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ASYMPTOTIC ANALYSIS OF DEFORMATIONS OF THE SLIGHTLY ORTHOTROPIC SPHERICAL LAYER UNDER NORMAL PRESSURE

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Abstract. *The deformation of the orthotropic spherical layer under normal pressure applied on the outer and inner surfaces is analyzed. The layer is assumed to be slightly orthotropic, it permits to apply asymptotic methods. The equations of zeroth and first approximations are derived. For the shell, which is much softer in the transverse direction than in the tangential plane, one gets singularly perturbed boundary value problem. Solving this problem in the zeroth approximation the asymptotic formula for the change of the relative layer thickness under normal pressure is obtained. Also the effect of Poisson ratio and the layer thickness on the deformation is studied. For the cases of the thick and thin layers the last formula may be simplified. The asymptotic results well agree with the exact solution. The developed formulas are used in analysis of the scleral shell under intraocular pressure and may also be used in solution of the inverse problem, i.e. in analysis of the stress-strain state of a human eye under injection. The solution of the problem helps to estimate the mechanical parameters of the sclera, i.e. to find the ratio of the tangential and transversal Young moduli using clinical data for the sclera thickness change.*

1 INTRODUCTION

The 3D problem for deformation of orthotropic spherical layer under normal pressure is considered. Such model may be used, for example, to describe the changes of the stress strain state of the external human eye shell under intraocular injections. For isotropic spherical layer this problem, known as Lame problem, is discussed, for example, in [1]. For transverse isotropic spherical layer the analytical solution has been obtained in [2, 3] and asymptotic solution in [4]. The stress-strain state of a two-layered transversely isotropic spherical shell is discussed in [5].

2 PROBLEM STATEMENT

2.1 Equilibrium equations in displacements

Consider orthotropic spherical layer with the internal radius R_1, external radius R_2 and thickness $h = R_2 - R_1$ (see Fig. 1).

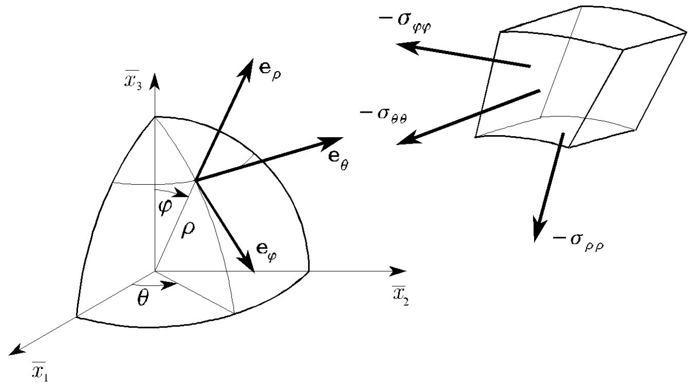

Figure 1: Orthotropic spherical layer

The position of the point of the spherical layer in given by spherical coordinates: ρ — radial coordinate, φ — meridional coordinate, θ — circumferential coordinate. Equilibrium equations for the spherical layer have the form [1]

$$\begin{aligned}
\frac{\partial \sigma_{\rho\rho}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \sigma_{\rho\varphi}}{\partial \varphi} + \frac{1}{\rho \sin\varphi}\frac{\partial \sigma_{\rho\theta}}{\partial \theta} + \frac{\cos\varphi}{\rho \sin\varphi}\sigma_{\rho\varphi} + \frac{1}{\rho}(2\sigma_{\rho\rho} - \sigma_{\varphi\varphi} - \sigma_{\theta\theta}) + f_\rho &= 0, \\
\frac{\partial \sigma_{\varphi\rho}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} + \frac{1}{\rho \sin\varphi}\frac{\partial \sigma_{\varphi\theta}}{\partial \theta} + \frac{3}{\rho}\sigma_{\varphi\rho} + \frac{\cos\varphi}{\rho \sin\varphi}(\sigma_{\varphi\varphi} - \sigma_{\theta\theta}) + f_\varphi &= 0, \\
\frac{\partial \sigma_{\theta\rho}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \sigma_{\theta\varphi}}{\partial \varphi} + \frac{1}{\rho \sin\varphi}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{3}{\rho}\sigma_{\theta\rho} + \frac{2\cos\varphi}{\rho \sin\varphi}\sigma_{\theta\varphi} + f_\theta &= 0,
\end{aligned}$$

here $\sigma_{\rho\rho}$, $\sigma_{\varphi\varphi}$ and $\sigma_{\theta\theta}$ — normal stress, $\sigma_{\rho\varphi}$, $\sigma_{\rho\theta}$, $\sigma_{\varphi\theta}$ — tangential stress, f_ρ, f_φ, f_θ — projections of the external forces in corresponding directions.

We consider axisymmetric problem without external forces. In this case the displacements do not depend on angle θ, and tangential stresses $\sigma_{\rho\theta}$, $\sigma_{\varphi\theta}$ and deformations $\varepsilon_{\varphi\theta}$, $\varepsilon_{\rho\theta}$ are equal to zero.

