Available online at www.eccomasproceedia.org
Eccomas Proceedia COMPDYN (2025) 1723-1730

ECCOMAS
Proceedia

COMPDYN 2025
10th ECCOMAS Thematic Conference on
Computational Methods in Structural Dynamics and Earthquake Engineering
M. Papadrakakis, M. Fragiadakis (eds.)
Rhodes Island, Greece, 15-18 June 2025

DOMAIN DECOMPOSITION OF FINITE ELEMENT MODEL AND OPTIMIZATION OF THE INTERFACE - CURVED SURFACES ANALYSIS

Radim Dvořák[1,2], **José A. González**[3], **Radek Kolman**[1]

[1]Institute of Thermomechanics of the CAS, v. v. i.
Dolejškova 1402/5, 182 00 Praha 8, Czech Republic
e-mail: {radimd,kolman}@it.cas.cz

[2] CTU in Prague Faculty of Transportation Sciences, Department of Mechanics and Materials
Na Florenci 25, 110 00 Praha 1, Czech Republic
e-mail: dvorara9@fd.cvut.cz

[3] Escuela Técnica Superior de Ingeniería, Universidad de Sevilla
Camino de los Descubrimientos s/n, 410 92 Sevilla, Spain
e-mail: japerez@us.es

Abstract. *Domain decomposition (DD) methods based on Lagrange multipliers are introduced. DD methods provide an effective approach for model size reduction by dividing it into smaller parts. The localized Lagrange multipliers method is discussed, along with an optimization technique for the automatic construction of interface operators to couple non-matching 3D meshes. The core of the method lies in using least-squares approximation to optimally position additional interface nodes, enabling the problem to be solved without modifying coupled subdomain meshes while passing the patch test. Presented method offers the key advantages of avoiding the computation of complex surface integrals on boundary mesh intersections. Error analysis for the curved interface problems is included as a benchmark of the method.*

Keywords: Non-matching meshes, Localized Lagrange multipliers, Direct least squares, Optimization, Interface coupling, Non-planar interface.

ISSN:2623-3347
Peer-review under responsibility of the organizing committee of COMPDYN 2025.
doi: 10.7712/120125.12523.26028

1 INTRODUCTION

In computational mechanics, connecting non-conforming meshes is a common challenge, particularly when neighboring subdomains are meshed separately or use different finite element interpolations. The primary difficulty lies in accurately representing interface effects, which involves ensuring continuity of the displacement field and precise stress transfer across the non-conforming boundary.

In addition to the Mortar method (MM) that is normally used by the scientific community [1, 2, 3], which is widely used for various problems such as curved interfaces [4, 5], contact problems [6], and isogeometric analysis [7]; we also recognize the method of localized Lagrange multipliers (LLM) [8]. The LLM method is a generalization of the MM which brings both advantages (straightforward cross point solution - point of the incidence of 3 and more subdomains) and disadvantages (more unknowns).

Both methods (LLM and MM) may achieve the same accuracy when conventional shape functions are used for the discretization. However, classical LLM are often used [9, 10] with Dirac delta shape functions for Lagrange multipliers, which enormously simplifies the computation cost, but, at the same time, interrupts the accuracy. Several methods have been developed during the last decade [11, 12, 13, 14, 15], in an attempt to improve the performance of classical LLM. In this paper, we take the most recent and most promising one [15], which is based on optimization techniques, and we perform error analysis for non-planar interfaces.

The paper is organized as follows. In Section 2 we present the weak formulation of the coupling problem. Section 3 describes the discretization using FEM. The most recent LLM method is described in Section 4, where the optimization idea is briefly introduced. Finally, we present several benchmarks in Section 5 and close the paper with conclusions in Section 6.

