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BENDING STIFFNESS OF A MULTILAYERED PLATE

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Abstract. *A thin elastic multilayered plate consisting of alternating hard and soft isotropic layers is studied. One of the face planes is subject to a normal pressure and the other face plane is free. A formula for the deflection of an infinite plate under a doubly periodic external force is delivered using asymptotic expansions. This formula is also applied for a rectangular plate with Navier boundary conditions on its edges. The maximal deflection is accepted as a measure of the plate stiffness. The purpose of the present paper is to obtain an expression for the plate deflection and find an optimal distribution of hard and soft layers assuming that their total thicknesses are given. The Monte Carlo method is used for finding the optimal distribution of layers.*

1 INTRODUCTION

The 2D classical equation for a plate was obtained by Sophie Germain in 1808 with the purpose of explaining Chladni figures. The equation of a plate bending can be obtained on the basis of the Kirchhoff–Love (KL) hypotheses [1, 2]. The more involved and sometimes more exact equation, which takes into account the transversal shear, follows from the Timoshenko–Reissner (TR) hypotheses [3, 4].

The 2D models of plates and shells are chiefly based on the 3D equations of the theory of elasticity. The methods of unknown functions expansions in series of Legendre polynomials in the thickness direction were used in [5, 6]. Numerous investigations [7–9] were devoted to the derivation of 2D equations by using asymptotic expansions in power series in the small parameter $\mu = h/L$, which is equal to the dimensionless plate thickness (h and L are, respectively, the thickness and the typical wave length in the tangential directions). A different approach [10,11] rests on the direct derivation of 2D equations of plates and shells without referring to a 3D media. An account of general problems of the plates theory may be found in the books [12–14].

The present paper is concerned with a thin plate of constant thickness made of a linearly elastic material that is transversally isotropic and heterogeneous in the thickness direction. For a transversally isotropic material the accepted simplification [15], for which a 3D system of sixth order of the theory of elasticity splits into systems of second and fourth orders, is possible. Asymptotic expansions in powers of the small thickness parameter μ are constructed, the bending equation of second-order accuracy (the SA model) is obtained. The origin of this paper is the paper [16], in which an isotropic homogeneous plate is studied, and the paper [8], in which a heterogeneous plate is briefly examined. For a multi-layer plate with alternating hard and soft layers, an explicit formula based on the SA model for the maximum deflection is delivered. The results of the SA model are compared with the KL classical model and also with the exact numerical solution. The ratio $\eta = E_2/E_1$ of the soft and hard Young moduli changes in a very wide range ($0.0001 \leq \eta \leq 1$).

The aforementioned formula is obtained for an infinite in the tangential directions plate subject to a doubly periodic external load. After employing the Fourier expansions this formula is applied to a rectangular plate with the Navier boundary conditions.

For a multi-layer plate with alternating hard and soft layers, the distribution of layer thicknesses, which gives the maximum bending stiffness, is found. Here it is assumed that the summary thicknesses of the hard and of the soft layers are given. This problem is attacked by the Monte Carlo method. For a very small ratio η of elastic moduli, the three-layer plate with the soft layer lying between two hard layers was found not to be optimal. In this case, the plate with the maximum bending stiffness is multi-layered one with the number of layers $n > 3$.

2 EQUILIBRIUM EQUATIONS AND THEIR SIMPLIFICATION

Consider the linear bending problem of a thin plate made of a transversally isotropic heterogeneous material. The 3D equilibrium equations are

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0, \qquad i,j = 1,2,3, \qquad 0 \leq x_3 = z \leq h, \tag{1}$$

where x_j are the Cartesian coordinates, f_i are the projections of the external load intensity, the summation being carried out over repeating subscripts.

The stresses σ_{ij} are related to the strains ε_{ij} as follows:

$$\begin{aligned}
&\sigma_{11} = E_{11}\varepsilon_{11} + E_{12}\varepsilon_{22} + E_{13}\varepsilon_{33}, && \sigma_{12} = G_{12}\varepsilon_{12},\\
&\sigma_{22} = E_{12}\varepsilon_{11} + E_{11}\varepsilon_{22} + E_{13}\varepsilon_{33}, && \sigma_{13} = G_{13}\varepsilon_{13},\\
&\sigma_{33} = E_{13}\varepsilon_{11} + E_{13}\varepsilon_{22} + E_{33}\varepsilon_{33}, && \sigma_{23} = G_{13}\varepsilon_{23},\\
&\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \qquad \varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}, \quad etc.
\end{aligned} \tag{2}$$

Here, $E_{11} = E_{12} + 2G_{12}$, u_i are deflections. The elastic moduli E_{ij}, G_{ij} are independent of the tangential coordinates x_1, x_2, but they may depend on the transversal coordinate $x_3 = z$. For functionally gradient materials the moduli are continuous functions in z, and for multi-layered plates they are piecewise continuous functions.

