

USE OF THE CONTINUOUS WAVELET TRANSFORM FOR PROCESSING SEISMIC STRUCTURAL RESPONSES

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Abstract. *The Continuous Wavelet Transform (CWT) method is now commonly employed for monitoring the health of civil engineering structures subjected to various types of excitation. This article begins with a brief overview of the CWT method's key features, ensuring it is used effectively and in line with the desired objectives. It then discusses the use of a progressive Morse-class mother wavelet, which has excellent time-frequency localisation properties. The positions and amplitudes of the CWT modulus maxima, known as 'ridges' and 'skeletons' respectively, are shown to carry significant information about the response signal. Two example applications are presented: an elasto-plastic single-degree-of-freedom oscillator, which is often used in seismic applications, and the well-known Sherman Oaks building, which has been subjected to a sequence of six damaging earthquakes over the last forty years. Despite the non-stationary nature of the examined response signals, the CWT method has proven highly efficient in analysing structural response signals and detecting nonlinearities in the vibration response.*

Keywords: wavelet transform, seismic structural vibrations, building oscillations, modal identification, damage detection, singularity detection

1 Introduction

The Continuous Wavelet Transform (CWT) method is a powerful tool for signal processing with many applications in engineering. In recent years, it has become more popular due to the availability of ready-to-use wavelet libraries in many high-level computing packages and libraries, such as Python and Matlab. However, using freely available software without understanding the fundamentals of the method can sometimes lead to disappointing, or difficult-to-interpret results.

Wavelet transforms are widely used in many fields of science and engineering. Among the large number of books on wavelet transforms, few are dedicated to applications in civil engineering structures [1, 2, 3]. Chatterjee's book [4] details several applications of wavelets in relation to civil engineering problems, demonstrating their importance in analysing non-stationarities in seismic ground motions, studying bridge vibrations caused by vehicle passage and identifying structural damage. Identifying time-varying frequencies and amplitudes in real-life structural vibrations is an essential part of structural health monitoring data processing. Structural responses (especially transient) can be often seen as a combination of amplitude and phase modulated signals.

Shortly after the introduction of the wavelet transform in the 1980s, several authors began working on methods for extracting and characterising the amplitude and phase of such signals from their continuous wavelet transforms, based on the concept of asymptotic behaviour. The absolute value of the CWT of these signals tends to concentrate near the 'ridges' of the transform. The set of maxima of the absolute value of the CWT along these ridges forms the skeleton of the CWT of the signal.

The concepts of 'ridge' and wavelet 'skeleton' has been first proposed by several authors (see, for example, [5, 6]) and have been then widely used since the late 1990s to process transient signals of civil engineering structures. One of the earliest applications was the study of the free oscillations of buildings after impact. The accelerometric responses of buildings subjected to non-destructive shocks (e.g. the impact of a shovel) were processed using the CWT in [7] for the assessment of modal characteristics. Analysing the accelerometric responses of a Plexiglas plate to after-shocks (impact hammer) in a laboratory setup, the free vibration responses of a non-proportionally damped plate were subsequently processed and the modal frequencies, damping ratios and complex mode shapes were accurately identified [8]. More recently, the method was applied to the post-shock responses of a masonry wall during shear compression tests. These tests correspond to the free responses of non-proportionally damped systems integrated in noise with the occurrence of complex modes [9, 10].

Several applications of the CWT method based on the free oscillations of nonlinear structures have also been carried out. The first application concerns accelerometric after-shock responses (impact hammer) on a beam with a non-linear interface [11, 12]. This beam was characterized by a concentrated nonlinearity, whose first mode and related super-harmonics were seen to simulate a Duffing oscillator. Another application focused on the response of a French railway bridge to high-speed train crossings [13]. The first three instantaneous modal frequencies calculated from the CWT analysis showed significant variations over time that indicated a strong nonlinear dynamics of this bridge.

Another very important capacity of the method is related to the detection of anomalies. This is in particular interesting in structural engineering, where damaging events result in short-time,

non-stationary vibration characteristics that are often difficult to analyze using classical modal approaches. The Wavelet Transform methods are better suited for analyzing signals generated by time variant systems (as compared to Fourier transform) and makes the method a very interesting tool for damage detection. The Discrete Wavelet Transform (DWT) is commonly used in the literature for structural damage detection. A multi-resolution analysis approach [14] allows to identify the occurrence of high amplitude *spikes* in the high frequency components of the signal. These spikes can be associated in theory with stiffness changes as shown in [15, 16], among others. High frequency effects can be identified using either CWT or DWT, however they may be due to a variety of reasons and hence they can hardly be linked to structural defects. Recently, the CWT wavelet-based transmissibility function has been proposed for damage detection [17, 18].

This article demonstrates how the CWT can be used to analyse the vibration responses of civil engineering structures for structural health monitoring purposes. The main focus is on the seismic response of the buildings. The structure of the paper is as follows: Section 2 will review the key features of the CWT method, followed by a detailed presentation of the Cauchy–Paul progressive mother wavelet. One significant advantage of this wavelet is its ability to link the CWT of a signal with that of its derivatives. Section 3 will demonstrate the application of the CWT method to analyse two examples of structural responses. The first example concerns the simulated response of an elasto-plastic oscillator with kinematic hardening. The second example is the nine-storey Sherman Oaks building. It is a fully instrumented building that was damaged by a sequence of six earthquakes.

2 Essential features of the CWT method - Presentation of an analytic mother wavelet

Having presented an introduction to the CWT method in [19], we will here first review some of the basic properties of the method. Next, we offer some suggestions to help identify the most suitable mother wavelet for a given problem. Finally, we provide a thorough analysis of the analytic Cauchy–Paul mother wavelet.

2.1 Some basic properties of the CWT

The Continuous Wavelet Transform (CWT) is a scalar product of a finite energy signal $u(t) \in L^2(\mathbb{R})$ and the child wavelets $\psi_{(b,a)}(t)$, where $\psi \in L_1(\mathbb{R}) \cap L^2(\mathbb{R})$ is the mother wavelet:

$$T_\psi[u](b, a) = \langle u(t), \psi_{(b,a)}(t) \rangle. \quad (1)$$

There are two standard forms of the above definition, depending on the norm chosen for the child wavelets: the L_1 norm where the child wavelets are defined by $\psi_{(b,a)}(t) = \frac{1}{a} \psi\left(\frac{t-b}{a}\right)$ and the L_2 norm with the child wavelets $\psi_{(b,a)}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right)$. The L_1 norm is preferred for time-localised signals with different amplitudes. The objective is to ensure that the wavelet transform modulus accurately replicates the amplitude of the analysed signal.

In the case of the L_2 norm, the square of the wavelet transform modulus should reproduce the energy of the analysed signal:

$$\int_{-\infty}^{+\infty} |u(t)|^2 dt = \frac{2}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} |T_\psi[u](b, a)|^2 db \frac{da}{a^2},$$

where $C_\psi = \int_0^{+\infty} |\hat{\psi}(f)|^2 \frac{df}{f}$ is the admissibility coefficient of the mother wavelet. We have found that time-localised signals are generally better described by their amplitude. For this purpose, we opted for the L_1 normalisation of $1/a$.

Using a mother wavelet ψ , the CWT method allows to decompose a response signal $u(t)$ into the frequency-time domain as follows:

$$T_\psi[u](b, a) = \frac{1}{a} \int_{-\infty}^{+\infty} u(t) \psi^* \left(\frac{t-b}{a} \right) dt, \quad (2)$$

where ψ^* is the conjugate of ψ . The definition of CWT in the time domain Eq. 2 can be rewritten in the Fourier domain:

$$T_\psi[u](b, a) = \int_{-\infty}^{+\infty} \hat{u}(f) \hat{\psi}^*(af) e^{2i\pi fb} df, \quad (3)$$

where $\hat{u}(f) = \int_{-\infty}^{+\infty} u(t) e^{-2i\pi ft} dt$ is the Fourier transform of $u(t)$. Eq. 3 allows for an easy numerical computation by means of the fast Fourier transform algorithm.

The CWT of a signal $u(t)$ can also be used to compute the CWT of its derivative $\frac{du(t)}{dt}$. From Eq. (3) and $\frac{d\hat{u}}{dt}(f) = 2i\pi f \hat{u}(f)$, it can be deduced [20]:

$$T_\psi \left[\frac{du}{dt} \right](b, a) = -\frac{1}{a} T_{\frac{d\psi}{dt}}[u](b, a), \quad (4)$$

where $d\psi(t)/dt$ can play the role of a mother wavelet.

For a successful decomposition with the CWT of the signal, two issues need to be resolved and controlled: (i) the wavelet **time-frequency resolution**, and (ii) the **edge effect problem** that arises due to the finite signal length, to the discretization of measured signals and to the nature (convolution product) of the CWT [21]. Concerning the first issue, higher the resolution in time, leads to lower resolution in the frequency domain.