Thus, system of equations becomes

$$\begin{aligned}
\frac{\partial \sigma_{\rho\rho}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \sigma_{\rho\varphi}}{\partial \varphi} + \frac{\cos\varphi}{\rho \sin\varphi}\sigma_{\rho\varphi} + \frac{1}{\rho}(2\sigma_{\rho\rho} - \sigma_{\varphi\varphi} - \sigma_{\theta\theta}) &= 0, \\
\frac{\partial \sigma_{\rho\varphi}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} + \frac{3}{\rho}\sigma_{\rho\varphi} + \frac{\cos\varphi}{\rho \sin\varphi}(\sigma_{\varphi\varphi} - \sigma_{\theta\theta}) &= 0.
\end{aligned} \qquad (1)$$

The displacements of the spherical layer are given with projections of the displacement vector $(w,\, u,\, v)$ in directions ρ, φ and θ correspondingly. For axisymmetrilc problem $v = 0$. The relations for deformations and displacements of the spherical layer are the following [1]

$$\varepsilon_{\rho\rho} = \frac{\partial w}{\partial \rho}, \;\; \varepsilon_{\varphi\varphi} = \frac{1}{\rho}\frac{\partial u}{\partial \varphi} + \frac{w}{\rho}, \;\; \varepsilon_{\theta\theta} = \cot\varphi \frac{u}{\rho} + \frac{w}{\rho}, \;\; \varepsilon_{\rho\varphi} = \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial w}{\partial \varphi} - \frac{u}{\rho} + \frac{\partial u}{\partial \rho}\right). \tag{2}$$

Next consider constitutive relations for stresses and deformations. For the orthotropic media they contain 9 independent elastic moduli: E_ρ, E_φ, E_θ — Young moduli, $\nu_{\varphi\rho}$, $\nu_{\theta\rho}$, $\nu_{\theta\varphi}$, — Poisson ratios, $G_{\rho\varphi}$, $G_{\varphi\theta}$, $G_{\rho\theta}$ — shear moduli [6]

$$\begin{aligned}
\varepsilon_{\rho\rho} &= \frac{1}{E_\rho}\sigma_{\rho\rho} - \frac{\nu_{\rho\varphi}}{E_\varphi}\sigma_{\varphi\varphi} - \frac{\nu_{\rho\theta}}{E_\theta}\sigma_{\theta\theta}, \qquad \varepsilon_{\rho\theta} = \sigma_{\rho\theta}/G_{\rho\theta},\\
\varepsilon_{\varphi\varphi} &= -\frac{\nu_{\varphi\rho}}{E_\rho}\sigma_{\rho\rho} + \frac{1}{E_\varphi}\sigma_{\varphi\varphi} - \frac{\nu_{\varphi\theta}}{E_\theta}\sigma_{\theta\theta}, \quad \varepsilon_{\varphi\theta} = \sigma_{\varphi\theta}/G_{\rho\theta},\\
\varepsilon_{\theta\theta} &= -\frac{\nu_{\theta\rho}}{E_\rho}\sigma_{\rho\rho} - \frac{\nu_{\theta\varphi}}{E_\varphi}\sigma_{\varphi\varphi} + \frac{1}{E_\theta}\sigma_{\theta\theta}, \quad \varepsilon_{\rho\varphi} = \sigma_{\rho\varphi}/G_{\rho\varphi}.
\end{aligned} \tag{3}$$

Due to symmetry of relations (3) the following equalities are valid

$$E_\varphi \nu_{\varphi\rho} = E_\rho \nu_{\rho\varphi}, \quad E_\varphi \nu_{\varphi\theta} = E_\theta \nu_{\theta\varphi}, \quad E_\theta \nu_{\theta\rho} = E_\rho \nu_{\rho\theta}. \tag{4}$$

Introduce new constants [6]

$$\begin{aligned}
\nu^*_{\varphi\theta} &= \frac{\nu_{\varphi\theta} + \nu_{\varphi\rho}\nu_{\rho\theta}}{1 - \nu_{\theta\rho}\nu_{\rho\theta}}, \quad \nu^*_{\varphi\rho} = \frac{\nu_{\varphi\rho} + \nu_{\varphi\theta}\nu_{\theta\rho}}{1 - \nu_{\theta\rho}\nu_{\rho\theta}}, \quad \nu^*_{\theta\varphi} = \frac{\nu_{\theta\varphi} + \nu_{\theta\rho}\nu_{\rho\varphi}}{1 - \nu_{\varphi\rho}\nu_{\rho\varphi}},\\
\nu^*_{\theta\rho} &= \frac{\nu_{\theta\rho} + \nu_{\theta\varphi}\nu_{\varphi\rho}}{1 - \nu_{\varphi\rho}\nu_{\rho\varphi}}, \quad \nu^*_{\rho\varphi} = \frac{\nu_{\rho\varphi} + \nu_{\theta\varphi}\nu_{\rho\theta}}{1 - \nu_{\varphi\theta}\nu_{\theta\varphi}}, \nu^*_{\rho\theta} = \frac{\nu_{\rho\theta} + \nu_{\varphi\theta}\nu_{\rho\varphi}}{1 - \nu_{\varphi\theta}\nu_{\theta\varphi}},
\end{aligned} \tag{5}$$

