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AN APROXIMATE ANALYTICAL SOLUTION FOR NONLINEAR FGM SHELL STRUCTURE WITH VARIABLE IN TIME PARAMETERS

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Abstract. *This paper deals with the problem of dynamic behavior of thin geometrically imperfect shell structures made of functionally gradient material (FGM) with time dependent parameters. Hybrid asymptotic approach is used to obtain an approximate analytical solution of the problem. The material properties are graded in the thickness direction according to the given law distribution and initial conditions. The non-linear strain-displacement relationships based upon the von Karman theory for moderately large normal deflections. Discussed problem leads to a singular non-linear second order non homogeneous differential equation with variable in time coefficients. An analytical solution by hybrid perturbation-WKB-Galerkin (P-WKB-G) method in some parameters of structure is compared with direct numerical integration results of initial equation of the problem.*

1 INTRODUCTION

Thin walled structures made of functionally graded materials (FGM) with metal inner surface and ceramic in outer surface widely used, for example, in modern air-space systems. In recent years important studies have been researched about vibration and stability FGM plates and shells with using mostly by numerical approaches [1-4, 12].

The present research devoted to an approximate analytical solution of nonlinear dynamic problem of FGM imperfect shallow shells with time dependent parameters (for example, with thickness depending on time) on the basis of hybrid (P-WKB-G) asymptotic method, which was successfully applied in some mechanical problems [5-11,15].

2 BASIC CONCEPT OF THE HYBRID APPROACH TO SOLUTION OF NONLINEAR PROBLEMS

To solve the non-linear differential equations with variable coefficients the approach is applied in three stages. On the first step the solution is determined using perturbation method by forming an expansion in parameter near the non-linear term of initial equation and we obtain the related system of linear non-homogeneous equations with variable coefficients. On the second step the solutions of linear system are determined using the (one or two-term) WKB-approximation by forming an expansion in parameter. On the third step the correction functions are obtained by classical Galerkin procedure.

In this paper an approximate analytical solution of nonlinear dynamic behavior of shallow shells with time-dependent thickness, which is described by singular nonlinear non homogeneous equation with variable coefficients was found.

3 THE NONLINEAR DYNAMIC BEHAVIOR OF IMPERFECT SHALLOW SHELLS WITH VARIABLE IN TIME THICKNESS

An approximate analytical non-linear analysis is given on the basic system of equations in terms of the stress and deflection following to the paper [1]. Suppose the FGM shallow shell is simply supported at its edges and subjected to a transverse load $q_0(t)$ and compressive edge loads $r_0(t)$, $p_0(t)$. We assume that modulus of elasticity and the mass density changes in the thickness direction, while the Poisson ratio is assumed to be constant and thickness of shell is function of time.

Appling Bubnov-Galerkin procedure with assumption that initial imperfection of middle surface of shell has the form:

$$w_0(x_1, x_2) = f_0 \sin\frac{m\pi x_1}{a}\sin\frac{n\pi x_2}{b} \tag{1}$$

where f_0 is given amplitude, the non-linear second – order ordinary differential equation for function $f(t)$ with deflection function $w=(x_1, x_2, t)$:

$$w(x_1, x_2, t) = f(t)\sin\frac{m\pi x_1}{a}\sin\frac{n\pi x_2}{b} \tag{2}$$

that are correspond to simply support boundary conditions, is given in the form [1]:

$$\varepsilon^2 \frac{d^2 f}{dt^2} + f\left(1 + 2 f_0 \bar{A}_2(t) - \bar{A}_3(t) f_0^2 - \bar{A}_1(t)\right) + f^2\left(-3\bar{A}_2(t)\right) + f^3 \bar{A}_3(t) = Q_0 - \bar{A}_0(t) + f_0 - \bar{A}_2(t) f_0^2 \quad (3)$$

where

$$\varepsilon^2 = \frac{1}{\omega_{mn}^2},\ \omega_{mn}^2 = \frac{1}{\rho_1(t)}\left[\frac{\left(E_1 E_3 - E_2^2\right)}{E_1\left(1-\nu^2\right)} \cdot \frac{\left(m^2 + n^2\lambda^2\right)\pi^2}{a^4} + \frac{E_1\left(k_1 n^2 \lambda^2 + k_2 m^2\right)^2}{\left(m^2 + n^2\lambda^2\right)^2}\right],\ \bar{A}_i = \frac{A_i}{\omega_{mn}^2}$$