2 WEAK FORMULATION

We treat the domain of interest Ω (see Fig. 1) as a system of two free floating subdomains $\Omega_i\,(i = 1, 2)$ connected by displacement constraints [9, 16]. The virtual work of the system is then represented as

$$\delta\mathscr{W}_T = \delta\mathscr{W}_1 + \delta\mathscr{W}_2 + \delta\mathscr{W}_\Gamma\,, \tag{1}$$

where

$$\delta\mathscr{W}_i = \int_{\Omega_i} \delta\boldsymbol{\varepsilon}_i : \boldsymbol{\sigma}_i \, d\Omega - \int_{\Omega_i} \delta\boldsymbol{u}_i \cdot \boldsymbol{b}_i \, d\Omega - \int_{\Gamma_i} \delta\boldsymbol{u}_i \cdot \boldsymbol{t}_i \, d\Gamma \qquad (i = 1, 2) \tag{2}$$

and

$$\delta\mathscr{W}_\Gamma = \int_\Gamma \delta \left\{ \boldsymbol{\ell}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_\Gamma) + \boldsymbol{\ell}_2 \cdot (\boldsymbol{u}_2 - \boldsymbol{u}_\Gamma) \right\} \, d\Gamma \tag{3}$$

represent the virtual work of the elastic substructure and of the interface constraints, respectively, where $\boldsymbol{\sigma}_i(\boldsymbol{x})$ denotes the stress tensor, $\boldsymbol{t}_i(\boldsymbol{x})$ is identified as the traction vector, $\boldsymbol{u}_i(\boldsymbol{x})$ describes the displacement field, $\boldsymbol{u}_\Gamma$ represents the displacements of the interface Γ and $\boldsymbol{\ell}_i$ are fields of Lagrange multipliers.

3 DISCRETIZATION BY FEM

If one introduce conventional FE shape functions for the approximation of unknown fields in (3), i.e., subdomain displacement $\boldsymbol{u}_i$, interface displacement $\boldsymbol{u}_\Gamma$ and Lagrange multipliers field $\boldsymbol{\ell}_i$, and we involve classical Localized Lagrange multipliers method [9, 10] (see Fig. 2),

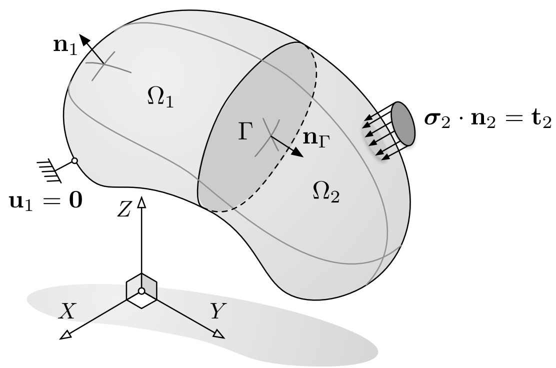


Figure 1: The domain Ω partitioned into two subdomains Ω_i with a shared interface Γ.

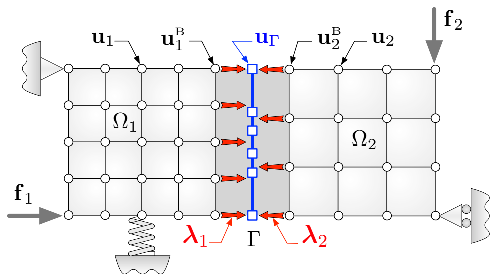


Figure 2: The classical LLM method - Lagrange multipliers are here represented as concentrated nodal forces at subdomain nodes

we arrive to the discrete system

$$\begin{bmatrix} \mathbf{K}_1 & \mathbf{0} & \mathbf{B}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_2 & \mathbf{0} & \mathbf{B}_2 & \mathbf{0} \\ \mathbf{B}_1^{\mathsf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{L}_1 \\ \mathbf{0} & \mathbf{B}_2^{\mathsf{T}} & \mathbf{0} & \mathbf{0} & -\mathbf{L}_2 \\ \mathbf{0} & \mathbf{0} & -\mathbf{L}_1^{\mathsf{T}} & -\mathbf{L}_2^{\mathsf{T}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \\ \mathbf{u}_\Gamma \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \tag{4}$$