For an isotropic material

$$E_{11}=E_{33}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \quad E_{12}=E_{13}=\frac{E\nu}{(1+\nu)(1-2\nu)}, \quad G_{12}=G_{13}=G=\frac{E}{2(1+\nu)}, \tag{3}$$

where E and ν are the Young's modulus and the Poisson ratio, respectively.

We set the homogeneous boundary conditions on the face planes $z = 0$ and $z = h$

$$\sigma_{i3} = 0, \quad i = 1, 2, 3. \tag{4}$$

If the surface forces are given, then they are included in the body forces by using the Dirac delta-function.

Introduce new unknown functions u, v, σ, τ as

$$\begin{aligned}
u &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \qquad & v &= \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1},\\
\sigma &= \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2}, \qquad & \tau &= \frac{\partial \sigma_{13}}{\partial x_2} - \frac{\partial \sigma_{23}}{\partial x_1}.
\end{aligned} \tag{5}$$

For a transversally isotropic material the system (1), (2) is split into two subsystems [15]:

$$\frac{\partial \tau}{\partial z} + G_{12}\Delta v + m_1 = 0, \quad \tau = G_{13}\frac{\partial v}{\partial z}, \qquad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \quad m_1 = \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}; \tag{6}$$

$$\begin{aligned}
&\sigma_{33} = E_{13}u + E_{33}\frac{\partial w}{\partial z}, && \sigma = G_{13}\left(\frac{\partial u}{\partial z} + \Delta w\right), && w = u_3,\\
&\frac{\partial \sigma}{\partial z} + E_0\Delta u + \frac{E_{13}}{E_{33}}\Delta\sigma_{33} + m = 0, && \frac{\partial \sigma_{33}}{\partial z} + \sigma + f_3 = 0, && m = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2},
\end{aligned} \tag{7}$$

with

$$E_0 = E_{11} - \frac{E_{13}^2}{E_{33}} = \frac{E}{1-\nu^2}.$$

System (6) of the second differential order in z describes the boundary layer and will not be studied here.

The fourth order system (7) describes the plate bending. The 2D plate model is obtained here by using asymptotic expansions in powers of the small parameter $\mu = h/L$ [8,17].

We introduce the dimensionless variables (denoted with hats)

$$\{u_1, u_2, w, z\} = h\{\hat{u}_1, \hat{u}_2, \hat{w}, \hat{z}\}, \quad \{x_1, x_2\} = L\{\hat{x}_1, \hat{x}_2\}, \quad u = \mu\hat{u},$$
$$\{\sigma_{ij}, \sigma, E_{ij}, G_{ij}, E_0\} = E_*\{\hat{\sigma}_{ij}, \hat{\sigma}, \hat{E}_{ij}, \hat{G}_{ij}, c_0\}, \quad f_i = \frac{E_*}{h}\hat{f}_i, \quad E_* = \frac{1}{h}\int_0^h E_0(z)dz, \tag{8}$$

and rewrite system (7) in the dimensionless form

$$\frac{\partial\hat{w}}{\partial\hat{z}} = -\mu c_\nu \hat{u} + c_3\hat{\sigma}_{33}, \qquad \frac{\partial\hat{u}}{\partial\hat{z}} = -\mu\hat{\Delta}\hat{w} + c_g\hat{\sigma},$$
$$\frac{\partial\hat{\sigma}}{\partial\hat{z}} = Y_3(\hat{z}) = -\mu^2 c_0\hat{\Delta}\hat{u} - \mu c_\nu\hat{\Delta}\hat{\sigma}_{33} - \hat{m}, \qquad \frac{\partial\hat{\sigma}_{33}}{\partial\hat{z}} = Y_4(\hat{z}) = -\mu\hat{\sigma} - \hat{f}_3, \quad 0 \le \hat{z} \le 1, \tag{9}$$

with

$$c_\nu = \frac{E_{13}}{E_{33}} = \frac{\nu}{1-\nu}, \quad c_3 = \frac{E_*}{E_{33}}, \quad c_g = \frac{E_*}{G_{13}},$$
$$\hat{\sigma} = \frac{\partial\hat{\sigma}_{13}}{\partial\hat{x}_1} + \frac{\partial\hat{\sigma}_{23}}{\partial\hat{x}_2}, \quad \hat{m} = \frac{\partial\hat{f}_1}{\partial\hat{x}_1} + \frac{\partial\hat{f}_2}{\partial\hat{x}_2}, \quad \hat{\Delta} = \frac{\partial^2}{\partial\hat{x}_1^2} + \frac{\partial^2}{\partial\hat{x}_2^2}. \tag{10}$$

Here, L is the typical value of the wave length in the tangential directions, and E_* is the average value of the modulus E_0. The dimensionless coefficients c_0, c_ν, c_g, c_3 are the given functions of $\hat{z}$. From the boundary conditions (4) we get

$$\hat{\sigma} = \hat{\sigma}_{33} = 0 \quad at \quad \hat{z} = 0 \quad and \quad \hat{z} = 1. \tag{11}$$

In what follows, the hat sign will be omitted.