$$A_1(t) = \frac{\pi^2 h(t)}{a^2}\left(m^2 r_0 + n^2 \lambda^2 p_0\right),\ A_2(t) = \frac{16 E_1(t) m n \lambda^2 \left(k_1 n^2 \lambda^2 + k_2 m^2\right)}{3a^2\left(m^2 + n^2\lambda^2\right)^2},\ A_3(t) = \frac{512 E_1(t) m^2 n^2 \lambda^4}{9a^4\left(m^2 + n^2 \lambda^2\right)^2} \quad (4)$$

$$E_1(t) = \left(E_m + \frac{E_c - E_m}{k+1}\right) h(t),\ \rho_1 = \left(\rho_m + \frac{\rho_c - \rho_m}{k+1}\right) h(t)$$

k_1, k_2 – curvatures of middle surface shell in x_1 and x_2 directions.
Initial differential equation (3) we rewrite in the form:

$$\varepsilon^2 \frac{d^2 f}{dt^2} + B_1(t) f + \mu\left(B_2(t) f^2 + B_3(t) f^3\right) = \bar{Q}_0(t) \quad (5)$$

where

$$B_1(t) = 1 + 2 f_0 \bar{A}_2(t) - \bar{A}_3(t) f_0^2 - \bar{A}_1(t)$$

$$B_2(t) = \frac{-3}{\mu} \bar{A}_2(t)$$

$$B_3(t) = \frac{1}{\mu} \bar{A}_3(t) \quad (6)$$

$$\bar{Q}_0(t) = Q_0 - \bar{A}_0(t) + f_0 - \bar{A}_2(t) f_0^2$$

ε, μ are parameters.

According to perturbation method with respect to parameter of nonlinearity μ, solution of differential equation (5) we obtain in the form of two terms approximation:

$$f(t) = \varphi_0(t) + \mu \varphi_1(t) \quad (7)$$

Substituting (7) into equation (5) and acquainted the terms with the same order of nonlinear parameter we obtain the system equations for unknown functions $\varphi_0(t)$ and $\varphi_1(t)$:

$$\mu^0 : \varepsilon^2 \varphi_0''(t) + B_1(t)\varphi_0 = \bar{Q}_0 \tag{8}$$

$$\mu^1 : \varepsilon^2 \varphi_1''(t) + B_1(t)\varphi_1 = -B_2(t)\varphi_0^2 - B_3(t)\varphi_0^3 \tag{9}$$

Ordinary singular differential equation with variable in time coefficient B_1 is solved by two terms WKB-approximation [5]:

$$\varphi_0^0(t) = 1/B_1(t)^{0.25}\left(c_1 \sin K(t) + c_2 \cos K(t)\right) \tag{10}$$

where

$$K(t) = \int \varepsilon^{-1} B_1^{0.25}(t)\,dt \tag{11}$$

Particular solution of equation (9) can be present in the form:

$$\varphi_0^p(t) = 1/B_1(t)^{0.25}\left[\bar{c}_1(t)\sin K(t) + \bar{c}_2(t)\cos K(t)\right] \tag{12}$$

where

$$\begin{aligned} \bar{c}_1(t) &= \varepsilon \int \frac{\bar{Q}_0(t)\cos K(t)}{B_1^{-0.25}(t)}dt \\ \bar{c}_2(t) &= -\varepsilon \int \frac{\bar{Q}_0(t)\sin K(t)}{B_1^{-0.25}(t)}dt \end{aligned} \tag{13}$$

The solution of equation (5) in the first approximation is:

$$\varphi_0(t) = \varphi_0^0(t) + \varphi_0^p(t) = B_1^{0.25}\left[\sin K(t)\left(c_1 + \bar{c}_1(t)\right) + \cos K(t)\left(c_2 + \bar{c}_2(t)\right)\right] \tag{14}$$

Second term in (9) for ordinary equation is obtained from:

$$\varepsilon^2 \varphi_1''(t) + B_1(t)\varphi_1 = 0 \tag{15}$$

where

$$K(t) = \int \varepsilon^{-1} B_1^{0.25}(t)\,dt \tag{16}$$

Particular solution of equation (5) can be present in the form:

$$\varphi_0^p(t) = 1\big/ B_1(t)^{0.25}\left[\bar{c}_1(t)\sin K(t) + \bar{c}_2(t)\cos K(t)\right] \tag{17}$$

where

$$\bar{c}_1(t) = \varepsilon\int \frac{\bar{Q}_0(t)\cos K(t)}{B_1^{-0.25}(t)}dt$$
$$\bar{c}_2(t) = -\varepsilon\int \frac{\bar{Q}_0(t)\sin K(t)}{B_1^{-0.25}(t)}dt \tag{18}$$