Referring to the conventional frequency analysis of constant- Q filters, Le and Argoul [22] introduced a parameter Q_m called “quality factor” and can be used to control the time-frequency resolution of the CWT. This quality factor Q_m is defined as the ratio of the mean value f_ψ^{mean} (in terms of root mean squares with L_2 norm) over to two times the standard deviation Δf_ψ of $\hat{\psi}$, i.e. it can be calculated with the aid of the expression:

$$Q_m = \frac{f_\psi^{mean}}{2\Delta f_\psi}. \quad (5)$$

The *edge effect* problem is a common issue of the CWT. The results of the CWT method are always distorted in a specific area of the time-frequency domain. This distortion cannot be removed; however, it can be controlled by properly adjusting the quality factor Q_m , as discussed in [19, 22]. The difficulty then lies in selecting a Q_m for each eigenfrequency f_j of the signal. In other words, the Q_m choice should minimized edge effects and also help identifying closely

spaced eigenfrequencies. For this reason, Le and Argoul [22] introduced two real coefficients c_t and c_f greater than, or equal, to 1. These coefficients facilitate the extension of the time-frequency localisation domain and ultimately provide limits for Q_m :

$$Q_m^{(min)} = c_f \frac{f_j}{2df_j} \leq Q_m \leq \frac{L\pi f_j}{c_t} = Q_m^{(max)}, \quad (6)$$

where L is the total length of the measurement time and df_j is the frequency distance from which the effect of the modal coupling must be avoided. It has been demonstrated in [22] that, in case of an increased Q_m value, the consequence is an amplification of the edge effect, which in turn results in a contraction of the useful time interval for modal identification. Conversely, the mode decoupling becomes more pronounced.

It is common practice to plot the scalogram — the absolute value of the CWT — as a function of time and frequency since frequency is inversely proportional to scale. Thus, a small scale a corresponds to a compressed child wavelet and rapidly changing details, and finally to high frequency f and vice versa. Although this general correlation exists, there is no precise relationship between scale and frequency; introducing the peak frequency f_ψ^0 , we propose the following relation:

$$a = \frac{f_\psi^0}{f}, \quad (7)$$

where f_ψ^0 is the modulus of the Fourier transform of the mother wavelet ψ is maximum. Other relationships between a and f may be also adopted. For example $a = f_\psi^{mean}/f$, where f_ψ^{mean} is the mean frequency of ψ for the L_2 norm [23]. The term wavelet modulus maxima describes any point (b_r, a_r) in the time-scale-domain such that $|T_\psi[u](b, a)|$ is locally maximum (strictly) at $(b, a) = (b_r, a_r)$, either the right or the left neighborhood.

A ridge is defined as the curve $a_r(b_r)$ in the time-frequency plane of the points at which the Wavelet Transform Modulus is Maxima (WTMM) [1, 24, 25].

Finally it is important to note that temporal sampling of the signal and wavelet transform representation are directly related. To represent the spectrum of a signal within a given frequency interval $[-F, F]$, the signal must be sampled at a step size dt smaller than the Nyquist step: $\frac{1}{2F}$: $dt \leq \frac{1}{2F}$. Therefore, the maximum frequency F_{max} for representing the scalogram is given by: $F_{max} = 1/2dt$.

2.2 Analytic mother wavelets - The Cauchy-Paul wavelet

A mother wavelet is an oscillating function of zero mean, denoted $\psi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. A mother wavelet has a certain degree of regularity and is of finite length. This explains the term ‘wavelet’, which means ‘small wave’. A mother wavelet must first satisfy the necessary **admissibility condition**: $C_\psi < +\infty$; which requires that, for functions of $L_2(\mathbb{R})$, $\psi(t)$ has zero mean. The mother wavelet can be subject to regularity conditions, such as possessing **multiple zero moments**. Thus, a mother wavelet admits m zero moments if, for all integers $k \in [0, m-1]$, the following condition is met: $\int_{-\infty}^{+\infty} t^k \psi(t) dt = 0$. A mother wavelet with m zero moments is, by definition, orthogonal to all polynomials of degree up to and including $m-1$. Therefore, this leads to the properties of decay and convergence of both the function $\psi(t)$ and its Fourier transform, towards zero at infinity. These properties are useful for detecting singularities in signals and for determining their local Lipschitz regularity [25]. **Analyticity** is another interesting

property of the mother wavelet, as it allows the phase and amplitude information of a signal to be separated. An **analytic**, or progressive, mother wavelet is a complex function for which the Fourier transform is zero at negative frequencies. This is the case for Morse wavelets [26], which do not support negative frequencies. Analytic wavelets can also be used to characterize singular phenomena, however, they have been more commonly chosen for analysing oscillatory phenomena. Lilly and Olhede [26] have demonstrated that many commonly used continuous analytical wavelets are not intrinsically distinct, but can be considered as special cases of a single, unified family: the generalised Morse wavelets. These wavelets are continuous, exactly analytic and depend on two parameters.

Among these mother wavelets, the single-parameter progressive Cauchy–Paul wavelets have been widely used in quantum mechanics [27] but also for 2D applications [28] when the features to be detected have a preferred direction, e.g. straight edges, filaments, oriented textures, velocity fields, etc. These wavelets, denoted by the generic notation: ψ_n , where n is an integer that controls the number of oscillations in the wavelets' shape and corresponds to the number of vanishing moments will be detailed in the following. The way they are normalised can also change, for example, time normalisation or frequency normalisation using the L_2 norm, the L_∞ norm, and so on. Some authors have used them with a typical normalisation: $\|\psi_n\|_\infty = 1$ for analysing different oscillatory mechanical responses of civil engineering structures [11, 22, 20, 21, 8, 29, 30].

If $\|\hat{\psi}_n\|_\infty = 2$ is chosen for normalising the mother wavelet, it is defined in the frequency domain as follows:

$$\text{for } f \geq 0 \quad \hat{\psi}_n(f) = 2 \times \left(\frac{2\pi e}{n} \right)^n f^n e^{-2\pi f}, \quad (8)$$

$$\text{for } f < 0 \quad \hat{\psi}_n(f) = 0. \quad (9)$$

The time expression of this mother wavelet is then given by

$$\psi_n(t) = \frac{n!}{\pi} \left(\frac{e}{n} \right)^n \left(\frac{i}{t+i} \right)^{n+1}. \quad (10)$$

Its mean frequency $f_{\psi_n}^{mean}$ for L_2 norm is $f_{\psi_n}^{mean} = \frac{2n+1}{4\pi}$. Its peak frequency $f_{\psi_n}^0$ corresponding to the maximum of $\hat{\psi}_n$ depends on n : $f_{\psi_n}^0 = n/2\pi$, thus $a = n/2\pi f$. The mean and peak frequencies are slightly different, indicating the asymmetrical nature of the shape of $\hat{\psi}_n(f)$. The standard deviation Δf_{ψ_n} of $\hat{\psi}_n$ is then: $\Delta f_{\psi_n} = \frac{\sqrt{2n+1}}{4\pi}$ and the quality factor becomes $Q_m = \frac{\sqrt{2n+1}}{2}$.

The real part of $\psi_n(t)$ and its Fourier transform $\hat{\psi}_n(f)$ are drawn in Figures 1 a and b, respectively. It can be easily shown that:

$$\hat{\psi}_{n+1}(\nu) = \frac{2\pi e}{n+1} \left(\frac{n}{n+1} \right)^n \nu \hat{\psi}_n(\nu). \quad (11)$$

Consequently, using the formula $\widehat{\frac{du}{dt}}(\nu) = 2i\pi\nu \widehat{u}(\nu)$, we obtain:

$$\begin{aligned} T_{\psi_{n+1}}[u](b, a) &= \int_0^{+\infty} \widehat{u}(\nu) \widehat{\psi_{n+1}}^*(a\nu) e^{2i\pi\nu b} d\nu \\ &= \frac{ie}{n+1} \left(\frac{n}{n+1}\right)^n a \int_0^{+\infty} \widehat{\frac{du}{dt}}(\nu) \widehat{\psi_n}^*(a\nu) e^{2i\pi\nu b} d\nu \\ &= \frac{ie}{n+1} \left(\frac{n}{n+1}\right)^n a T_{\psi_n}\left[\frac{du}{dt}\right](b, a). \end{aligned} \quad (12)$$

To plot the scalogram of the signal as a function of the scalogram of its velocity, we propose replacing a by $\frac{n+1}{2\pi f}$ in the previous equation 12. It should be noted that the calculation of $T_{\psi_n}\left[\frac{du}{dt}\right](a, b)$ is not performed for $a = \frac{n}{2\pi f}$ but for $a = \frac{n+1}{2\pi f}$!