where $\mathbf{u}_i$ is subdomain displacement vector, $\mathbf{u}_\Gamma$ interface displacement vector, $\boldsymbol{\lambda}_i$ Lagrange multipliers, $\mathbf{f}_i$ the vector of external forces, $\mathbf{K}_i$ stiffness matrix, $\mathbf{B}_i$ is the Boolean assembly matrix selecting boundary nodes and $\mathbf{L}_i$ is the coupling operator.

Within the conventional LLM, the coupling operator $\mathbf{L}_i$ is given by the formula

$$\mathbf{L}_i = \mathbf{N}_\Gamma \left(\mathbf{x}_i^{\mathrm{B}}\right) \quad \text{for} \quad i = 1, 2 \tag{5}$$

where $\mathbf{N}_\Gamma$ are shape function for interface displacement and $\mathbf{x}_i^{\mathrm{B}}$ are the locations of the boundary nodes (see Fig. 2). However, it is beneficial to use the DLS approach (direct least squares) for the construction of $\mathbf{L}_i$ instead of [14]. By doing so, one does not have to define and evaluate shape functions $\mathbf{N}_\Gamma$. The operator is given by direct pseudo-inversion

$$\mathbf{L}_i = \left(\mathbf{N}_i \left(\mathbf{x}_\Gamma^p\right)\right)^+ \qquad \text{for} \quad i = 1, 2 \tag{6}$$

where $\mathbf{N}_i$ are subdomain displacement shape functions and $\mathbf{x}_\Gamma^p$ are projected interface nodal position on the subdomain surface boundary.

4 OPTIMIZATION OF THE INTERFACE

The classical LLM method [9, 10] together with DLS [14] construction of the interface operator leads to a major reduction of the "degrees of freedom" of the analysis. At this point, the only parameters that interrupt the uniqueness of the solution are the total number and position of the interface nodes. Depending on the rules of the interface node configuration, we recognize several methods which can be found in [11, 12, 13, 14]. Listed methods are accurate for 2D problems; however, in 3D the accuracy is only a matter of chance - there are no general rules to satisfy specific level of accuracy.

Optimization techniques offer massive improvements to existing methods [15]. The general approach consists of the following steps:

Step 1. Choose existing method (rule) for initial interface configuration,

Step 2. define optimization parameters (usually interface nodes coordinates) and objective function to minimize (e.g. stress error within the patch test [17]) and

Step 3. solve the optimization problem.

In particular, the authors have already presented the optimization scheme for the classical LLM method [9, 10] with the DLS approach [14], optimizing the locations of the interface nodes, where the initial position was given by the rule of Barlow points [14]. The strategy followed to construct accurate coupling operators is to propose analytical solutions, expressed in terms of analytical stresses $\bar{\boldsymbol{\sigma}}(\mathbf{x})$, that our operators should be able to transmit properly through the interface.

The interface Γ is then defined by an analytically parameterized surface with surface coordinates $\mathbf{x}_\Gamma$ and normal $\mathbf{n}$ where the interface nodes should be located. The tractions produced by the analytical stresses on the surface Γ can be calculated as $\bar{\boldsymbol{\lambda}} = \bar{\boldsymbol{\sigma}}\mathbf{n}$ and the cost function to minimize becomes:

$$\mathbf{x}_\Gamma^* = \underset{\mathbf{x}_\Gamma \in \Gamma}{\operatorname{argmin}} \parallel \mathbf{L}^\mathsf{T} \bar{\boldsymbol{\lambda}} \parallel_2 \tag{7}$$

corresponding to a minimization of the equilibrium error produced by the analytical multipliers acting on the curved interface. Detailed analysis and step-by-step algorithm can be found in [15].