3 ASYMPTOTIC SOLUTION OF THE BOUNDARY VALUE PROBLEM (9), (11)

Assume that the dimensionless external forces f_3, m are of the order of unity. Then the orders of the unknown functions are

$$\sigma_{33} = O(1), \quad \sigma = O(\mu^{-1}), \quad u = O(\mu^{-3}), \quad w = O(\mu^{-4}). \tag{12}$$

The right-hand sides of Eqs. (9) are small, and the method of iterations [10,18] is used. To construct the solution of second-order accuracy we seek it as

$$w = \mu^{-4}w_0 + \mu^{-2}w_2, \quad u = \mu^{-3}u_0 + \mu^{-1}u_2, \quad \sigma = \mu^{-1}\sigma_0 + \mu\sigma_2, \quad \sigma_{33} = \sigma_{33,0} + \mu^2\sigma_{33,2}. \tag{13}$$

The arbitrary functions $w^0(x_1, x_2)$ and $u^0(x_1, x_2)$ appear after the integration in z of the first two equations (9). These functions are found from the compatibility conditions of the remaining two equations (9) and the boundary conditions (11) [17]

$$\langle Y_3(z)\rangle = 0, \quad \langle Y_4(z)\rangle = 0, \qquad \langle Z(z)\rangle \equiv \int_0^1 Z(z)dz. \tag{14}$$

In the zero approximation we get

$$w_0 = w_0(x_1, x_2), \quad u_0 = (a - z)\Delta^2 w_0, \quad a = \langle zc_0(z)\rangle,$$
$$\sigma_0 = \varphi_1(z)\Delta^2 w_0^0, \quad \varphi_1(z) = \int_0^z c_0(z)(z-a)dz,$$
$$D\Delta^2 w_0 = F_3, \qquad D = \langle (z-a)^2 c_0(z)\rangle, \quad F_3 = \langle f_3(z)\rangle, \tag{15}$$
$$\sigma_{33,0} = -\frac{F_3}{D}\varphi_2 - \varphi_3, \quad \varphi_2(z) = \int_0^z \varphi_1(z)dz, \quad \varphi_3(x_1, x_2, z) = \int_0^z f_3(x_1, x_2, z)dz,$$

where $z = a$ is the position of the plate neutral layer, D is the bending stiffness of a plate with the variable elastic moduli, F_3 is the full transversal force. The equation $D\Delta^2 w^0 = F_3$ corresponds to the classical KL model.

In the second approximation the solution is more unwieldy. Here, we give only the function w_2, which depends on z. At $z = 0$ it satisfies the equation

$$D\Delta^2 w_2(0) = A\Delta F_3 + N(\Delta f_3) - M, \qquad A = A_g - A_\nu, \tag{16}$$

where

$$\begin{aligned} N(\Delta f_3) &= \int_0^1 c_\nu(z)(a-z)\left(\int_0^z \Delta f_3 dz_1\right)dz, \quad M = \int_0^1 (a-z)m(z)dz, \\ A_g &= \frac{1}{D}\int_0^1 c_0(z)(z-a)\int_0^z c_g(z_1)\int_0^{z_1} c_0(z_2)(z_2-a)dz_2dz_1dz, \\ A_\nu &= \frac{1}{D}\int_0^1 (z-a)\int_0^z\int_0^{z_1}\left(c_\nu(z)c_0(z_2)+c_0(z)c_\nu(z_2)\right)(z_2-a)dz_2dz_1dz. \end{aligned} \tag{17}$$

The full deflection of the reference plane $z = 0$ satisfies the equation

$$D\mu^4\Delta^2 w(0) = F_3 + \mu^2(A\Delta F_3 + L(\Delta f_3) - M) + O(\mu^4), \tag{18}$$

in which the coefficients D and A depend of the elastic moduli distribution in the plate thickness, the summands $L(\Delta f_3)$ and M depend on the distribution of external transversal and tangential loads, respectively. The summands A_g and A_ν take into account the transversal shear and the Poisson's strains, respectively.