We can then extend the relationship to the second-order derivative $\frac{d^2u}{dt^2} = \ddot{u}$; from previous relationship of Eq. 12, we obtain:

$$T_{\psi_{n+2}}[u](b, a) = -e^2 a^2 \frac{n^n}{(n+2)^{n+2}} T_{\psi_n}[\ddot{u}](a, b). \quad (13)$$

As previously, to plot the scalogram of the signal as a function of the scalogram of its acceleration, we propose replacing a by $\frac{n+2}{2\pi f}$ in the previous Eq. 13.

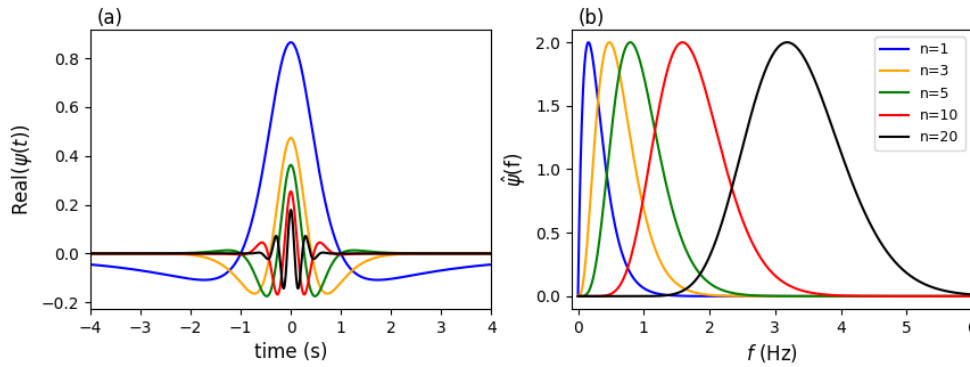


Figure 1: The Cauchy-Paul mother wavelet proposed here: (a) $\Re\{\psi_n\}(t)$ in the time domain, (b) $\hat{\psi}_n(f)$ in the Fourier domain.

3 Numerical examples on the use of the CWT technique for processing structural responses

This section demonstrates the application of the CWT method to a nonlinear single-degree-of-freedom (SDoF) system, and to the Sherman Oaks building in the Los Angeles neighbourhood of the same name. The latter is a well-known instrumented building that has been hit by several damaging earthquakes.

3.1 Application to the numerical response of an elasto-plastic oscillator subjected to transient loading

The first application example concerns a nonlinear elasto-plastic SDoF (Single-Degree of-Freedom) system with kinematic hardening, subjected to a sinusoidal harmonic force that ceases abruptly after nine and a quarter load cycles (Fig. 2a). As shown below, this fundamental example, despite its apparent simplicity, provides interesting findings on the dynamic response of elasto-plastic structures. Savi and Pacheco [31], have also presented the dynamic analysis of an elasto-plastic oscillator with kinematic and isotropic hardening. They have shown that kinematic hardening may cause a superharmonic steady state response where plastic variables assume periodic responses.

Upon the termination of the harmonic force, the system is left to oscillate freely (Fig. 2a). The maximum amplitude of the force is F_{max} and the instant t_0 at which it is suddenly stopped is shown with a red dashed line. If the amplitude of the applied force is $F_{max} = 2W$, its expression is:

$$F(t) = 2W \sin(2\pi f_h t) (\Theta(t) - \Theta(t - t_0)), \quad (14)$$

where $\Theta(t)$ is the Heaviside function. Moreover, if $T_h = 5s$ is the period of the applied harmonic force, the time instant t_0 is equal to $t_0 = 9.25T_h = 46.25s$. Regarding the dynamic properties of the oscillator, the Eigen-frequency is equal to: $f_n = 0.5\text{Hz}$, while the damping ratio is set to $\xi = 2\%$.

The force-displacement capacity curve of the SDoF oscillator is elastic-perfectly plastic. In this work it is assumed that the yield force is equal to 50% of the building weight, thus: $F_y = 0.5W$ where $W = mg$ is the total weight of the structure and g is the acceleration of gravity. The yield displacement u_y is obtained as: $u_y = F_y / k$. If the stiffness k of the structure is function of its Eigenfrequency f_n , i.e $k = 4\pi^2 m f_n^2$, after some manipulations it is found that: $u_y = F_y / (4\pi^2 m f_n^2) = 0.5g / (4\pi^2 f_n^2) \approx 5 / (4\pi^2 f_n^2)$. Following this approach, the mass can be set to any arbitrary value.

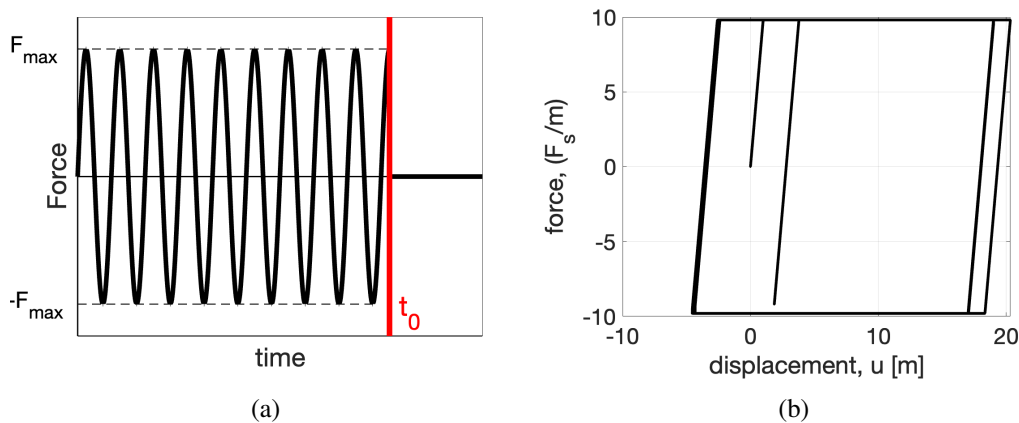


Figure 2: (a) Applied force $F(t)$, (b) Force-displacement hysteretic curve of the SDoF oscillator considered.

The normalized equation of motion of the SDoF system becomes:

$$\ddot{u}(t) + 4\pi\xi f_n \dot{u}(t) + F_s(t)/m = F(t)/m \Leftrightarrow \ddot{u}(t) + 0.04\pi \dot{u}(t) + F_s(t)/m = 2g \sin(0.4\pi t) (\Theta(t) - \Theta(t - t_0)), \quad (15)$$

where $F_s(t)/m = 4\pi^2 f_n^2 u = \pi^2 u$ for the linear problem. For the nonlinear problem $F_s(t)/m = \pi^2 (u(t) - u_p(t)) + b\pi^2 u_p(t)$, where b is the hardening ratio and $u_p(t)$ is the vector of plastic displacements. Zero initial velocity $\dot{u}(0)$ and initial displacement $u(0)$ were assumed as initial conditions.

The hysteretic model is bilinear with pure kinematic hardening. Thus, yielding occurs when $|F_s(t) - \alpha| - F_y \leq 0$, where α is the shift due to hardening. During elasto-plastic analysis, the accumulated force α shifts the yield surface, reducing the compressive strength. The hysteretic loop obtained is shown in Figure 2b. In CWT analysis, the abrupt stiffness changes due to force reversals is expected to produce ridges parallel to the frequency axis that represent stiffness drops at each yield point and a harmonic generation during asymmetric cycles. More details about the hysteretic model can be found in [31]. Moreover, Eq. 15 is a stiff nonlinear ordinary differential equation (ODE) with a discontinuous second term. It is widely agreed that stiff nonlinear ODEs give rise to problems that explicit methods can not solve effectively. Therefore, it is more difficult to solve numerically. Stiff problems typically require computationally more expensive techniques, such as implicit Runge–Kutta methods [32].

Figure 3 shows four sub-figures that compare acceleration and displacement response histories for both the elastic and the elasto-plastic oscillator. Figs 3a and 3b present the numerical signals of the acceleration and the displacement response histories. Steady and transient components can be seen in both the accelerometric and the displacement responses. The transient component depends on the initial conditions at $t = t_0$ and dies out quickly according to the modal damping ratio ξ . Moreover, when subjected to a Fast Fourier Transform (FFT), the signals obtained by solving Eq. 15 over the time interval $[0, 60]$ are periodised. The periodised signals are discontinuous, especially in the case of displacement responses of the elasto-plastic oscillator. To avoid this issue, we recommend extending the input signal symmetrically. This can be achieved using the “ExtendSignal” option in the Matlab toolbox cwt, or by using the command “flip”. Figures 3c and 3d present acceleration and displacement signals of the elasto-plastic oscillator after transforming them to “zero mean” and having them “flipped”.