In the framework of previous publication we demonstrated robustness of the method - its capability of solving non-matching mesh problems and also the combination with non-matching geometry scenario. We also outlined the problem of non-planar interfaces - we bring fundamental error analysis within this work.

5 CURVED INTERFACE ERROR ANALYSIS

Within the following section, we provide formulation of the benchmark, assumptions and the results.

5.1 Definition of the problem

The curved interface approximates the elevation function

$$h(x, y) = A \sin(\pi x)\sin(\pi y)\,, \qquad (x, y) \in [0, 1] \times [0, 1]\,, \tag{8}$$

where A is the amplitude of the curvature (see Fig. 3).

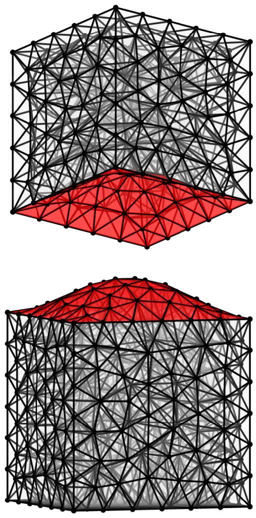


Figure 3: Mesh of the problem with highlighted interface

5.2 Simplified projection

As it occurs in other methods for interface coupling with non-conforming interfaces, the projection process is expected to affect accuracy; especially when high curvature of the interface is combined with a coarse discretization of the solid boundaries. Further, we use simplified projection of interface nodes which follows vertical (z) direction (see Fig.3), i.e., we do not take into account the curvature, when mapping interface nodes onto the highlighted boundary surfaces.

5.3 Results

The error is measured by means of following terms

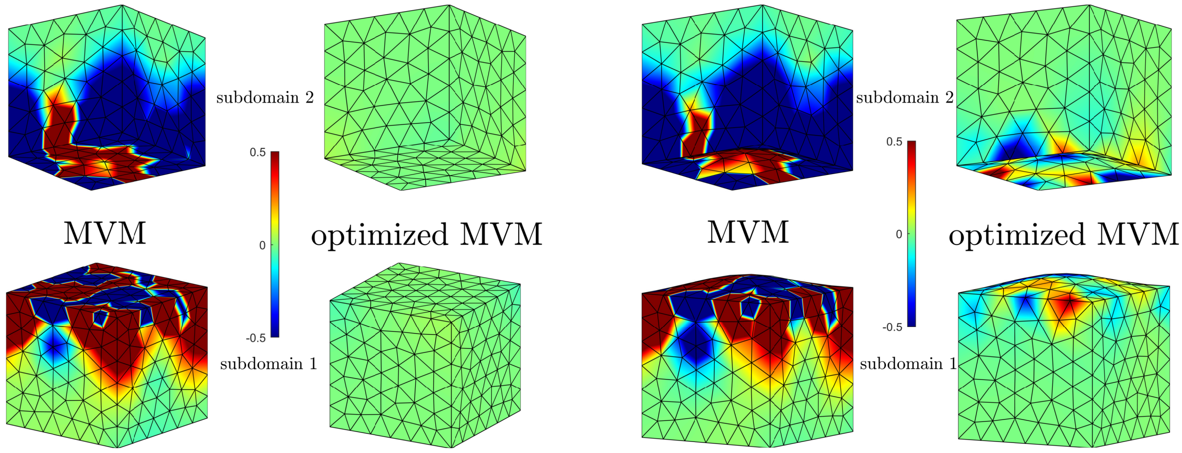
$$\epsilon_{\mathbf{u}} = \frac{100}{\bar{u}_m} \times \big(\mathbf{u}(\mathbf{x}) - \bar{\mathbf{u}}(\mathbf{x})\big) , \quad \epsilon_{\sigma} = 100 \times (\boldsymbol{\sigma}_z(\mathbf{x}) - \mathbf{1}) , \tag{9}$$

where $\mathbf{u}(\mathbf{x})$ and $\boldsymbol{\sigma}_z(\mathbf{x})$ stand for the resultant displacement and z-stress vectors from the optimized interface, respectively; $\bar{\mathbf{u}}(\mathbf{x})$ refers to the reference displacement derived via the Mortar method, with $\bar{u}_m = \max\{\bar{\mathbf{u}}(\mathbf{x})\}$ representing its characteristic displacement. Additionally, $\mathbf{1}$ denotes the reference unit stress vector, serving as the solution for the Mortar method.

