# WAVE TRANSMISSION THROUGH NONLINEAR IMPACTING METAMATERIAL UNIT

## Arnab Banerjee[1], Raj Das[2], and Emilio P. Calius[3]

[1] PhD student, Mechanical Engineering Dept., University of Auckland, New Zealand
e-mail: aban991@aucklanduni.ac.nz

[2] Senior Lecturer, Mechanical Engineering Dept., University of Auckland, New Zealand
email: r.das@auckland.ac.nz

[3] Senior Scientist, Callaghan Innovation, Auckland, New Zealand
email: Emilio.Calius@callaghaninnovation.govt.nz

**Keywords:** Nonlinear metamaterial, Mechanical metamaterial, Acoustic metamaterial, Nonlinear oscillation, attenuation bandwidth, wideband metamaterial, mass-in-mass lattice

**Abstract.** *Metamaterials are an increasingly well-known type of generalized composites that can exhibit unconventional behaviors and responses due to its exciting frequency dependent properties which are not commonly encountered in natural materials. Elastodynamic metamaterials based on mechanical structures have a range of potential applications of importance in sound, vibration and seismic engineering. However, the effectiveness of metamaterials is limited to a relatively narrow frequency band as they are generally based on linear resonance mechanisms. These linear metamaterials do not perform well when under the broadband excitation spectra that are common in real life applications.*

*Mathematically, the 1-D metamaterial is represented by a series of periodic resonating spring and mass lattice structures. In this paper, the bandwidth of a metamaterial is examined by introducing nonlinearity in the resonating frequency. The performance of nonlinear metamaterial is well investigated in the field of electromagnetic wave propagation although the existing literature on nonlinear mechanical metamaterial is very limited and investigated only bistable and monostable type resonating metamaterial. Till date, any research on the impacting resonating metamaterial has not been reported, in spite of its excellent vibration insulation and resonating bandwidth enhancement properties. Therefore, as a first step towards exploring the impacting metamaterial, we have analytically estimated the bandwidth enhancement of a one dimensional impacting mechanical metamaterial unit over a linear metamaterial. First, the steady state response of an impacting system is computed analytically and then the transmission loss in the frequency domain is estimated using MATLAB. From the analysis, it is found that the resonating bandwidth of an impacting system is wider than that of an equivalent linear system. Therefore, it can be effectively used as a wideband mechanical filter, acoustic insulator or shield.*

# 1 INTRODUCTION

Natural materials have their own excitation independent material properties, such as mass density, Poisson's ratio and Young's modulus. Therefore, most of the natural materials react in phase with the excitation under external simulation; whereas, metamaterial behaves in out of phase [1]. Metamaterials are a special kind of artificial composite, designed to have exotic behaviours, such as negative mass [2-16], negative Poisson's ratio [17-22] and negative Young's modulus [4], in certain range of frequencies due to the out of phase response of multiple resonating units which are present in a metamaterial. Negative mass is achieved due to the out of phase motion of resonating unit during resonance [4, 6, 10, 11], for an example, the periodic arrangement of steel balls in a silicon rubber matrix can act as a negative effective mass system [23]. In this discussion, the definition of dynamic effective mass is according to the Newton's 2nd law of motion, which is the ratio of force and acceleration, instead of the amount of matter present in a body. An acoustic metamaterial is used as a waveguide and sound insulator. The elastic metamaterial has an extensive application in the field of vibration insulation [13]. Elastic wave propagation through a medium depends on the impedance of the medium ($Z_0$), which is the product of the mass density ($\rho$) and the P-wave ($v_p$) and S-wave ($v_s$) velocity of the medium. On the other hand, the P-wave velocity depends on the Young's modulus ($E$) and Poisson's ratio ($v$) of the medium. So, the frequency dependent negative material property changes the impedance of the medium which reflects the incident waves back. Due to the out phase response of resonating units, effective mass which is the ratio of momentum and velocity becomes negative [6, 10, 13] which refracts the propagating waves opposite to the conventional direction.