Figures 4a and 4b, show the Fourier spectra of the elastic and the elasto-plastic oscillators, respectively. For the elastic SDoF oscillator, it can be seen that the sudden stop of force at t_0 results in an impact that amplifies the acceleration response. However, this amplification is not visible in the displacement plot (Fig. 3b). Moreover, two characteristic frequencies can be identified in Fig. 4a: that of the Eigenfrequency of the structure at $f_n = 0.5\text{Hz}$ (period $T_n = 2\text{s}$) and the harmonic force frequency at $f_h = 0.2\text{Hz}$ ($T_h = 5\text{s}$). The Fourier transform of the displacement signal identifies the frequency of the harmonic force, but the Eigen-frequency of the isolator is very weak in both the elastic and the elasto-plastic cases. Note that all signals were zero-measured and flipped before applying the FFT algorithm.

Figure 5 shows the phase plots for both the elastic and the elasto-plastic oscillators. To make the plots more clear, only the forced oscillation part of the response history is shown, i.e. $t < t_0$. Phase plots show the velocity over displacement response histories and illustrate the evolution of the dynamic behavior of the oscillators. For the elastic oscillator, the trajectories remain closed and elliptical, reflecting the absence of permanent deformations and the conservation of energy

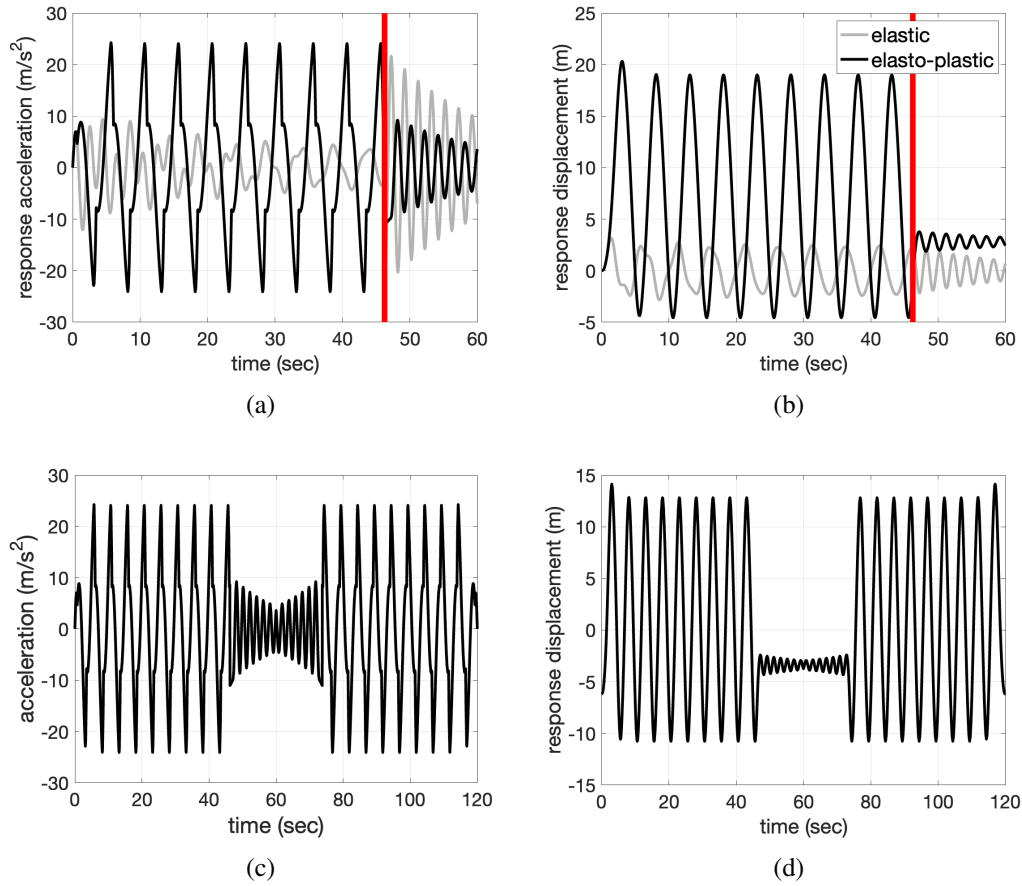


Figure 3: Elastic and the elasto-plastic SDoF oscillator: Time representation of raw numerical signals: (a) acceleration response history, (b) displacement response history and of flipped and zero mean signals for the elasto-plastic oscillator: (c) acceleration response history, (d) displacement response history.

in the system. In contrast, the elasto-plastic oscillator exhibits wider loops, which highlight the effects of yielding, energy dissipation, and the accumulation of residual displacements.

Figure 6a and b show the scalograms of the acceleration signals plotted in Figure 3a, for the elastic and the elasto-plastic oscillators, respectively. The resolution parameter value adopted is equal to $Q_m = 10$, while the maximum value using Eq. 6 is equal to $Q_m^{(max)} = 104.72$ and the low bound is $Q_m^{(min)} = 1.5$. The CWT method has identified both the Eigen-frequencies of the structure and of the harmonic load. The maximum amplitude (skeleton) of the CWT after load removal is reduced compared to that before the removal. In addition, during the harmonic loading phase ($t < t_0$), we observe energy dissipation, shown as ridges at higher frequencies. This is not visible in the case of the elastic oscillator (Fig. 6a). These ridges correspond to superharmonics of the periodic response of the system to harmonic excitation and are not visible on the linear case (Fig. 6a).

The scalogram of the displacement signal is shown in Fig. 6c for the elastic case and in Fig. 6d for the elasto-plastic oscillator. Most of the observations already seen in Fig. 6 can be verified, especially for the elastic case. In the case of elasto-plastic response (Fig. 6d), the superharmonics are less visible except for the first superharmonic. Such information is easier to extract from

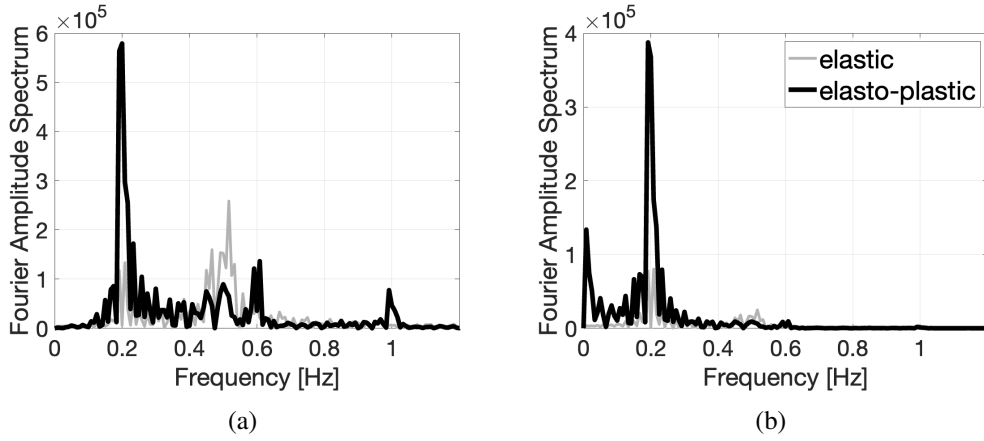


Figure 4: Elastic and the elasto-plastic SDoF oscillators: Fourier representation of flipped and zero mean signals: (a) Fourier spectrum of the acceleration response, (b) Fourier spectrum of the displacement response. The calculation of the Fourier transform was conducted on the time series, subsequent to the removal of their mean.

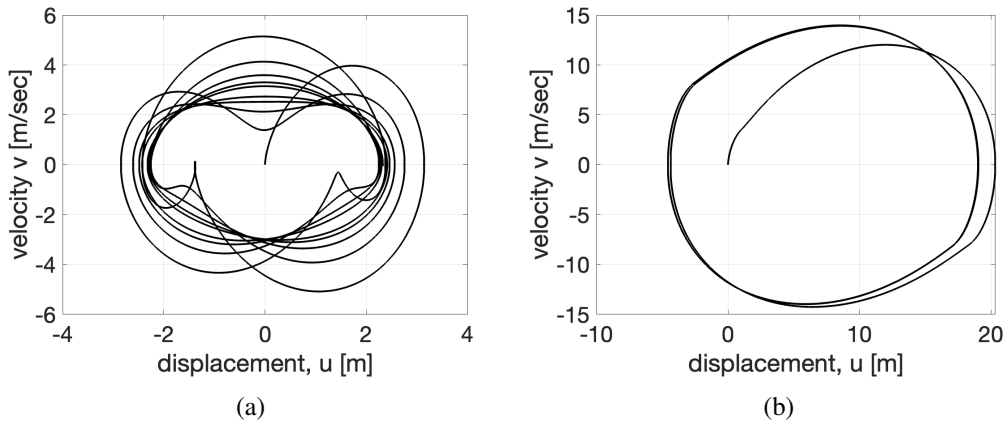


Figure 5: Elastic and elasto-plastic oscillators - Phase plane diagrams: (a) elastic behaviour, (b) elasto-plastic behaviour. Only the forced oscillation part of the response history is displayed: $\forall t \in [0, t_0]$.

accelerations that tend to amplify the high frequencies as opposed to the displacement signals. Moreover, the free response amplitude of the CWT modulus is considerably smaller compared to that the harmonic part. This is in agreement with the response histories of Fig. 3b, since due to the normalization chosen the module of the CWT provides the amplitude of the signal.