$$\begin{aligned}
E^*_\varphi = E_\varphi / \left(1 - \nu^*_{\varphi\theta}\nu_{\theta\varphi} - \nu^*_{\varphi\rho}\nu_{\rho\varphi}\right), \qquad E^*_\theta = E_\theta / \left(1 - \nu^*_{\theta\varphi}\nu_{\varphi\theta} - \nu^*_{\theta\rho}\nu_{\rho\theta}\right),\\
E^*_\rho = E_\rho / \left(1 - \nu^*_{\rho\theta}\nu_{\theta\rho} - \nu^*_{\rho\varphi}\nu_{\varphi\rho}\right),
\end{aligned} \tag{6}$$

where

$$E^*_\rho \nu^*_{\rho\varphi} = E^*_\varphi \nu^*_{\varphi\rho}, \quad E^*_\varphi \nu^*_{\varphi\theta} = E^*_\theta \nu^*_{\theta\varphi}, \quad E^*_\theta \nu^*_{\theta\rho} = E^*_\rho \nu^*_{\rho\theta}. \tag{7}$$

Substituting (3) and (2) into (1) and taking into account (5)–(7) we get equilibrium equations in displacements as

$$\begin{aligned}
c_0\frac{\partial^2 w}{\partial \rho^2} + c_1\frac{\partial w}{\partial \rho} + c_2\frac{\partial^2 w}{\partial \varphi^2} + c_3\frac{\partial w}{\partial \varphi} + c_4 w + c_5\frac{\partial^2 u}{\partial \rho\,\partial \varphi} + c_6\frac{\partial u}{\partial \rho} + c_7\frac{\partial u}{\partial \varphi} + c_8 u &= 0,\\
d_0\frac{\partial^2 u}{\partial \rho^2} + d_1\frac{\partial u}{\partial \rho} + d_2\frac{\partial^2 u}{\partial \varphi^2} + d_3\frac{\partial u}{\partial \varphi} + d_4 u + d_5\frac{\partial^2 w}{\partial \rho\,\partial \varphi} + d_6\frac{\partial w}{\partial \rho} + d_7\frac{\partial w}{\partial \varphi} + d_8 w &= 0,
\end{aligned} \tag{8}$$

where expressions for c_i and d_i are listed in Appendix.

2.2 Boundary conditions

We consider the quarter of the spherical layer, i.e. domain $0 \leqslant \varphi \leqslant \dfrac{\pi}{2}$ and $R_1 \leqslant \rho \leqslant R_2$.

On the boundary $\varphi = 0$ and $\varphi = \dfrac{\pi}{2}$ we assume

$$u(\rho, 0) = u\left(\rho, \frac{\pi}{2}\right) = 0, \quad \frac{\partial u}{\partial \varphi}(\rho, 0) = \frac{\partial u}{\partial \varphi}\left(\rho, \frac{\pi}{2}\right) = 0, \quad \frac{\partial w}{\partial \varphi}(\rho, 0) = \frac{\partial w}{\partial \varphi}\left(\rho, \frac{\pi}{2}\right) = 0. \quad (9)$$

On the boundary $\rho = R_1$ and $\rho = R_2$ we suppose that the normal pressure $P_1,\ P_2$ is given

$$\sigma_{\rho\rho}\left(R_1, \varphi\right) = -P_1, \qquad \sigma_{\rho\rho}(R_2, \varphi) = -P_2 \qquad (10)$$

where

$$\sigma_{\rho\rho} = E_\rho^* \left(\frac{\partial w}{\partial \rho} + \frac{\nu_{\rho\varphi}^* + \nu_{\rho\theta}^*}{\rho} w + \frac{\nu_{\rho\varphi}^*}{\rho} \frac{\partial u}{\partial \varphi} + \frac{\nu_{\rho\theta}^* \cot(\varphi)}{\rho} u \right).$$

Equations (8) and boundary conditions (9)–(10) provide the boundary value problem.