The stress error (9) is plotted for two particular cases as a color map in order to demonstrate the accuracy of the author's method [15] with comparison to the non optimized configuration [14] (see Fig. 4).

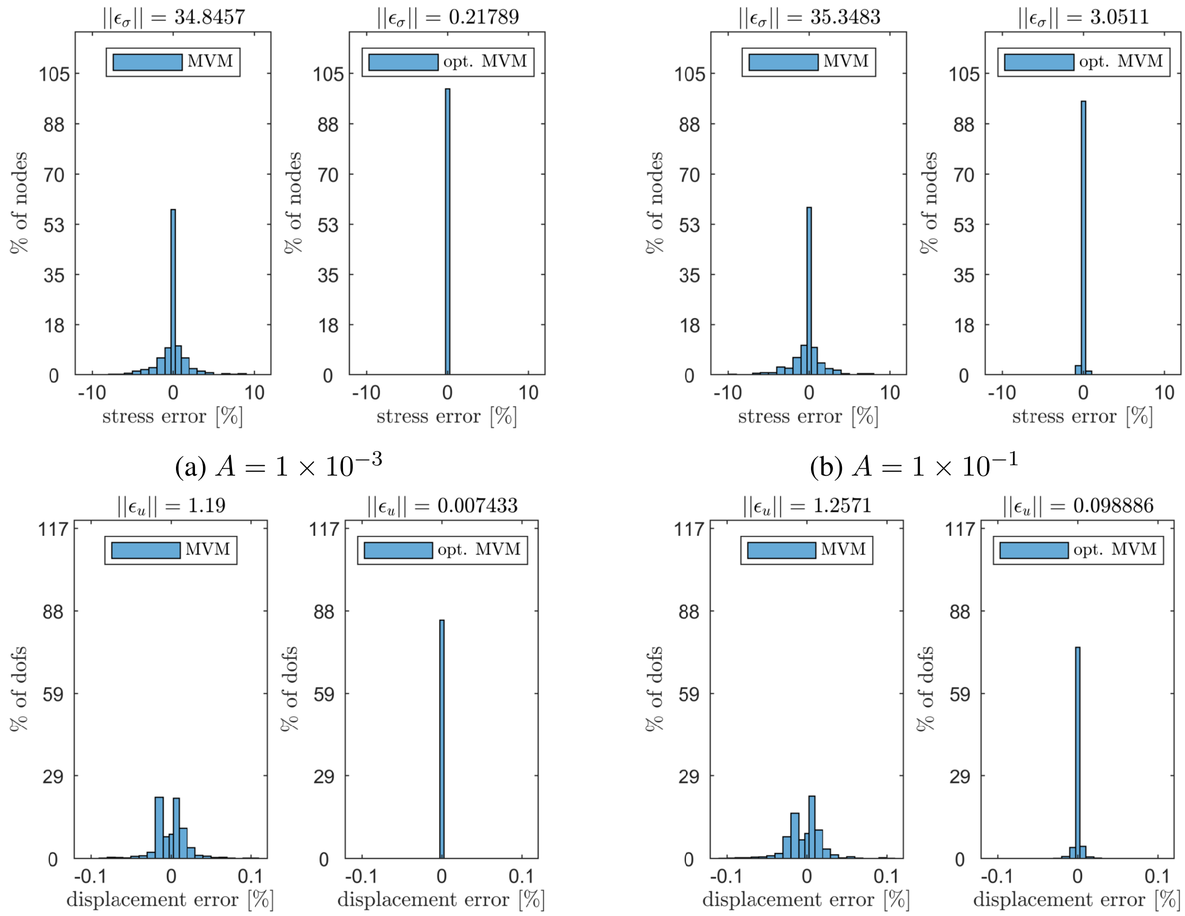
Detailed histograms of the errors (9) are plotted (see Fig. 5). The values shown at the headings are the Euclidean norms of error vectors (9).