In Figure 7a we have identified the ridges between 0.01 and 2 Hz, at which the system vibrates due to nonlinear response. During the harmonic loading phase ($t < t_0$), we observe energy dissipation at higher frequencies. These ridges appear at frequencies that are odd multiples of the excitation frequency $f_h = 0.2\text{Hz}$: $(2p + 1)f_h$, where the cases $p = 1, 2, 3, 4$ are visible in Figure 7b. These ridges correspond to the superharmonics of the periodic response due to harmonic excitation. This is in contrast to the linear case where the response to a harmonic force is harmonic at the same excitation frequency, as shown in the Fig. 6a. It should also be noted that after t_0 , only the two first super-harmonics seem to persist in time. It is evident,

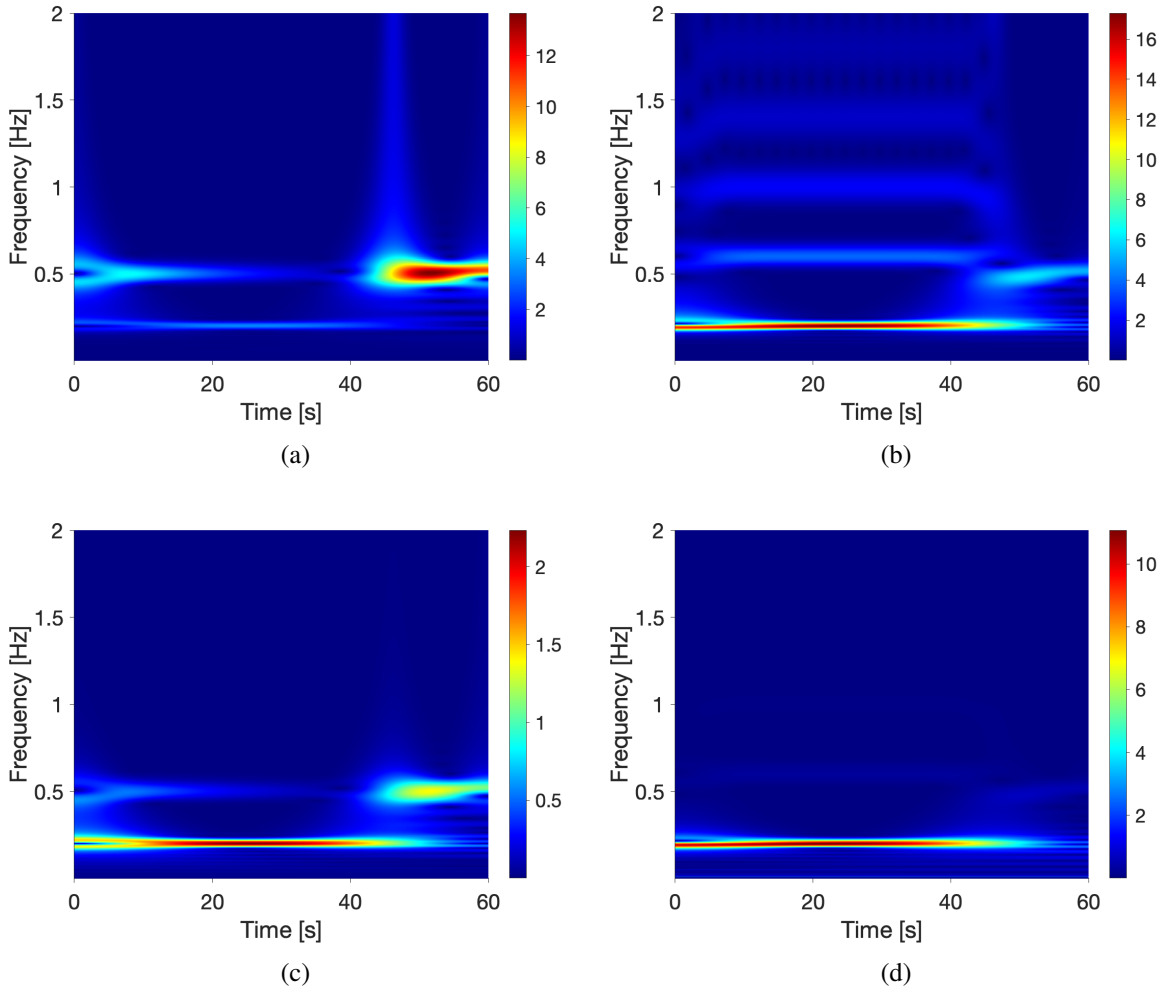


Figure 6: Elastic and elasto-plastic oscillators - Scalograms of the acceleration signal $|T_\psi[\ddot{u}](a, b)|$ shown in Figure 3a: (a) elastic behaviour, (b) elasto-plastic behaviour ($Q_m = 10$), and Scalogram of the displacement signal $|T_\psi[u](a, b)|$ shown in Figure 3b: c elastic behaviour and d elasto-plastic behaviour ($Q_m = 10$).

that understanding the frequency-amplitude relationship is pivotal to understanding nonlinear vibrating systems.

Figure 7b repeats the CWT calculations of Figure 6b, increasing the resolution of the time axis by reducing the resolution parameter Q_m to 2 and displaying the log of the modulus. Several “time discontinuities” at higher frequencies can be clearly seen in the time-frequency plane. A time discontinuity induces a vertical line in the time-frequency plane corresponding to sudden changes in either the load, or the stiffness, of the system. Some researchers [15, 16] called them “spikes”, indicating features such as a jump, or discontinuity, in the time series data as. Previous authors used the Discrete Wavelet Transform method to identify them. Note that the spike due to the sudden change in load at t_0 is stronger and appears in the scalogram of Figure 7b. On the other hand, discontinuities/spikes due to the change of the stiffness appear only in the elasto-plastic case (Fig. 7b), are milder and take place at increments of $0.5T_h$, that is, at every peak of the applied load (Fig. 2a).

All identified ridges between 0.01 and 2 Hz are shown in Figure 8 in time-amplitude and

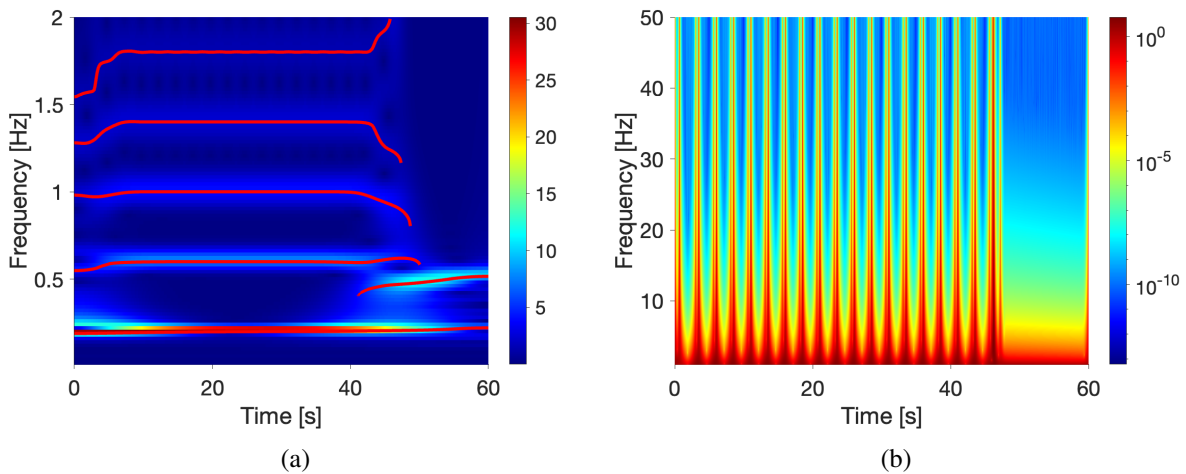


Figure 7: Elasto-plastic oscillator - Focus of the scalogram of the acceleration signal $|T_\psi[\ddot{u}](a, b)|$: (a) First ridges at low frequencies between 0.01 and 2 Hz ($Q_m = 10$), and (b) in $\log_{10}$ scale, at higher frequencies between 2 and 50 Hz in order to study the observed spikes ($Q_m = 2$).

amplitude-frequency ordinates for the elasto-plastic oscillator (Fig. 6b). Ridges with small mean amplitude are shown with gray lines, while the major ridges are shown in colour. It is evident, that most of the energy is concentrated on few ridges, while shorter ridges contain energy that is associated to the higher frequencies.