The solution of equation (5) in the first approximation is:

$$\varphi_0(t) = \varphi_0^0(t) + \varphi_0^p(t) = B_1^{0.25}\left[\sin K(t)\left(c_1 + \bar{c}_1(t)\right) + \cos K(t)\left(c_2 + \bar{c}_2(t)\right)\right] \tag{18}$$

Second term in (9) for ordinary equation is obtained from:

$$\varepsilon^2\varphi_1''(t) + B_1(t)\varphi_1 = 0 \tag{19}$$

in the form:

$$\varphi_1^0(t) = 1\big/ B_1(t)^{0.25}\left(d_1\sin K(t) + d_2\cos K(t)\right) \tag{20}$$

Corresponding particular solution of equation (9) is

$$\varphi_1^p(t) = 1\big/ B_1(t)^{0.25}\left[\bar{d}_1(t)\sin K(t) + \bar{d}_2(t)\cos K(t)\right] \tag{21}$$

where

$$\bar{d}_1(t) = \varepsilon\int \frac{\left(-B_2(t)\varphi_0^2 - B_3(t)\varphi_0^3\right)\bar{Q}_0(t)\cos K(t)}{B(t)^{-0.25}}dt$$
$$\bar{d}_2(t) = -\varepsilon\int \frac{\left(-B_2(t)\varphi_0^2 - B_3(t)\varphi_0^3\right)\bar{Q}_0(t)\sin K(t)}{B_1(t)^{-0.25}}dt \tag{22}$$

Finally, we have obtained the solution of nonlinear problem on the basis of perturbation – (two-term) WKB method:

$$f(t)=\varphi_0(t)+\mu\varphi_1(t)=\sin K(t)\left(c_1+\bar{c}_1(t)\right)+\cos K(t)\left(c_2+\bar{c}_2(t)\right)+$$
$$+\mu\left(\sin K(t)\left(d_1+\bar{d}_1(t)\right)+\cos K(t)\left(d_2+\bar{d}_2(t)\right)\right)=$$
$$=B_1(t)^{0.25}\left\{sinK(t)\left[c_1+\bar{c}_1(t)+\mu\bar{d}_1(t)\right]+cosK(t)\left[c_2+\bar{c}_2(t)+\mu\bar{d}_2(t)\right]\right\} \tag{23}$$

The initial conditions are:

$$\varphi(0)=1$$
$$\varphi'(0)=0 \tag{24}$$

In order to obtain hybrid asymptotic solution of initial nonlinear equation we rewrite equation (3) as

$$\varepsilon^2\frac{d^2 f}{dt^2}+B_1(t)\,f=\bar{Q}_0(t)-\mu\left(B_2(t)\,f^2+B_3(t)\,f^3\right)=\bar{Q}_0(t)-\mu N(t) \tag{25}$$

Approximate analytical (P-WKB-G) solutions of equation (3) are given in the following forms (26) - (29).

3.1 Hybrid asymptotic solution of linear homogeneous problem

$$f_0^0(t)=exp(\varphi(t))\left(cos(K(t))(-0.5973)+sin(K(t))(-0.8508)\right) \tag{26}$$

where according to [5]:

$$\varphi(t)=\int\frac{B_1(0)-B_1(1)}{4\int_0^1 B_1(t)^{1.5}\,dt}B_1(t)^{0.5}\,dt$$
$$K(t)=\int\sqrt{\frac{1}{\varepsilon^2}-\frac{B_1(1)-B_1(0)}{4\int_0^1 B_1(t)^{1.5}\,dt}B_1(t)^{0.5}}\,dt \tag{27}$$

Some numerical calculations for the shell with variable in time parameters and comparison of approximate analytical solutions with direct numerical integration of initial nonlinear nonhomogeneous equation with variable coefficients are given on Figures 1-7.