The full deflection $w(z)$ of the arbitrary plane z is expressed through $w(0)$ as

$$w(z) = w(0) + \mu^2\Delta w(0)\int_0^z c_\nu(z)(z-a)dz. \tag{19}$$

4 MULTILAYERED PLATE UNDER A NORMAL PRESSURE

Consider a multi-layered plate with hard and soft isotropic homogeneous layers under a normal pressure F_3 acting on the plane $z = 0$. In this case, in Eq. (18) $L(\Delta f_3) = \langle c_\nu(a-z)\rangle \Delta F_3$ and $M = 0$, and so Eq. (18) assumes the form

$$D\mu^4\Delta^2 w(0) = F_3 + \mu^2 A_1\Delta F_3, \quad A_1 = A + \int_0^1 c_\nu(z)(z-a)dz. \tag{20}$$

If the plate is homogeneous one-layered, then the elastic moduli c_0, c_g, c_ν are constant, and so the integrals in (17) can be taken, and so in Eq. (20) we have

$$D\mu^4\Delta^2 w(0) = F_3 + \mu^2 A_1\Delta F_3, \quad A_1 = \frac{2c_\nu - c_g}{10} = -\frac{1}{5}, \quad D = \frac{1}{12}. \tag{21}$$

Using the neutral plane $z = 1/2$ as a reference one due to Eq. (19) we get

$$D\mu^4\Delta^2 w(1/2) = F_3 + \mu^2 A_2\Delta F_3, \quad A_2 = \frac{3c_\nu - 4c_g}{40} = \frac{3\nu - 8}{40(1-\nu)}. \tag{22}$$

This value A_2 was obtained in [16] and was repeated in [8].

Consider a plate of thickness h consisting of $n = 2n_0 + 1$ homogeneous isotropic layers of thickness $h_k,\ k = 1, 2, ..., n$ ($h = \sum h_k$). Let E_1, ν_1 and E_2, ν_2 ($E_2/E_1 \le 1$) be the Young's moduli and the Poisson ratios of hard layers with odd numbers and of soft layers with even numbers, respectively. We set

$$z_0 = 0, \quad z_k = \sum_{i=1}^{k} h_i, \quad k = 1, \ldots, n. \tag{23}$$

The elastic moduli are known to be piecewise functions in z, and so we put

$$\begin{aligned} e_k &= \frac{E_1}{1-\nu_1^2}, \quad c_k = \frac{\nu_1}{1-\nu_1}, \quad g_k = \frac{E_1}{2(1+\nu_1)}, \qquad k \ \ is\ odd, \\ e_k &= \frac{E_2}{1-\nu_2^2}, \quad c_k = \frac{\nu_2}{1-\nu_2}, \quad g_k = \frac{E_2}{2(1+\nu_2)}, \qquad k \ \ is\ even. \end{aligned} \tag{24}$$

Calculating the integrals in Eqs. (15) and (17) we find the coordinate $z = a$ of the neutral layer, the bending stiffness D according to the KL model, and the coefficients A_g and A_ν in Eqs. (17) as follows:

$$\begin{aligned} a &= \frac{1}{2}\sum_{k=1}^{n} e_k(z_k^2 - z_{k-1}^2)\left(\sum_{k=1}^{n} e_k h_k\right)^{-1}, \quad D = \frac{1}{3}\sum_{k=1}^{n} e_k(\hat{z}_k^3 - \hat{z}_{k-1}^3), \quad \hat{z}_k = z_k - a, \\ A_g &= \frac{1}{D}\sum_{k=1}^{n}\left(\frac{e_k f_{1k}}{2}(\hat{z}_k^2 - \hat{z}_{k-1}^2) + \frac{f_{2k}}{3g_k}(\hat{z}_k^3 - \hat{z}_{k-1}^3) + \frac{e_k}{30g_k}(\hat{z}_k^5 - \hat{z}_{k-1}^5)\right), \\ A_\nu &= \frac{1}{D}\sum_{k=1}^{n}\left(\frac{e_k f_{3k}^c + c_k f_{3k}^e}{2}(\hat{z}_k^2 - \hat{z}_{k-1}^2) + (c_k f_{4k}^e + e_k f_{4k}^c)(\hat{z}_k^3 - \hat{z}_{k-1}^3) + \frac{e_k c_k}{15}(\hat{z}_k^5 - \hat{z}_{k-1}^5)\right), \end{aligned} \tag{25}$$

where

$$\begin{aligned} f_{1k} &= \sum_{i=1}^{k-1}\left(\frac{f_{2i}h_i}{g_i} + \frac{e_i}{6g_i}(\hat{z}_i^3 - \hat{z}_{i-1}^3)\right) - \frac{f_{2k}h_k}{g_k} - \frac{e_k}{6g_k}\hat{z}_k^3, \\ f_{2k} &= \frac{1}{2}\sum_{i=1}^{k-1} e_i(\hat{z}_i^2 - \hat{z}_{i-1}^2) - \frac{1}{2}e_k\hat{z}_k^2, \\ f_{3k}^e &= \sum_{i=1}^{k-1}\left(f_{4i}^e h_i + \frac{e_i}{6}(\hat{z}_i^3 - \hat{z}_{i-1}^3)\right) - f_{4k}^e h_k - \frac{e_k}{6}\hat{z}_k^3, \\ f_{4k}^e &= \frac{1}{2}\sum_{i=1}^{k-1} e_i(\hat{z}_i^2 - \hat{z}_{i-1}^2) - \frac{1}{2}e_k\hat{z}_k^2, \quad (e \to c). \end{aligned} \tag{26}$$

Here, $(e \to c)$ means that the similar formulae also hold for the moduli c_k.