2.3 Restrictions on elastic constants

As we mention above the descriptions of the orthotropic material requires 9 independent elastic moduli. However, due to the positive definiteness of the elastic potential its coefficient must satisfy Sylvester criterion, from which the following inequalities may be found [6]

$$\begin{aligned} &\nu_{\rho\varphi} < \sqrt{E_\varphi / E_\rho}, \qquad \nu_{\rho\theta} < \sqrt{E_\theta / E_\rho}, \qquad \nu_{\varphi\theta} < \sqrt{E_\theta / E_\varphi}, \\ &\nu_{\rho\varphi}\nu_{\varphi\theta}\nu_{\theta\rho} < 1/2\left(1 - \nu_{\rho\varphi}^2 E_\rho / E_\varphi - \nu_{\varphi\theta}^2 E_\varphi / E_\theta - \nu_{\theta\rho}^2 E_\theta / E_\rho\right). \end{aligned} \qquad (11)$$

3 SLIGHTLY ORTHOTROPIC MATERIAL. PERTURBATION METHOD

Consider that the elastic modulus in the circumferential direction is little different from the modulus in the meridional direction. We assume that the following relations for the material elastic constants are valid

$$\begin{aligned} &E_\rho = E_1,\ E_\varphi = E,\ E_\theta = E(1+\mu),\ \nu_{\theta\varphi} = \nu,\ \nu_{\varphi\rho} = \nu_{\theta\rho} = \nu_1. \\ &G_{\varphi\theta} = G + \mu G' = \frac{E}{2(1+\nu)} + \mu G',\ G_{\rho\varphi} = G_1,\ G_{\rho\theta} = G_1 + \mu G'', \end{aligned}$$

where $\mu \ll 1$. For $\mu = 0$ the material becomes transverse isotropic.

In this case the limitations on elastic modulii (11) are

$$\nu < \frac{1}{1+\mu}, \qquad \nu_1 < \sqrt{\frac{E_1}{E}} \cdot \min\left(1, \sqrt{\frac{1}{1+\mu}}, \sqrt{\frac{1+\mu-\nu^2}{(1+\mu)(2+\mu+2\nu)}}\right). \qquad (12)$$

The domain of the material parameters (μ, ν, ν_1) is plotted in Fig. 2

Solution of equation (8) we seek in the form

$$w(\rho, \varphi) = w_0(\rho) + \mu w_1(\rho, \varphi) + O(\mu^2), \qquad u(\rho, \varphi) = \mu u_1(\rho, \varphi) + O(\mu^2).$$

Equation of zeroth approximation for function $w_0(\rho)$ have been considered in [4]. This is equation for the transversely isotropic layer. We report shortly the results of [4] in next section.

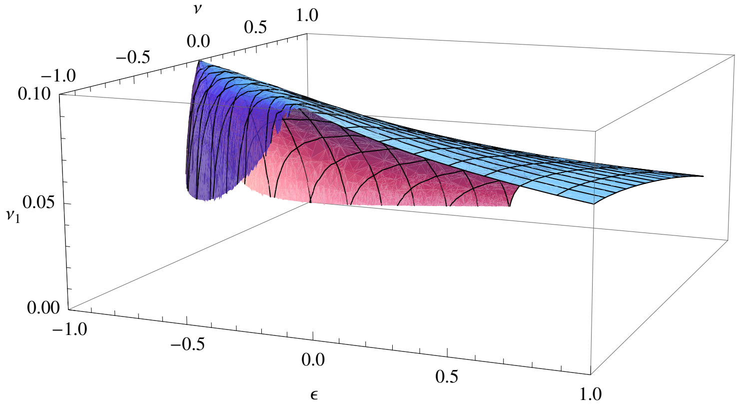

Figure 2: The domain of the elastic moduli (μ, ν, ν_1) for $E/E_1 = 0.01$.

4 EQUATION OF ZEROTH APPROXIMATION

For zeroth approximation the equilibrium equations for the normal displacement may be reduced to a single equation in the form

$$w_0''(\rho) + \frac{2}{\rho}w_0'(\rho) - 2\frac{E(1-\nu_1)}{E_1(1-\nu)\rho^2}w_0(\rho) = 0, \tag{13}$$

and boundary conditions are

$$2EE_1\nu_1 w(R_i) + E_1^2(1-\nu)w'(R_i)R_i = -P_iR_i\left(E_1(1-\nu) - 2E\nu_1^2\right), \qquad i = 1, 2.$$

The restrictions on elastic constants becomes

$$\nu < 1, \qquad \nu_1 < \sqrt{\frac{E_1}{E}} \cdot \min\left(1, \sqrt{\frac{1-\nu}{2}}\right),$$

and since for all known materials $\nu > -1$ then $\sqrt{\dfrac{1-\nu}{2}} < 1$. This problem has an exact analytical solution obtained in [2, 3], but formulas for the displacement w and stress functions are rather complicated. That is why it is more convenient to use an asymptotic approach to analyze the effect of the parameters of the spherical layer on the stress-strain state of the layer.