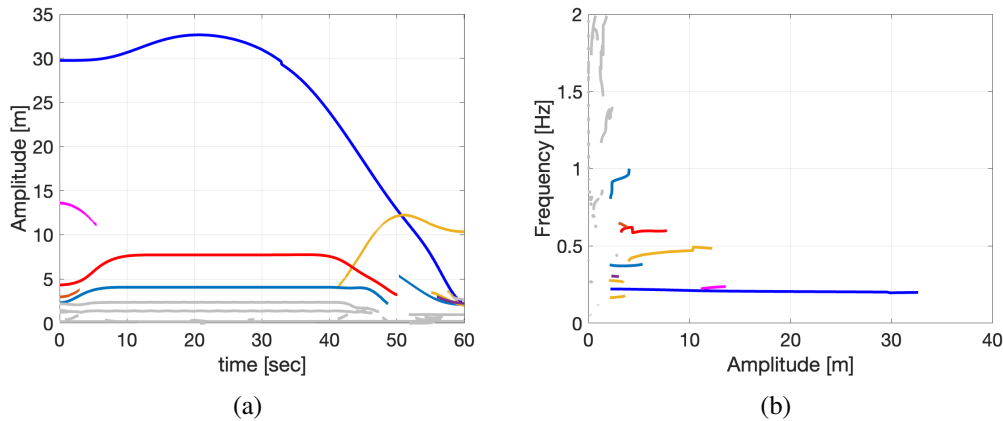


Figure 8: Elasto-plastic oscillator - Major ridges identified between 0.01 and 2 Hz: (a) skeleton - time vs amplitude, (b) amplitude versus instantaneous frequency.

3.2 Application of the CWT method on the 13-storey Sherman Oaks building damaged from multiple earthquakes

The building under study is the 13-storey Sherman Oaks building, which was hit by six damaging earthquakes between 1987 and 2010. This building was initially designed in 1964 as an office building and has 13 stories and two basement floors. The building is shown in Figure 9. The lateral load resisting system consists of moment resisting concrete frames in the upper sto-

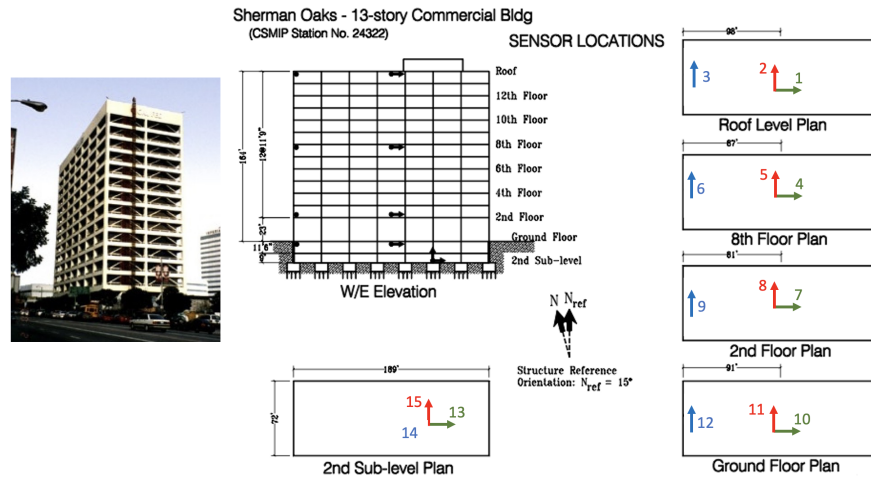


Figure 9: Sherman Oaks building: 13-storey steel frame, side view and instrumentation of the building [33].

ries and concrete shear walls in the basements. The building was founded on alluvial sediments which has lead to a foundation system of grade beams and friction piles. The vertical load carrying system consisted of 11.5 cm thick one-way concrete slabs supported by concrete beams, girders and columns. More details about the building can be found in the California Strong Motion Instrumentation Program (CSMIP) website (Station No. 24322) [33].

The building has been instrumented with 15 sensors Figure 9b. Specifically, three sensors were positioned at stories 2, 8, 13, the ground floor and the second basement. The post-processed acceleration response signals can be found online in the database of CSMIP [33]. Three sensors were positioned at every storey thus allowing to determine the displacements at every location within the slab, which can be assumed rigid. The sensor locations and the numbering of the channels are shown in Figure 9.

Figure 10 shows the sequence of earthquakes that stuck the building from 1987 until 2010. The first three events caused damage, while after the devastating 1994 Northridge earthquake, the building was restored and upgraded. The largest accelerations were recorded in the North-South (N-S) direction (Fig. 9b). Especially in the case of the 1994 Northridge earthquake, the recorded accelerations reached $0.88g$ at the ground floor level. After that earthquake, the building was retrofitted, while the events that followed and were recorded were of relatively low amplitude.

CWT is a powerful tool for modal identification and structural damage detection, but its use requires attention in the case of structural response due to seismic loading. Figure 10 shows the acceleration response signals at the roof of the building along the N-S direction, the direction of the largest recorded accelerations (Fig. 9). The signals were downloaded from the CSMIP database were baseline are were corrected; a bandpass filter was also applied. This processing is a common practice in seismic engineering, while some databases provide also the “uncorrected” signals.

Figures 11a-f show the scalograms of the modulus of the six response acceleration signals of Figure 10. The time-frequency plots of the six earthquakes differ considerably. Some signals trigger a wide spectrum of frequencies, while there are signals where the energy is concentrated in a certain band of frequencies. High mode effects can be seen on the scalograms as short-duration, high-frequency pulses, e.g. Figure 11a, c, d and f and $t = 0 \div 10$ sec and frequencies

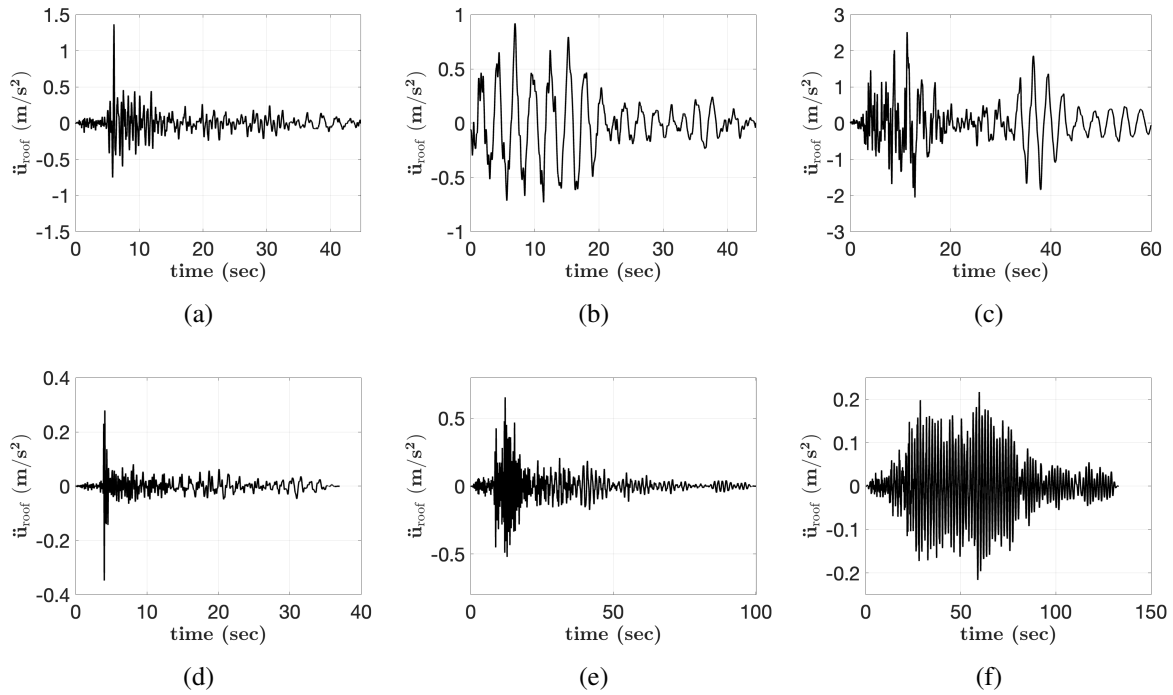


Figure 10: Sherman Oaks building - Applied acceleration time histories recorded at the Sherman Oaks building (N-S direction): (a) 1987 Whittier, (b) 1992 Landers, (c) 1994 Northridge, (d) 2007 Chatsworth, (e) 2008 Chino Hills, (f) 2010 Callexico.

above 1 Hz. Such high frequency pulses, that look like short-time impacts, correspond to strong and short-duration pulses that are hidden within the seismic signal.