Assume, at first, that the plate is infinite ($-\infty < x_1, x_2 < \infty$), and consider a doubly periodic normal pressure $F_3 = F_3^0 \sin r_1 x_1 \sin r_2 x_2$. Then the deflection $W(x_1, x_2, z) = W(z)\sin r_1 x_1 \sin r_2 x_2$ is also doubly periodic. We rewrite Eq. (20) as

$$Dr^4 W(0) = (1 - r^2 A_1)F_3^0, \tag{27}$$

where

$$A_1 = A_g - A_\nu - \frac{1}{2}\sum_{k=1}^{n} c_k(\hat{z}_k^2 - \hat{z}_{k-1}^2), \qquad r^2 = r_1^2 + r_2^2, \tag{28}$$

and the coefficients $D,\ A_\nu,\ A_g$ are given in Eqs. (25).

In the dimensionless form (with hats)

$$W(0) = hw(0), \quad D = h^3\hat{D}, \quad A_1 = h^2\hat{A}_1, \quad \mu = rh \tag{29}$$

Equation (27) gives the deflection of the second-order accuracy (SA)

$$w(0)^{SA} = w^{KL}(1 - \mu^2\hat{A}_1), \quad w^{KL} = \frac{F_3^0 h}{\mu^4\hat{D}}, \tag{30}$$

where w^{KL} is the deflection in the KL model.

A comparison of the expressions $\mu = rh$ and $\mu = h/L$ leads to the conclusion that in this problem the typical wave length L should be taken as

$$L = \frac{1}{r} = \frac{1}{\sqrt{r_1^2 + r_2^2}}. \tag{31}$$

To discuss the errors of the approximate expressions (30) for w^{KL} and $w(0)^{SA}$ we require some numerical examples. We consider a multi-layered plate with $n = 11$ layers and $h = 1$, $\nu_1 = \nu_2 = 0.3$, $F_3^0 = 1$, and take $h_k = 1/18,\ E_k = 1,\ k = 2i - 1$, i=1,. . . ,6, for the hard layers, and $h_k = 2/15,\ E_k = \eta,\ k = 2i,\ i = 1, \ldots, 5$, for the soft layers. Such a plate will be referred to as a plate with uniform thickness distribution. We take two values of the relative thickness parameter: $\mu = 0.3776$ and $\mu = 0.2221$. The ratio $\eta = E_2/E_1$ of the elastic moduli will vary in the range $0.0001 \le \eta \le 1$. We compare the approximate values (30) with the exact value w^e found by numerical solution of Eqs. (9). The results are given in Table 1. Columns 2–4 correspond to $\mu = 0.3776$, and columns 5–7, to $\mu = 0.2221$.

1	2	3	4	5	6	7
η	w^e	w^{KL}/w^e	w^{SA}/w^e	w^e	w^{KL}/w^e	w^{SA}/w^e
1.0	553	0.9717	0.9994	4529	0.9902	0.9999
0.31	936	0.9539	0.9992	7578	0.9837	0.9999
0.1	1270	0.8899	0.9985	9830	0.9532	0.9998
0.031	1684	0.7322	0.9975	11587	0.8882	0.9996
0.01	2700	0.4703	0.9987	14723	0.7198	0.9998
0.0031	5746	0.2231	1.0110	23692	0.4510	1.0036
0.001	14258	0.0800	1.0441	51150	0.2098	1.0192
0.00031	41793	0.0308	1.1156	134232	0.0800	1.0557
0.0001	110822	0.0116	1.3054	374835	0.0287	1.1337

Table 1: Error of the approximate models for some values of η with $\mu = 0.3776$ and $\mu = 0.2221$.

The relative error of the KL and SA models is found by comparing the results in columns 3,4,6,7 with one. The exactness of the KL model is of first asymptotical order with respect to μ, and it is acceptable for applications only for $0.1 \le \eta \le 1$. The SA model of second asymptotical order may be used in the very wide range $0.001 \le \eta \le 1$. For $0.1 \le \eta \le 1$ the SA model is essentially more exact than the TR model. For $0.0001 \le \eta \le 0.001$ the 2D models are unacceptable.