4.1 Asymptotic analysis

For the layer, which is much softer in the transverse direction than in the tangential surface we assume $E_1 = \varepsilon^2 E$. $\nu_1 = \varepsilon^\alpha \nu_1^*$, where $\varepsilon \ll 1$, $\nu_1^* \approx 1$, $\alpha \geq 1$. In the further analysis $\alpha = 1$. The equation for the normal displacement becomes

$$\varepsilon^2 w_0''(\rho)\rho^2 + 2\varepsilon^2\rho w_0'(\rho) - (a + b\varepsilon)w_0(\rho) = 0, \tag{14}$$

where $a = \dfrac{2}{1-\nu}$; and $b = -\dfrac{2\nu_1^*}{1-\nu}$, and the boundary conditions are

$$\varepsilon f w(R_i) + \varepsilon^2 g w'(R_i)R_i = -cp_iR_i, \qquad i = 1, 2. \tag{15}$$

Here $p_i = P_i/E$, $c = (1 - \nu - 2\nu_1^{*2})$, $f = 2\nu_1^*$, $g = 1 - \nu$. To solve singularly perturbed equation (14) we follow [7] and seek a solution in the form

$$w(\rho) = e^{\frac{1}{\varepsilon}\int_0^\rho \lambda(t)dt}(w_0(\rho) + \varepsilon w_1(\rho) + \cdots). \tag{16}$$

After substitution in (14) and equating the coefficients at ε^k to zero we obtain a system of equations for the unknowns $\lambda(\rho), w_0(\rho), w_1(\rho), \dots$

$$\begin{aligned}&w_0(\rho)(-a+\rho^2\lambda^2(\rho))=0,\\&2\rho^2 w_0'(\rho)\lambda'(\rho)+w_0(\rho)(-b+2\rho\lambda'(\rho)-\rho^2\lambda''(\rho))=0,\\&\dots\end{aligned}\tag{17}$$

The solution of the first equation gives $\lambda(\rho)=\pm\dfrac{\sqrt{a}}{\rho}$ and the solution of the second equation is

$$w_0(\rho)=\rho^\beta,\qquad \beta=\frac{1}{2}\left(\pm\frac{b}{\sqrt{a}}-1\right).$$

So, for the first approximation we get

$$w(\rho)=A\rho^{\gamma_1}+B\rho^{\gamma_2},\tag{18}$$

where $\gamma_{1,2}=\pm\dfrac{\sqrt{a}}{\varepsilon}+\dfrac{1}{2}\left(\pm\dfrac{b}{\sqrt{a}}-1\right)$. Constants A and B may be found from the boundary conditions

$$A=-\frac{c}{\varepsilon}\frac{R_1G_1p_1-R_2G_2p_2}{F^+(G_1^2-G_2^2)},\qquad B=-\frac{c}{\varepsilon}\frac{G_1G_2(R_2G_1p_2-R_2G_2p_1)}{F^-(G_1^2-G_2^2)}$$

where $F^\pm=\sqrt{a}g\pm f$ and $G_i=R_i^{\sqrt{a}/\varepsilon}$. The change of the layer thickness is obtained using formula for the normal displacement

$$\Delta h=\frac{w(R_2)-w(R_1)}{R_2-R_1}$$

or

$$\Delta h=-\frac{c}{\varepsilon}\frac{p_1R_1(F^+G_2-F^-G_1)+p_2R_2(F^-G_2-F^+G_1)}{(F^-F^+(G_1+G_2))(R_2-R_1)}\tag{19}$$

4.2 Examples and applications

Further simplification of the formula for the change of the layer thickness is based on the assumptions on ratio of the values R_1 and R_2. For example, for the thick layer $R_1 \ll R_2$, then $\xi=R_1/R_2\ll 1$ ($\xi \ll 1$) and $G_1 \ll G_2$. In this case the first approximation for Δh has the form

$$\Delta h=-\frac{c}{\varepsilon}\frac{p_1\xi F^+ + p_2F^-}{(F^+F^-)(1-\xi)}.\tag{20}$$

For the thin layer $R_1\approx R_2$. In this case

$$\Delta h=-\frac{c}{\varepsilon\eta}\frac{p_1(1-\eta)(F^+-F^-(1-\eta)^\varkappa)+p_2(F^--F^+(1-\eta)^\varkappa)}{F^-F^+(1+(1-\eta)^\varkappa)},$$

where $R_1=(1-\eta)R_2$ ($\eta=1-\xi\ll 1$), $\varkappa=\sqrt{a}/\varepsilon$.

Let consider as an example the scleral shell with parameters [8] $\nu=0.48$, $P_1=p\times 133.3$ Pa, $P_2=0$ Pa, $E=14$ MPa, $\nu_1^*=0.5$, Here $p=20$ is the intraocular pressure in mm Hg. The change of the layer thickness as a function of ε is represented in Fig. 3 for thick layer ($\eta=0.7$) (left) and for thin layer ($\eta=0.04$) (right).