Figure 12 shows the identified ridges for the acceleration signal of the Northridge earthquake (Fig. 11c). The ten two strongest ridges are shown with colour, while other less significant ridges are shown with gray colour. It can be clearly seen that the ridges differ from those of the SDoF oscillator due to the different excitation. There are many ridges of varying length, while most of the energy is concentrated on a single ridge. Interestingly, there are many short-duration and high-frequency ridges which also contribute to the energy content of the signal.

Structural damage in buildings is typically measured from displacement-based response parameters. Specifically, the most common proxies of seismic damage are interstorey drifts and plastic hinge rotations, both directly obtained from the displacement response histories. Consequently, the use of time-frequency plots of displacement signals is a more expedient approach for the identification of structural damage resulting from substantial seismic loading. In this respect, Figures 13a-f show the corresponding displacement scalograms. The displacement signals were obtained through integrating twice the accelerations. In the displacement plots, the short-duration, high-frequency pulses have been filtered out, while the maximum modulus values $|T_{\psi_{n+2}}[u](b, a)|$ appear at the first, low frequency modes, which is common for seismic loading.

When the classical Cauchy mother wavelet is adopted, the modulus of the displacement scalogram can be directly obtained from the acceleration signal with the aid of Eq. 13. This is quite useful in seismic engineering applications, since the CWT can be directly applied on the unprocessed signal, i.e. on data that have not been baseline corrected and/or filtered. Note that

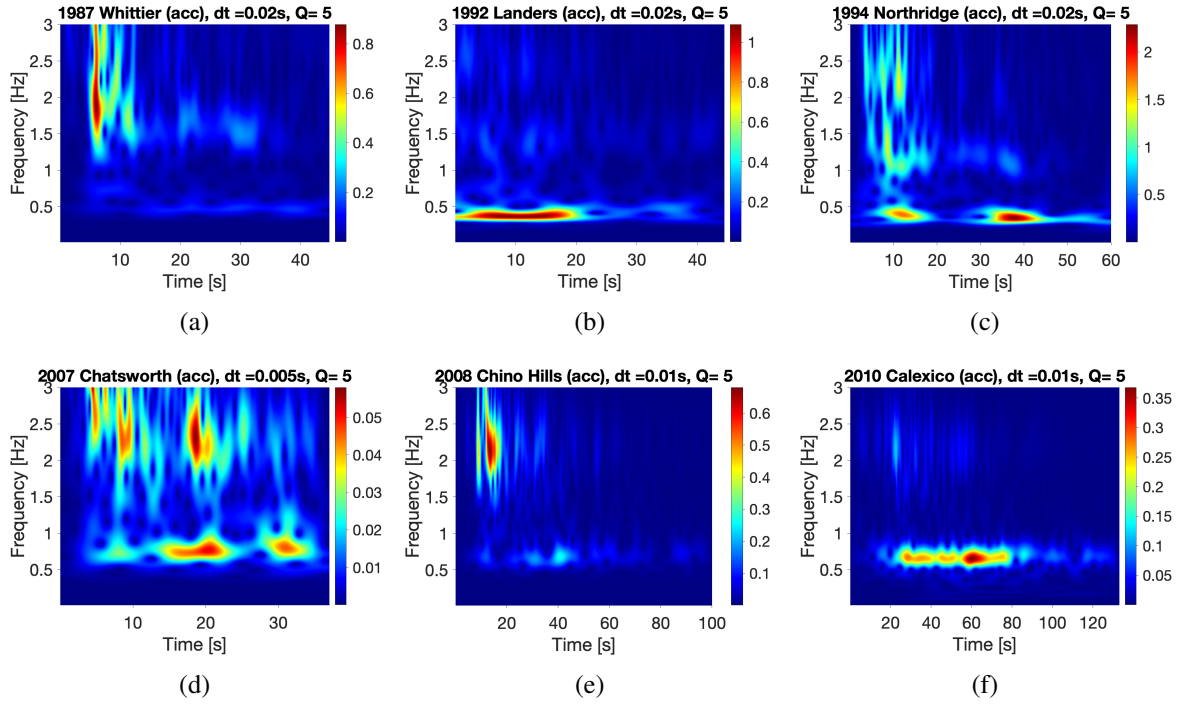


Figure 11: Sherman Oaks building - Scalograms of the response **acceleration**: (a) 1987 Whittier, (b) 1992 Landers, (c) 1994 Northridge, (d) 2007 Chatsworth, (e) 2008 Chino Hills, (f) 2010 Calexico Building roof, N-S direction, quality factor $Q_m = 5$.

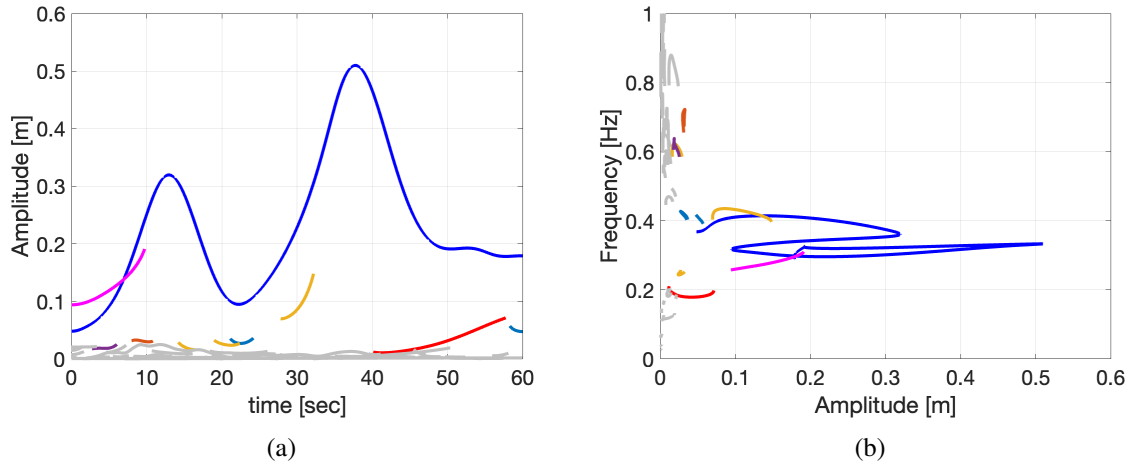


Figure 12: Sherman Oaks building - Amplitude-time and Frequency-amplitude plots of the ridges obtained for the 1994 Northridge earthquake (N-S): (a) time vs amplitude, (b) amplitude versus instantaneous frequency. The ten strongest ridges are shown with colour, while other less significant ridges are shown with gray colour.

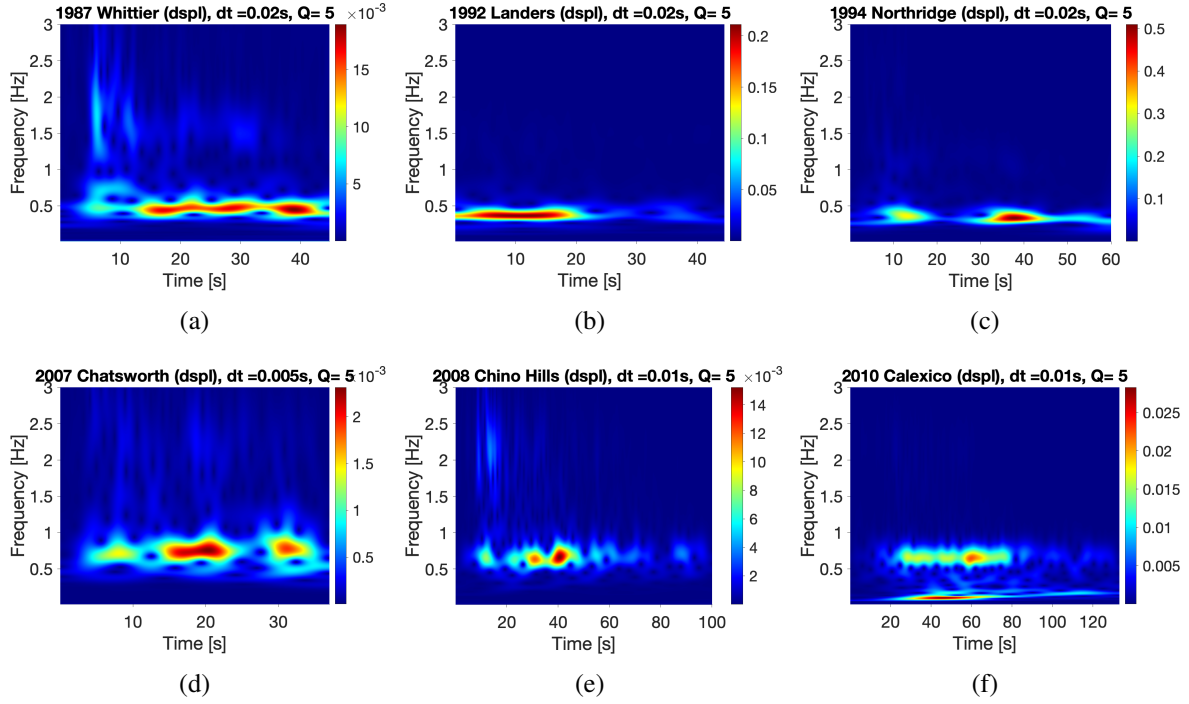


Figure 13: Sherman Oaks building - Scalograms of the response **displacement (integrated)**: (a) 1987 Whittier, (b) 1992 Landers, (c) 1994 Northridge, (d) 2007 Chatsworth, (e) 2008 Chino Hills, (f) 2010 Calexico Building roof, N-S direction, quality factor $Q_m = 5$.