The errors of the 2D models decrease simultaneously with the thickness parameter μ. This follows from the comparison of columns 3 and 6, and also of columns 4 and 7.

5 DEFLECTION OF A RECTANGULAR MULTILAYERED PLATE

Consider a rectangular multi-layer plate with $0 \le x_1 \le a_1,\ 0 \le x_2 \le a_2,\ 0 \le z \le h$ (Fig. 1). In the previous sections the boundary conditions at the plate edges were either not imposed or a plate was assumed to be infinite in the tangential directions. Now the following variant of boundary conditions will be accepted:

$$\begin{aligned} &u_2 = w = \sigma_{11} = 0 \qquad at \quad x_1 = 0, \quad x_1 = a_1, \\ &u_1 = w = \sigma_{22} = 0 \qquad at \quad x_2 = 0, \quad x_2 = a_2, \end{aligned} \tag{32}$$

(the so-called Navier conditions) and

$$\sigma_{13} = \sigma_{23} = 0,\ \sigma_{33} = -F_3^0 \quad at \quad z = 0; \qquad \sigma_{13} = \sigma_{23} = \sigma_{33} = 0 \quad at \quad z = h. \tag{33}$$

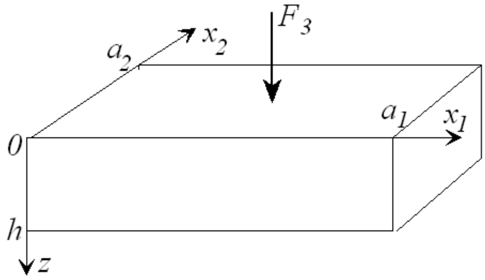

Fig. 1. A plate.

The functions

$$\begin{aligned} &u_1(x_1, x_2, z) = r_{1m}\, u(z) \cos(r_{1m}x_1) \sin(r_{2n}x_2), \\ &u_2(x_1, x_2, z) = r_{2n}\, u(z) \sin(r_{1m}x_1) \cos(r_{2n}x_2), \quad r_{1m} = \frac{m\pi}{a_1},\ r_{2n} = \frac{n\pi}{a_2}, \quad m, n = 1, 2, \ldots, \\ &w(x_1, x_2, z) = w(z) \sin(r_{1m}x_1) \sin(r_{2n}x_2), \end{aligned} \tag{34}$$

satisfy the boundary conditions (32). The boundary layer does not appear, because $v = \partial u_2/\partial x_1 - \partial u_1/\partial x_2 \equiv 0$ (see Eqs. (5)).

Eq. (30) for the deflection is delivered for a doubly periodic external force $F_3 = F_3^0 \sin(r_1 x_1) \sin(r_2 x_2)$ with $\mu = h\sqrt{r_1^2 + r_2^2}$. To use Eq. (30) for a constant force F_3 in the rectangle $0 \le x_1 \le a_1,\ 0 \le x_2 \le a_2$ we expand it in a double Fourier series

$$F_3 = F_3 \sum_{m,n=1,3,\ldots} \frac{16}{mn\pi^2} \sin(r_{1m}x_1) \sin(r_{2n}x_2), \tag{35}$$

and apply Eq. (30) to the each summand of (35). In this case

$$\mu = \mu_{mn} = h\sqrt{\left(\frac{m\pi}{a_1}\right)^2 + \left(\frac{n\pi}{a_2}\right)^2}, \quad m, n = 1, 2, \ldots \tag{36}$$

and so the full deflection is

$$w(x_1, x_2, 0) = \frac{F_3}{D} \sum_{m,n=1,3,\ldots} C_{mn} \sin(r_{1m}x_1) \sin(r_{2n}x_2), \qquad C_{mn} = \frac{16\,(1 - \mu_{mn}^2 A_1)}{mn\pi^2 \mu_{mn}^4}, \tag{37}$$

where D and A_1 are given in (25) and (28), respectively. The similar expression is valid for the deflection of the opposite face plane $w(x_1, x_2, h)$. In this case, the constant A_1 in Eq. (37) needs to be replaced by $A = A_g - A_\nu$.

The deflection of the line $x_2 = a_2/2,\ z = 0$ and the maximum deflection at $z = 0$ are, respectively,

$$\begin{aligned} w(x_1, a_2/2, 0) &= \frac{F_3}{D} \sum_{k=0}^{\infty} B_{2k+1} \sin(r_{1,2k+1} x_1), \quad B_{2k+1} = \sum_{j=0}^{\infty} (-1)^j C_{2k+1,2j+1}, \\ w(a_1/2, a_2/2, 0) &= \frac{F_3}{D} A_0, \quad A_0 = \sum_{k,j=0}^{\infty} (-1)^{k+j} C_{2k+1,2j+1} = \sum_{k=0}^{\infty} (-1)^k B_{2k+1}. \end{aligned} \tag{38}$$

All the series in (37) and (38) converge rapidly, the summands with C_{11} are much larger than the remaining summands (see Table 4).