The change of the layer thickness as a function of ε and $\xi=R_1/R_2$ is represented in Fig. 4

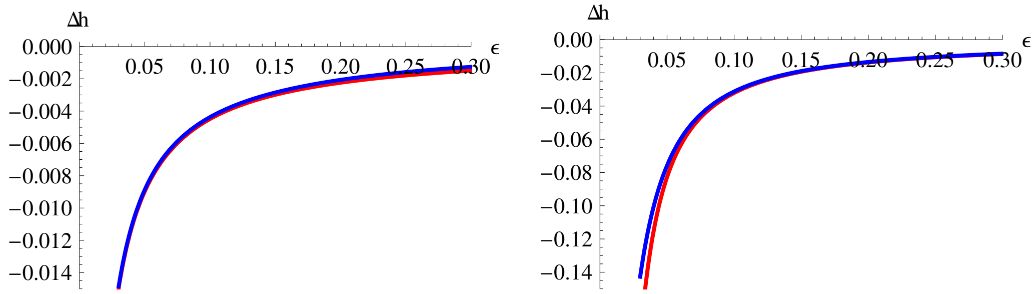

Figure 3: The change of the layer thickness vs. ε (the exact solution (blue line), the asymptotic solution (red line)).

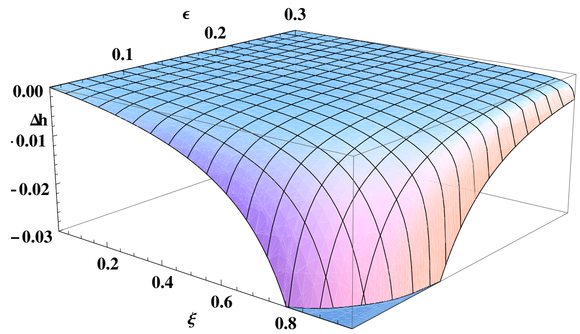

Figure 4: The change of the layer thickness as a function of ε and $\xi = R_1/R_2$.

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5 EQUATIONS OF FIRST APPROXIMATION

For the first approximation we get equations

$$m_0 \frac{\partial^2 w_1}{\partial \rho^2} + m_1 \frac{\partial w_1}{\partial \rho} + m_2 \frac{\partial^2 w_1}{\partial \varphi^2} + m_3 \frac{\partial w_1}{\partial \varphi} + m_4 w_1 + m_5 \frac{\partial^2 u_1}{\partial \rho \, \partial \varphi} + m_6 \frac{\partial u_1}{\partial \rho} + \\ + m_7 \frac{\partial u_1}{\partial \varphi} + m_8 u_1 + m_9 w_0 = 0, \tag{21}$$

$$n_0 \frac{\partial^2 u_1}{\partial \rho^2} + n_1 \frac{\partial u_1}{\partial \rho} + n_2 \frac{\partial^2 u_1}{\partial \varphi^2} + n_3 \frac{\partial u_1}{\partial \varphi} + n_4 u_1 + n_5 \frac{\partial^2 w_1}{\partial \rho \, \partial \varphi} + n_6 \frac{\partial w_1}{\partial \varphi} + \\ + n_7 \frac{\partial w_0}{\partial \rho} + n_8 w_0 = 0,$$

where expressions for m_i and n_i are listed in Appendix and boundary conditions

$$u_1\big|_{\varphi=0} = u_1\big|_{\varphi=\pi/2} = 0, \quad \frac{\partial u_1}{\partial \varphi}\bigg|_{\varphi=0} = \frac{\partial u_1}{\partial \varphi}\bigg|_{\varphi=\pi/2} = 0, \quad \frac{\partial w_1}{\partial \varphi}\bigg|_{\varphi=0} = \frac{\partial w_1}{\partial \varphi}\bigg|_{\varphi=\pi/2} = 0, \tag{22}$$

and

$$\sigma^1_{\rho\rho}(R_1, \varphi) = \sigma^1_{\rho\rho}(R_2, \varphi) = 0, \tag{23}$$

where

$$\sigma^1_{\rho\rho} = l_0 \frac{\partial w_1}{\partial \rho} + l_1 w_1 + l_2 \frac{\partial w_0}{\partial \rho} + l_3 w_0 + l_4 \frac{\partial u_1}{\partial \varphi} + l_5 u_1,$$

and

$$l_0 = 1, \quad l_1 = \frac{2}{\rho} \frac{E\nu_1}{E_1(1-\nu)}, \quad l_2 = \frac{E\nu_1^2}{(1-\nu)(E_1(1-\nu) - 2E\nu_1^2)}, \qquad (24)$$
$$l_3 = \frac{l_2}{\nu_1 \rho}, \quad l_4 = \frac{E\nu_1}{E_1(1-\nu)\rho}, \quad l_5 = l_4 \cot(\varphi).$$

So, the boundary value problem for the first approximation consists of equations (21) and boundary conditions (22)–(23). We note, that the change of the shear moduli (G',G'') do not effect equations and boundary conditions of the first approximation.

Figure 5 depicts deformed orthotropic layer for two different ratio of E_θ/E_φ. The layer parameters are taken as follows $R_2 = 12$ mm, $R_1 = 0.9R_2$, $E_\varphi = E = 14$ MPa, $E_\rho = E_1 = 1.26$ MPa, $\nu = 0.48$, $\nu_1 = 0.03$. The inner boundary $\rho = R_1$ is subjected to normal pressure $P_1 = 20 \times 133.3$ Pa.