Finally, we bring the representation (see Fig. 6) of relation between amplitude curvature A and the norms $||\epsilon_\sigma||$, $||\epsilon_{\mathbf{u}}||$ of the error vectors (9).



(a) $A = 1 \times 10^{-3}$ (b) $A = 1 \times 10^{-1}$

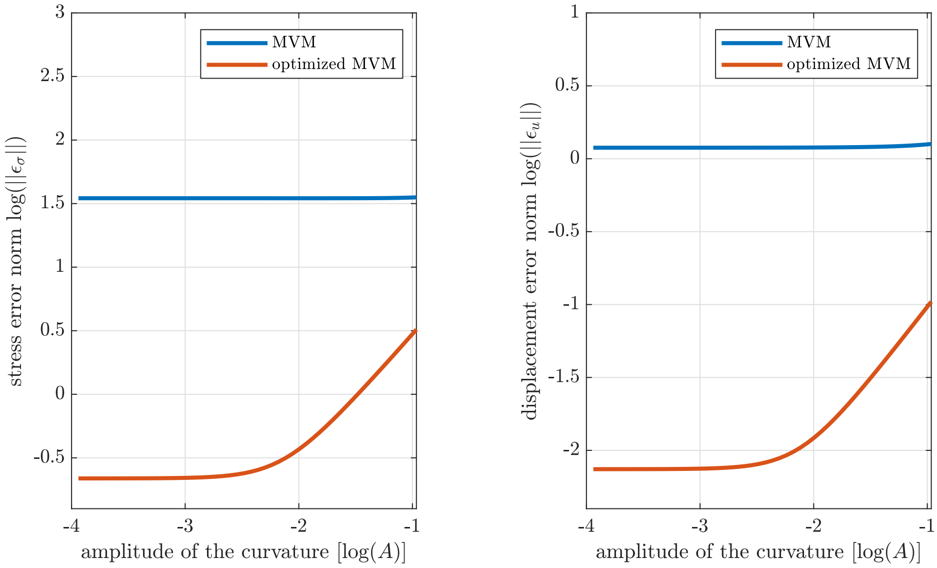
Figure 4: Z-stress error [%] given by (9) for two different amplitudes of the curvature



(a) $A = 1 \times 10^{-3}$ (b) $A = 1 \times 10^{-1}$

(c) $A = 1 \times 10^{-3}$ (d) $A = 1 \times 10^{-1}$

Figure 5: Histograms of the error (9) for $A = 1 \times 10^{-3}$ (left) and $A = 1 \times 10^{-1}$ (right) - comparison between optimized (opt. MVM) and un-optimized (MVM) interface



(a) Stress error (b) Displacement error

Figure 6: The amplitude of the curvature-the norms of the error vectors (9) relation

6 CONCLUSIONS

Brief performance and error analysis of a given interface-coupling method [15] was performed, within the problem of non-matching mesh of non-planar interface.

It has been shown that the analyzed method works well even when applying simplified projection within the non-planar interface (see Fig. 4). Both stress and displacement error are one order lower (see Fig. 5) compared to other available methods [14]. With decreasing curvature of the interfaces, the error significantly decreases (see Fig. 6) when using the optimized interface. Error improvement of the un-optimized interface is negligible.

We find the use of the optimization technique as a suitable approach for non-planar interfaces. We expect even better results with an accurate projection of the interface nodes.