As an example, we consider a multi-layer square plate with $n = 11$ and with the same data as in Section 4. We assume in addition that $a_1 = a_2 = 20$, and so $\mu = \mu_{11} = 0.2221$. For various values of $\eta = E_2/E_1$, the first coefficients $C_{2k+1,2j+1}$, B_{2k+1}, and A_0 are given in Table 2.

η	C_{11}	C_{13}	C_{33}	C_{15}	C_{35}	C_{55}	B_1	B_3	B_5	A_0
1	4528	191	62	31	19	10	4368	160	22	4230
0.1	9828	463	164	89	58	32	9454	388	63	9129
0.01	14720	1255	593	383	282	183	13848	1045	284	13087
0.001	52134	8717	4737	3251	2475	1675	46668	6455	2451	42669

Table 2: Coefficients $C_{2k+1,2j+1}$, B_{2k+1}, and A_0 for various values η.

The amplitude of deflection is proportional to A_0; it essentially depends on the ratio η (see Table 2), but the deflection mode depends slightly on η and is close to the function $\sin(\pi x_1/a_1)$ (see Fig. 2).

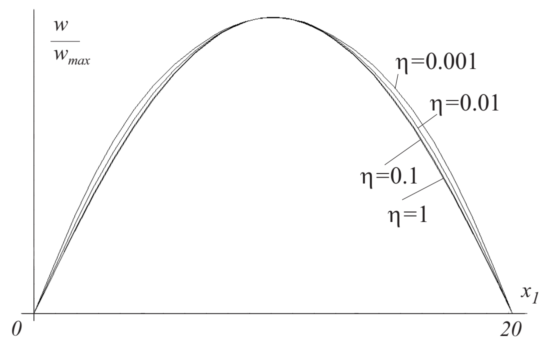

Figure 1: Deflection modes for various values of the ratio η of the Young moduli.

6 OPTIMAL DISTRIBUTION OF THE LAYER THICKNESSES

Consider a multilayer plate with the alternating hard and soft layers. Let the entire number of layers be $n = 2n_0 + 1$ with $n_0 + 1$ layers and n_0 soft layers. We set

$$h_h = \sum_{k=0}^{n_0} h_{2k+1}, \quad h_s = \sum_{k=1}^{n_0} h_{2k}, \quad h = h_h + h_s, \tag{39}$$

where h_h and h_s are the summary thicknesses of the hard and of the soft layers, respectively. Assume that the values h_h, h_s, also the elastic moduli of hard and of soft layers, and the thickness parameter μ are given. We seek the distribution of the layer thicknesses h_k satisfying Eqs. (39), for which the plate bending stiffness is maximum. As the bending stiffness we accept the inverse value the maximum deflection $f(h_1, h_2, \ldots, h_n) = w(0)^{SA}$ (see Eq. (30)).

In most cases a three-layer plate with

$$n = 3, \quad h_1 = h_3 = h_h/2, \quad h_2 = h_s \tag{40}$$

has the maximum stiffness. But if the ratio η of elastic moduli is very small, then the other distribution of thicknesses may lead to the maximum stiffness. Here we investigate this problem in details for the fixed values $\mu = 0.3776,\ \nu_1 = \nu_2 = 0.3$, for two changing parameters η and $s = h_h/h$, and for $n \leq 11$ (without loss of generality we may take $E_1 = 1$ and $h = 1$). The small parameter $\mu = 0.3776$ corresponds, in partial, to a rectangular plate with $a_1 = 10,\ a_2 = 15$.

To find the thickness distribution corresponding to the maximum stiffness for the fixed values η and s, we seek the minimum of function $f(h_k) = f(h_1, h_2, \ldots, h_n)$ in h_k satisfying Eqs. (39). To find h_k^0 that delivers the minimum to the function $f(h_k)$ we use the Monte Carlo method in combination with the method of iterations [18]. Thanks to Eqs. (39) the point h_k moves in the space R^{n-2}. Let $h_k^{(i)}$ be some point. We next seek the next point $h_k^{(i+1)}$ for which

$$f(h_k^{(i+1)}) < f(h_k^{(i)}) \tag{41}$$

as follows. We put $h_k^* = \max\{0, h_k^{(i)} + \xi_k\}$, where ξ_k are the uniformly distributed numbers in $[-\varepsilon, \varepsilon]$ (for example, $\varepsilon = 0.01$). Then we normalize the values h_k^* according to Eqs. (39)

$$h_{2i+1}^{**} = \frac{s\, h_{2i+1}^*}{\sum_{i=0}^{5} h_{2i+1}^*}, \qquad h_{2i}^{**} = \frac{(1-s) h_{2i}^*}{\sum_{i=1}^{5} h_{2i}^*}. \tag{42}$$

If $f(h_k^{**}) < f(h_k^{(i)})$, then we take $h_k^{(i+1)} = h_k^{**}$. Otherwise, we repeat the previous step with different random numbers ξ_k.