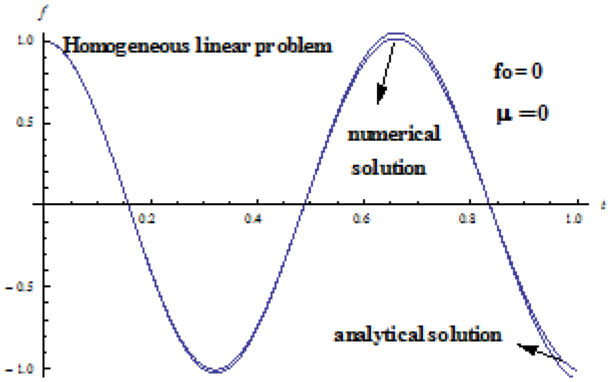

Figure 1.Comparison of hybrid asymptotic and numerical solutions for homogeneous linear problem

3.2 Hybrid asymptotic solution of linear nonhomogeneous problem

$$f_0(t) = f_0^0(t) + f_0^p(t) = exp(\varphi(t))\Big(cos(K(t))\big(c_1 + \bar{c}_1(t)\big) + sin(K(t))\big(c_2 + \bar{c}_2(t)\big)\Big) \tag{28}$$

$$c_1 = -0.5522$$

$$c_2 = -0.8826$$

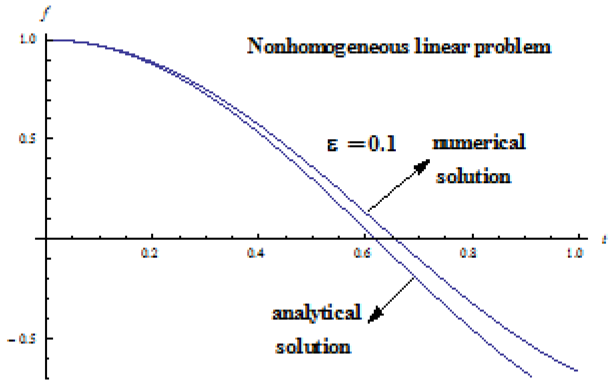

Figure 2.Comparison of hybrid asymptotic and numerical solutions for nonhomogeneous linear problem (one - term WKB-approximation)

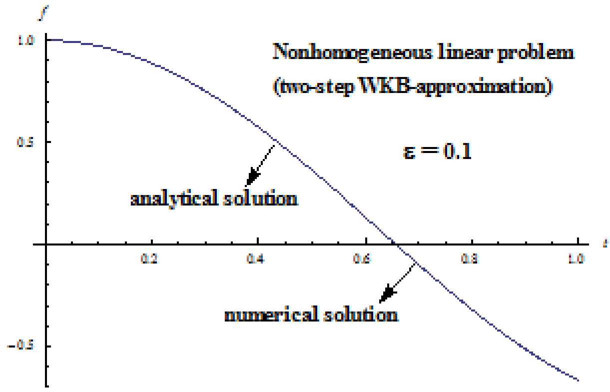

Figure 3. Comparison of analytical and numerical solutions for nonhomogeneous linear problem (two - terms WKB-approximation)

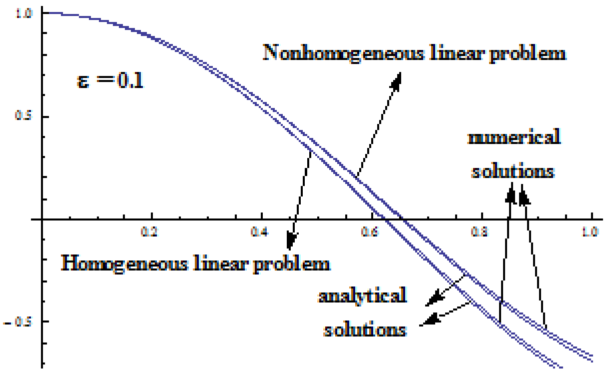

Figure 4. Comparison of analytical and numerical solutions

3.3 Hybrid asymptotic solution of nonlinear nonhomogeneous problem

$$f(t) = exp(\varphi(t))\left(cos(K(t))\left(c_1 + \bar{c}_1(t) + \mu \bar{d}_1(t)\right) + sin(K(t))\left(c_2 + \bar{c}_2(t) + \mu \bar{d}_2(t)\right)\right) \tag{29}$$

$$c_1 = -0.541$$

$$c_2 = -0.8566$$

where

$$\varphi(t) = -0.03872(1+0.1t)^{1.5}, \;\; K(t) = 16.6644(1+0.1t)^{1.5} \tag{30}$$

$$\begin{aligned} \bar{c}_1(t) &= -\int \frac{\bar{Q}_0(t)\,sinK(t)}{K'(t)} = \int series\left[-0.2(1+0.1t)^{0.5} cosK(t)\right]_0^8 dt \\ \bar{c}_2(t) &= \int \frac{\bar{Q}_0(t)\,cosK(t)}{K'(t)} = \int series\left[0.2(1+0.1t)^{0.5} sinK(t)\right]_0^8 dt \end{aligned} \tag{31}$$

$$\begin{aligned} \bar{d}_1(t) &= -\int \frac{N(t)\,sinK(t)}{K'(t)} = -\int series\left[\frac{30 f_0^2 - 10 f_0^3}{2.5(1+0.1t)^{0.5}} sinK(t)\right]_0^{22} dt \\ \bar{d}_2(t) &= \int \frac{N(t)\,cosK(t)}{K'(t)} = \int series\left[\frac{30 f_0^2 - 10 f_0^3}{2.5(1+0.1t)^{0.5}} cosK(t)\right]_0^{22} dt \end{aligned} \tag{32}$$

