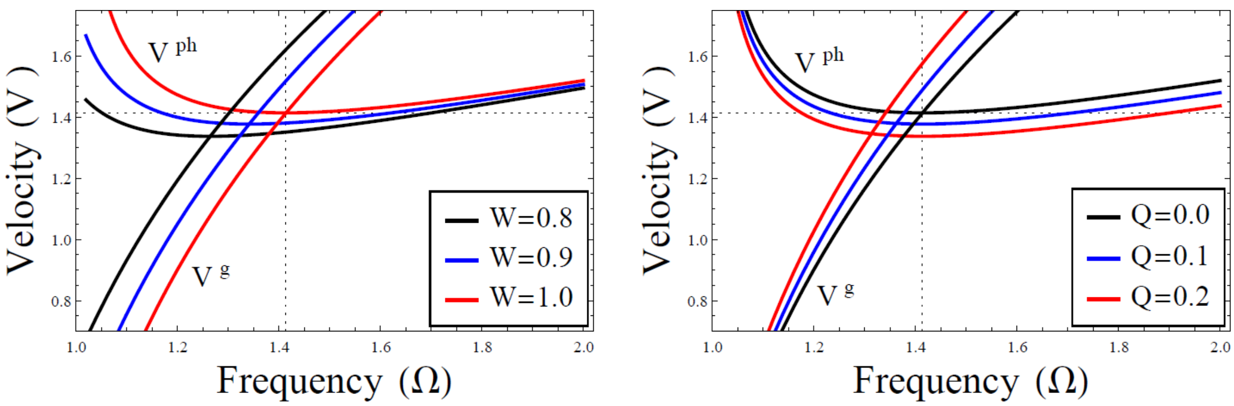
(a) Impact of Winkler's foundation when $P = 0$.

(b) Impact of Pasternak's foundation when $W = 1$.

Figure 3: Variation of the phase V^{ph} and group V^{g} velocities (18) and (19) against the frequency Ω.

4.2 Impact of Hetenyi's elastic foundation

Figure 4 equally studies the variational impact of dimensionless stiffness of the Winkler's foundation W on the phase and group velocities, (27) and (28) while disregarding the presence of the Hetenyi's parameter, that is, when $Q = 0$. In this regard, the discussion is the same with that of Figure 3 since both models coincide by disregarding the second parameters, P and Q; the phase velocity admits a local minimum $V^{ph} = \sqrt{1+W}\sqrt[4]{1-Q}$ at $\Omega = \sqrt{1+W}$ - coinciding with the results by Althobaiti et al. [5] when $W = 1$ and $Q = 0$. Further, Figure 4 (b) discussed the variational impact of varying Hetenyi's parameter Q on the phase and group velocities. Certainly, one notes from both velocity curves that an increase Q decreases the phase velocity, while at the same time increasing the group velocity.



(a) Impact of Winkler's foundation when $Q = 0$.

(b) Impact of Hetenyi's foundation when $W = 1$.

Figure 4: Variation of the phase V^{ph} and group V^{g} velocities (27) and (28) against the frequency Ω.

5 Concluding note

In conclusion, the present study examined the elastodynamics of a propagating edge wave on a semi-infinite orthotropic layer under the influence of both the Pasternak's and Hetenyi's elastic foundations; the famous two-parameter elastic foundations with rich literature. The consequential characteristic equations have been constructed and further resolved for the corresponding phase and group velocities. Having noted the suppression of the Winkler's parameter in the previous literature while non-dimensionalizing the characteristic equation, the present study endowed full independence to the Winkler's elastic parameter, and further analyzed its impact, together with the impacts of both the Pasternak's and Hetenyi's parameters on the dispersion of edge waves in the governing plate. Certainly, the Winkler's foundation parameter W has been noted to steadily increase both the phase and group velocities, while an increase in Pasternak's foundation parameter decreases the two velocities. In addition, an increase in the Hetenyi's foundation has been noted to decrease the phase velocity, while at the same time increasing the group velocity. Lastly, the presence study reduces to the work of Althobaiti et al [5] when both the second foundation parameters are disregarded. Moreover, future study intends to examine the relevance of material nonhomogeneity on the dispersion of edge waves in elastic plates [10], in addition to the consideration of multi-parameter foundation models, including the recent four-parameter viscoelastic foundations [11].

REFERENCES

[1] A.N. Norris. Flexural edge waves. *J. Sound Vib.* (1994), **171** 571–573.

[2] S. Althobaiti, D.A. Prikazchikov. The flexural edge waves of arbitrary profile in an orthotropic elastic plate. *Proc. Nat. Ac. Sci. Armenia. Mech.* (2016), **69**(1) 16–24.

[3] J. Kaplunov, D.A. Prikazchikov, G.A. Rogerson. The edge wave on an elastically supported Kirchoff plate (L). *J. Acoust. Soc. Am.* (2014), **136** 1487–1490.

[4] J. Kaplunov, A. Nobili. The edge waves on a Kirchoff plate bilaterally supported by a two-paramater elastic foundation. *J. Vib. Contr.* (2015), 1–9.

[5] S.N. Althobaiti, J. Kaplunov, D.A. Prikazchikov. An edge moving load on an orthotropic plate resting on a Winkler foundation. *Procedia Eng.* (2017), **199** 2579–2584.

[6] B. Erbas, J. Kaplunov, A. Nobili, G. Kilic. Dispersion of elastic waves in a layer interacting with a Winkler foundation. *J. Acoust Soc. Am.* **144** (2018), 2918–2925.

[7] R.I. Nuruddeen, A.M. Mubaraki. Propagation of damped SH waves on nonhomogeneous elastic composites involving viscoelasticity, generalized interface, rotation and mechanical loading. *Ain Shams Eng. J.* **15** (2024), 103053

[8] D. Zhou. A general solution to vibrations of beams on variable Winkler elastic foundation. *Comp. Struct.* **47**(1) (1993), 83-90.

[9] Y.H. Wang, L.G. Tham, Y.K. Cheung. Beams and plates on elastic foundations: a review. *Prog. Struct. Engng Mater.* **7** (2005), 174-182.

[10] O. Sahin. Vibration of a composite elastic beam on an inhomogeneous elastic foundation. *J. Appl. Math. Comp. Mech.* **19**(2) (2020), 107-119.

[11] M.H. Ghazwani, A. Alnujaie, P.V. Vinh. A general viscoelastic foundation model for vibration analysis of functionally graded sandwich plate with auxetic core. *Defence Technology*. (2025). DOI:10.1016/j.dt.2024.12.008