As the initial distribution we take $h_1^{(1)} = h_{11}^{(1)} = s/2,\ h_6^{(1)} = (1-s)$, assuming that the rest of $h_k^{(1)}$ is 0. Calculations show that it is enough to make 10^6 steps.

The curve tn Fig. 3 divides the plane (s, η) of the problem parameters into two parts; in one of which the three-layer plate has maximum stiffness.

Consider in more detail the case $s = 1/3,\ \mu = 0.3776$ with various values of the parameter η. For $\eta > \eta_* = 0.0122$ the three-layered plate (40) has maximal bending stiffness. In Table 3 for various η the minimal deflection, W_o, as found by the algorithm described above, is compared with the deflection, W_u, of a plate with the uniform thickness distribution (see Section 4) and with the deflection, W_3, of the three-layered plate (40). For $\eta > \eta_*$ the three-layered plate is optimal, $W_3 = W_o$. For $\eta < 0.006$ the stiffness of a plate with uniform thickness distribution is larger than the stiffness of a three-layer plate, $W_u < W_3$.

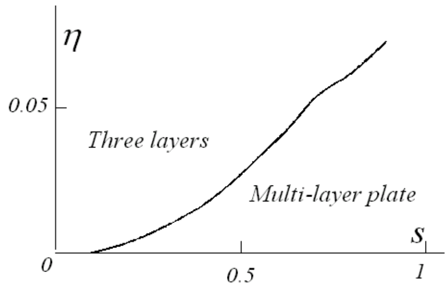

Figure 2: The plane (s, η) for $\mu = 0.3776$.

η	uniform, W_u	three-layered, W_3	optimal, W_o
0.001	15617	17812	11539
0.002	8446	9286	7815
0.005	4139	4170	4007
0.01	2697	2464	2458
0.0122	2434	2155	2155
0.02	1962	1607	1607
0.1	1268	901	901
0.5	785	663	663

Table 3: Comparison of the stiffness parameters for various multilayer plates with $\mu = 0.3776,\ s = 1/3$.

In Table 4 the optimal distribution of thicknesses for $s = 1/3,\ \mu = 0.3776$, and for four values of η is presented. The thicknesses of soft layers are shown in italic. For $\eta = 0.01$ the distribution is symmetric with respect to the mid-plane. With smaller η the distribution is not symmetric. The resulting asymmetry can be accounted for by the fact that the value W_o is the deflection of the plane $z = 0$ to which the external force is applied. At $\eta = 0.001$ the optimal plate degenerates to a three-layer plate with very dissimilar distribution of layers.

η	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}	h_{11}
0.01	0.149	*0.192*	0.007	*0.099*	0.010	*0.081*	0.010	*0.099*	0.007	*0.192*	0.149
0.005	0.094	*0.212*	0.025	*0.090*	0.048	*0.063*	0.047	*0.090*	0.025	*0.211*	0.095
0.002	0.030	*0.255*	0.024	*0.075*	0.213	*0.039*	0.027	*0.083*	0.011	*0.215*	0.029
0.001	0.331	*0.667*	0.002	–	–	–	–	–	–	–	–

Table 4: Distribution of thicknesses of the optimal plate with $s = 1/3,\ \mu = 0.3776$.

The similar results also hold for a rectangular plate with the Navier boundary conditions, because the first summand in the expression for $w(x_1, a_2/2, 0)$ in Eq. (38) is much larger than the rest summands.

The results of Section 6 are approximate, since they are based on the approximate equation (30). The error in Eq. (30) is estimated in Section 4 (see Table 1). The exact results may be obtained by numerical solution of Eqs. (9). But for the large number of iterations this approach takes a long time, though the result is almost the same.

7 CONCLUSIONS

- A two-dimensional linear model of second-order accuracy (SA) is used for analysis of a thin multi-layer plate bending.
- The formula for an infinite plate deflection subject to a doubly periodic external force is obtained. This formula is applied to a rectangular multi-layer plate with alternating hard and soft layers and with the Navier boundary conditions.
- The plate deflection depends on the distribution of thicknesses of hard and soft layers.
- The problem of the optimal distribution, which gives the maximum bending stiffness, is attacked using the Monte Carlo method.
- For a very small ratio η of the elastic moduli the three-layer plate with the soft layer lying between two hard layers was found not to be optimal. In this case, the plate with the maximum bending stiffness is the multi-layered one with the number of layers $n > 3$.
- The plate with $n = 11$ layers is examined.

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