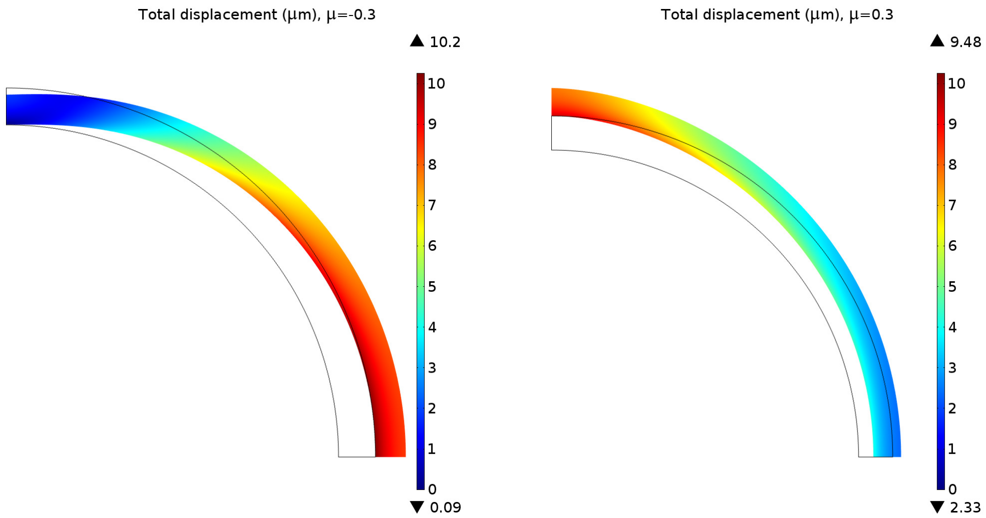

Figure 5: Deformed orthotropic layer with $E_\theta/E_\varphi = 0.7\,(\mu = -0.3)$ and $E_\theta/E_\varphi = 1.3\,(\mu = 0.3)$.

6 CONCLUSIONS

1. For transversely isotropic material the asymptotic formula describing the change of the layer thickness under normal pressure is obtained. Asymptotic relations show that for deformation of the spherical layer under the inner pressure the relative thickness varies more for the layer with smaller thickness.

2. The change of the scleral thickness under intraocular pressure (IOP) well agree with clinical data on decrease of scleral thickness in primary open-angle glaucoma (POAG) [8]. POAG is an eye disease, which is generally accompanied by the IOP increase.