ACKNOWLEDGEMENTS

This work has been supported by the grant 23-06220S of the Czech Science Foundation within institutional support RVO:61388998.

REFERENCES

[1] C. Bernardi, Y. Maday, A.T. Patera, Asymptotic and Numerical Methods for Partial Differential Equations with Critical Parameters - Domain Decomposition by the Mortar Element Method. *Springer Netherlands*, 978-94-011-1810-1, 269–286, 1993.

[2] Belgacem, B. Faker, The Mortar finite element method with Lagrange multipliers. *Numerische Mathematik*, **84**, 173–197, 1999.

[3] P. Seshaiyer, M. Suri, Uniform hp Convergence Results for the Mortar Finite Element Method. *Mathematics of Computation*, **69**, 521–546, 2000.

[4] B. Flemisch, M.A. Puso, B.I. Wohlmuth, A new dual mortar method for curved interfaces: 2D elasticity. *International Journal for Numerical Methods in Engineering*, **63**, 813–832, 2005.

[5] B. Flemisch, B.I. Wohlmuth, Stable Lagrange multipliers for quadrilateral meshes of curved interfaces in 3D. *Computer Methods in Applied Mechanics and Engineering*, **196**, 1589–1602, 2007.

[6] A. Popp, M. Gitterle, M.W. Gee, W. A. Wall, A dual mortar approach for 3D finite deformation contact with consistent linearization. *International Journal for Numerical Methods in Engineering*, **83**, 1428–1465, 2010.

[7] A. Seitz, P. Farah, J. Kremheller, B.I. Wohlmuth, W.A. Wall. A. Popp, Isogeometric dual mortar methods for computational contact mechanics. *Computer Methods in Applied Mechanics and Engineering*, **301**, 259–280, 2016.

[8] K.C. Park, C.A. Felippa, G. Rebel, A simple algorithm for localized construction of non-matching structural interfaces. *International Journal for Numerical Methods in Engineering*, **53**, 2117-2142, 2002.

[9] K.C. Park, C.A. Felippa, A variational framework for solution method developments in structural mechanics. *Journal of Applied Mechanics*, **65**, 242–249, 1998.

[10] K.C. Park, C.A. Felippa, U.A. Gumaste, A localized version of the metod of Lagrange multipliers and its applications. *Computational Mechanics*, **24**, 476–490, 2000.

[11] Y.U. Song, S.K. Youn, K.C. Park, A gap element for treating non-matching discrete interfaces. *Computational Mechanics*, **56**, 551–563, 2015.

[12] Y.U. Song, S.K. Youn, K.C. Park, Virtual gap element approach for the treatment of non-matching interface using three-dimensional solid elements. *Computational Mechanics*, **60**, 585–594, 2017.

[13] Y.U. Song, G.E. Jeong, S.K. Youn, K.C. Park, Virtual tetrahedral gap element to connect three-dimensional non-coincident interfaces. *Finite Elements in Analysis and Design*, **152**, 18–26, 2018.

[14] G.E. Jeong, Y.U. Song, S.k. Youn, K.C. Park, A new approach for nonmatching interface construction by the method of localized Lagrange multipliers. *Computer Methods in Applied Mechanics and Engineering*, **361**, 112728, 2020.

[15] R.Dvořák, J.A. González, On the automatic construction of interface coupling operators for non-matching meshes by optimization methods. *Computer Methods in Applied Mechanics and Engineering*, **432**, 117336, 2024.

[16] K.C. Park, C.A. Felippa, A variational principle for the formulation of partitioned structural systems. *International Journal for Numerical Methods in Engineering*, **47**, 395–418, 2000.

[17] R.L. Taylor, J.C. Simo, O.C. Zienkiewicz, A.C.H. Chan, The patch test — A condition for assessing FEM convergence. *International Journal for Numerical Methods in Engineering*, **22**, 39–62, 1986.