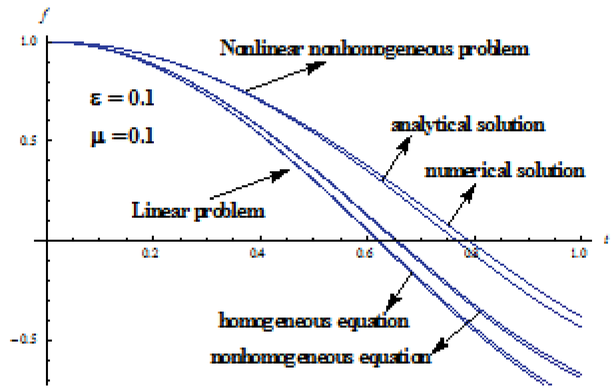

Figure5. Comparison of analytical and numerical solutions

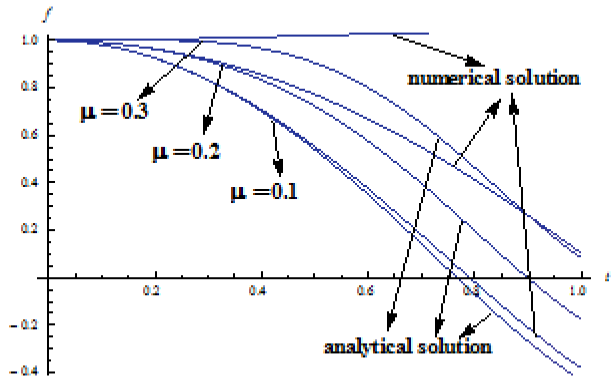

Figure 6. Influence of nonlinear parameter μ. (Comparison of analytical and numerical solutions at ε = 0.1)

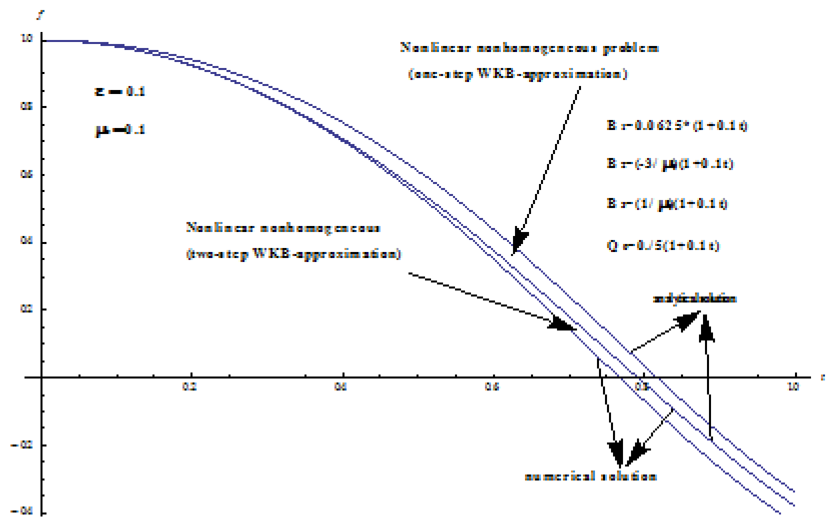

Figure7. Comparison of analytical and numerical solutions for nonhomogeneous linear and nonlinear problems (one-step and two-step WKB-approximations)

4 CONCLUSIONS

An approximate analytical solution for forced oscillations of geometrically non-linear FGM imperfect shallow cylindrical shells with time dependent parameters on the basis of hybrid perturbation-two-terms WKB approximation method are obtained. For some particular parameters of structure an analytical solutions are in a good enough correlations with direct numerical solutions of initial singular nonlinear differential equations with variable coefficients. In some cases one-term WKB-approximation gives good enough results for the practical purpose.

Researches will be devoted in future to study of the values of perturbation □ and singular ε parameters influence on dynamics of FGM shell behavior according to three – step (P-WKB-G) approach, different thickness function in time and functions of external forces as well.

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