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APPENDIX

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\begin{aligned}
&c_0 = 1, \;\; c_1 = \frac{2}{\rho}, \quad c_2 = \frac{1}{2\rho^2}\frac{G_{\rho\varphi}}{E^*_\rho}, \quad c_3 = c_2 \cot(\varphi), \\
&c_4 = \frac{1}{\rho^2}\left(\nu^*_{\rho\varphi} + \nu^*_{\rho\theta} - \frac{E^*_\varphi + 2\,\nu^*_{\varphi\theta}E^*_\varphi + E^*_\theta}{E^*_\rho}\right), \\
&c_5 = \frac{1}{\rho}\left(\nu^*_{\rho\varphi} + \frac{G_{\rho\varphi}}{2E^*_\rho}\right), \quad c_6 = \frac{\cot(\varphi)}{\rho}\left(\nu^*_{\rho\theta} + \frac{G_{\rho\varphi}}{2E^*_\rho}\right), \\
&c_7 = \frac{1}{\rho^2}\left(\nu^*_{\rho\varphi} - \frac{E^*_\varphi}{E^*_\rho}(1+\nu^*_{\phi\theta}) - \frac{G_{\rho\varphi}}{2E^*_\rho}\right), \\
&c_8 = \frac{\cot(\varphi)}{\rho^2}\left(\nu^*_{\rho\theta} - \frac{E^*_\varphi\nu^*_{\varphi\theta} + E^*_\theta}{E^*_\rho} - \frac{G_{\rho\varphi}}{2E^*_\rho}\right)
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&d_0 = 1, \;\; d_1 = \frac{2}{\rho}, \quad d_2 = \frac{2}{\rho^2}\frac{E^*_\varphi}{G_{\rho\varphi}}, \quad d_3 = d_2 \cot(\varphi), \\
&d_4 = -\frac{2}{\rho^2}\left(1 + \frac{E^*_\varphi\nu^*_{\varphi\theta}}{G_{\rho\varphi}} + \frac{E^*_\theta}{G_{\rho\varphi}}\cot^2(\varphi)\right), \\
&d_5 = \frac{1}{\rho}\left(1 + 2\frac{E^*_\rho\nu^*_{\rho\varphi}}{G_{\rho\varphi}}\right), \quad d_6 = \frac{2\cot(\varphi)}{\rho}\frac{E^*_\rho}{G_{\rho\varphi}}(\nu^*_{\rho\varphi} - \nu^*_{\rho\theta}), \\
&d_7 = \frac{2}{\rho^2}\left(1 + \frac{E^*_\varphi}{G_{\rho\varphi}}(1+\nu^*_{\varphi\theta})\right), \quad d_8 = \frac{2\cot(\varphi)}{\rho^2}\frac{E^*_\varphi - E^*_\theta}{G_{\rho\varphi}},
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&m_0 = 1, \;\; m_1 = \frac{2}{\rho}, \quad m_2 = \frac{G_{\rho\varphi}}{2\rho^2 E^*_{\rho_0}}, \quad m_3 = m_2\cot(\varphi), \\
&m_4 = -\frac{2}{\rho^2}\frac{E(1-\nu_1)}{E_1(1-\nu)}, \quad m_5 = \frac{1}{\rho}\left(\frac{G_{\rho\varphi}}{2E^*_{\rho_0}} + \frac{E\nu_1}{E_1(1-\nu)}\right), \\
&m_6 = m_5\cot(\varphi), \quad m_7 = -\frac{1}{\rho^2}\left(\frac{E(1-\nu_1)}{E_1(1-\nu)} + \frac{G_{\rho\varphi}}{2E^*_{\rho_0}}\right), \\
&m_8 = m_7\cot(\varphi), \quad m_9 = -\frac{1}{\rho^2}\frac{E(1-\nu_1)}{E_1(1-\nu)^2}, \\
&E^*_{\rho_0} = \frac{E_1^2(1-\nu)}{E_1(1-\nu) - 2E\nu_1^2}.
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
&n_0 = 1, \;\; n_1 = \frac{2}{\rho}, \quad n_2 = \frac{2E^*_{\varphi_0}}{G_{\rho\varphi}\rho^2}, \quad n_3 = n_2 \cot(\varphi),\\
&n_4 = -\frac{2}{\rho^2}\left(1 + \frac{E^*_{\varphi\theta_0}}{G_{\rho\varphi}} + \frac{E^*_{\varphi_0}}{G_{\rho\varphi}}\cot^2(\varphi)\right), \quad n_5 = \frac{1}{\rho}\left(1 + \frac{2E^*_{\rho\varphi_0}}{G_{\rho\varphi}}\right)\\
&n_6 = \frac{2}{\rho^2}\left(1 + \frac{2E^*_{\rho\varphi_0}}{G_{\rho\varphi}\nu_1}\right), \quad n_7 = -\frac{2\cot(\varphi)}{\rho}\frac{E^*_{\rho\varphi_0}}{G_{\rho\varphi}(1+\nu)}, \quad n_8 = \frac{n_7}{\nu_1\rho},\\
&E^*_{\varphi_0} = \frac{E(E_1 - E\nu_1^2)}{(1+\nu)(E_1(1-\nu) - 2E\nu_1^2)}, \quad E^*_{\rho\varphi_0} = \frac{EE_1\nu_1}{E_1(1-\nu) - 2E\nu_1^2},\\
&E^*_{\varphi\theta_0} = \frac{E(E_1\nu + E\nu_1^2)}{(1+\nu)(E_1(1-\nu) - 2E\nu_1^2)}.
\end{aligned}
\tag{28}
$$

REFERENCES

[1] T.M. Atanackovic, A. Guran, *Theory of Elasticity for Scientists and Engineers*. Birkhäuser, 2000.

[2] S.M. Bauer, L.A. Zamuraev, K.E. Kotliar, Model of the transversally isotropic spherical layer for estimation of intraocular pressure changes after intravitreal injections. *Russian Journal of Biomechanics*, **10**, 41-47, 2006.

[3] S.M. Bauer, E.B. Voronkova, Nonclassical shell theories in ocular biomechanics. *Advanced Structured Materials*. **45**, 81-97, 2015.

[4] S.M. Bauer, A.L. Smirnov, Axisymmetric Deformations of the Orthotropic Spherical Layer under Normal Pressure. *Vestnik St. Petersburg University. Mathematics,* **48**, 35-40, 2015.

[5] L.A. Karamshina, On deformation of transversally isotropic two-layered spherical shell, *Vestnik Sankt-Peterburgskogo Universiteta. Seriya 1: Matematika. Mekhanika. Astronomiya,* **2**, 133-138, 2011.

[6] V.A. Rodionova, V.F. Titaev, and K.F. Chernykh, *Applied Theory of Anisotropic Plates and Shells*. SPbU, 1996. (in Russian).

[7] S.M. Bauer, S.B. Filippov, A.L. Smirnov, P.E. Tovstik, R. Vaillancourt, *Asymptotic Methods in Mechanics of Solids*. Springer, 2015.

[8] K. Kotliar, M. Maier, S. Bauer, N. Feucht, C. Lohmann and I. Lanzl, Effect of intravitreal injections and volume changes on intraocular pressure: clinical results and biomechanical model. *Acta Ophthalmologica Scandinavica*, **85**, 777-781, 2007.