signal processing can distort or bias the signal at hand. Following Eq. 13, a different order of the Cauchy wavelet should be used for the acceleration signal in order to achieve the desired resolution for the displacement scalogram. More specifically, the formula receives as input $T_{\psi_n}[\ddot{u}](b, a)$ and outputs $T_{\psi_{n+2}}[u](b, a)$. The quality factor Q_m is given by the expression:

$$Q_m = \frac{n + 0.5}{\sqrt{2n + 1}} \Leftrightarrow n = 2Q_m^2 - \frac{1}{2}. \quad (16)$$

Setting $n \rightarrow n + 2$, we obtain $Q_m^{(n+2)}$. After some simple manipulations, it can be shown that:

$$Q_m^{(n)} = \sqrt{(Q_m^{(n+2)})^2 - 1}. \quad (17)$$

Therefore, if the desired Q_m is known and according to the left-handside of Eq. 13, is equal to $Q_m^{(n+2)}$, the Q_m that should be used for the CWT analysis of the wavelet is that of Eq. 17. Note that in order to apply Eq. 13, we have to update the Cauchy n parameter using Eq. 16. According to Eq. 13, when the frequency increases, the amplitude of the absolute value of $T_{\psi_{n+2}}[u](b, \alpha)$ decreases to the power of two of the frequency, thus the ridges of the higher frequencies that are visible on the acceleration scalogram will be less visible on the displacement scalogram.

Finally, Figure 15 examines the presence of high-frequency spikes on the seismic acceleration response signals. The frequency range is adjusted to $0 \div 50\text{Hz}$ and the quality factor is assumed equal to $Q_m = 2$ in order to improve the time resolution and make the spikes more clearly visible. As can be seen in Figure 15, spikes are visible in subplots Figs 15a and b, but not in c. As already discussed, the spikes represent discontinuities, hence they can be seen as proxies of damage during the excitation. However, they are not always clearly visible.

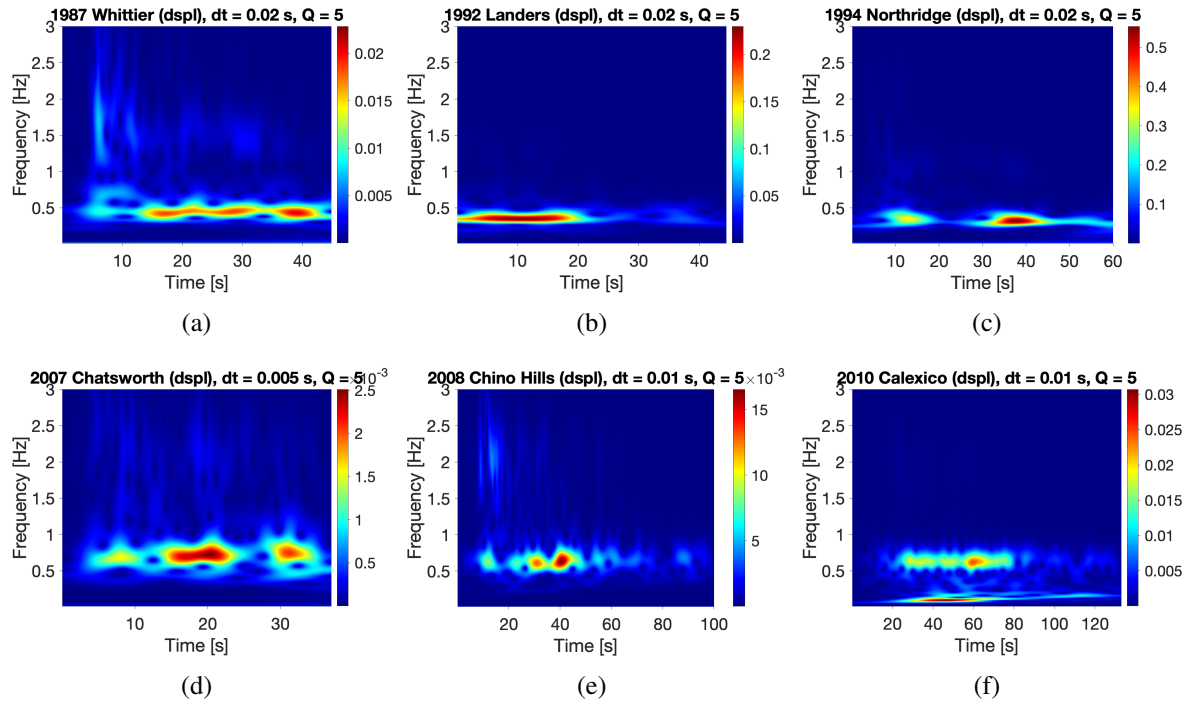


Figure 14: Sherman Oaks building - Scalograms of the response **displacement using the formula of Eq. 13**: (a) 1987 Whittier, (b) 1992 Landers, (c) 1994 Northridge, (d) 2007 Chatsworth, (e) 2008 Chino Hills, (f) 2010 Calexico Building roof. All plots refer to N-S direction, quality factor $Q_m = 5$.

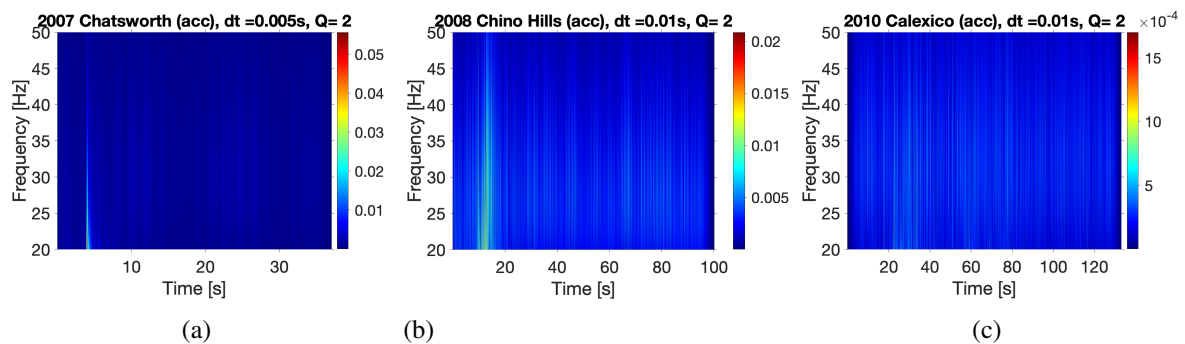


Figure 15: Sherman Oaks building - Spike identification. The presence of high-frequency spikes is examined at the frequency range 0 – 50Hz, a quality factor $Q_m = 2$ is assumed in order to improve the time resolution and make the spikes visible: (a) 2007 Chatsworth, (b) 2008 Chino Hills, (c) 2010 Calexico Building roof.

4 Conclusions

The Continuous Wavelet Transform (CWT) method is used to analyse how structures respond to earthquakes. This paper reviews some of the CWT's fundamental properties, along with suggestions for selecting the most appropriate mother wavelet for a given problem. Two example applications are then presented: a simulated elasto-plastic single-degree-of-freedom system subjected to an abruptly stopped periodic force, and a fully instrumented 13-storey building hit by a sequence of earthquakes. A key advantage of wavelet analysis is that it expresses the processed signal in the time-frequency domain. This makes it easy to estimate modal characteristics or detect nonlinear behaviour, such as locating and assessing damage severity over time. Thus, the CWT enables the straightforward detection of nonlinear structural behaviours, such as the presence of superharmonic components or the evolution of instantaneous frequencies over time. Another advantage of using the CWT with the proposed mother wavelet is that the CWT of displacement can be easily computed from the CWT of acceleration. These examples demonstrate that the CWT method can effectively identify the behaviour of buildings from their seismic responses, providing valuable information for structural assessments during earthquakes.

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