The metamaterial is used widely in the various field of application associated with the wave propagation. However, due to the dependency on the linear resonance, the attenuation bandwidth of the linear metamaterial is limited to a very narrow bandwidth. On the other hand, nonlinearity has a potential to widen the resonating bandwidth of an oscillator. Therefore the nonlinear metamaterial attracts the attention of the current researchers. The recent development in field of nonlinear electromagnetic metamaterial is condensed in this review article [24]. To widen the attenuation band of a mechanical metamaterial, Huang and Sun [5] proposed multi-resonance unit inside of the single block of the metamaterial. In contrary, an analytical study on nonlinear elastic dispersion for the elastic metamaterial was carried out by Khajehtuorian and Hossain [25]. This research concludes that the location and the length of the attenuation bandwidth of nonlinear elastic metamaterial depends on the amplitude of the wave. Nadkarni et al [26] analyzed a metamaterial containing bistable nonlinear resonating units for low and high amplitude wave motion and concluded that nonlinearity is proportional with the amplitude of wave and the nonlinearity enhanced the performance of metamaterial as a filter. Klatt and Haberman [27] proposed a multi-scale model for nonlinear metamaterial. Attenuation bandwidth of cubic [28, 29] and sinusoidal [30-32] nonlinear metamaterial is much wider than the linear metamaterial. Beside that, stress wave propagation through granular material or impacting chain of monoatomic lattice is also studied and found that it can attenuate much wider frequency band[33, 34]; thus, can be applicable in designing of helmet or protecting devices. Moreover, the resonating bandwidth of an energy harvester can be extended upto twice compared to an equivalent linear system by introducing impacting non-linearity [35, 36] although, the impacting resonating metamaterial has not been reported in any publication till now.

Impact or contact takes place when the two bodies come to very close vicinity. Starting from the time of Newton, plenty of literatures are available on the modelling of the impacting phenomena which are condensed and critically evaluated in this review article [37]. To model

the normal directional impact from a compliance based approach, nonlinear Hertzian element or Kelvin element can be used, where a small penetration is assumed for a finite duration. In the contrary, stereo-mechanical approach models the impact as an instantaneous event; thus, the velocity response becomes discontinuous. Modelling of this discontinuous response for a steady state vibration adds extra difficulty. A straight forward solution with standard numerical integration solver can solve the transient state response very easily but cannot converge to a steady state solution. Several techniques on the computation of the steady state response of an impacting device are proposed by various researchers[38]. In this paper, a solver designed by Nigm and Shabana [39] is used with stereo-mechanical unilateral impact model to solve the steady state response of the impacting metamaterial unit.

## 2    COMPUTATION OF STEADY STATE RESPONSE

The simplest resonating metamaterial can be mathematically conceptualized as a chain of mass-in-mass units. A single unit of the resonating metamaterial is depicted in Figure 1a) where the internal mass can vibrate without any obstruction; whereas, in Figure 1b) the motion of the internal mass is bounded by two stoppers. The impact with the stopper and the inner mass induced a nonlinear response to the system.
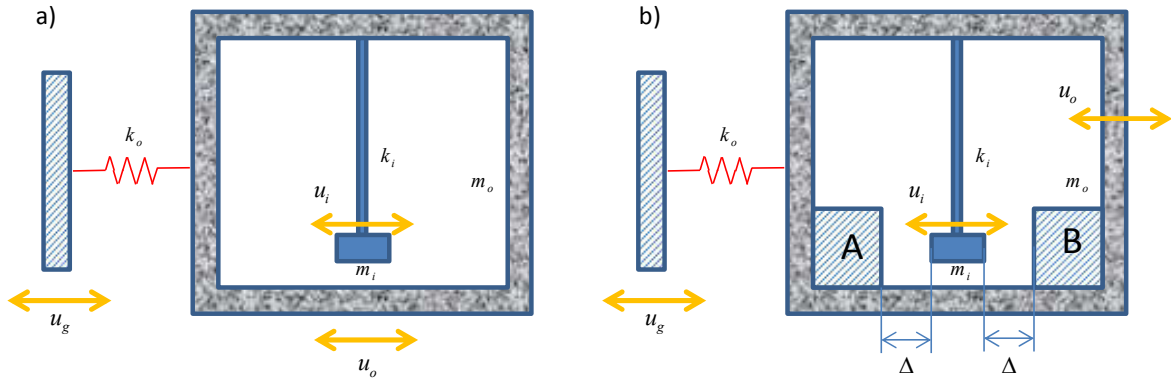


Figure 1: A single unit of metamaterial a) linear b) piece wise linear or impacting

The main aim of this paper is to compare the attenuation bandwidth for the linear and impacting metamaterial, subjected to a monochromatic displacement excitation from the base, as shown in Figure 1.

### 2.1    Linear system

The equation of motion of the linear system which is shown in Figure 1a) can be written as:

$$
\underbrace{\begin{bmatrix} m_o & 0 \\ 0 & m_i \end{bmatrix}}_{m} \underbrace{\begin{Bmatrix} \ddot{u}_o \\ \ddot{u}_i \end{Bmatrix}}_{\ddot{x}} + \underbrace{\begin{bmatrix} k_o + k_i & -k_i \\ -k_i & k_i \end{bmatrix}}_{k} \underbrace{\begin{Bmatrix} u_o \\ u_i \end{Bmatrix}}_{x} = \underbrace{\begin{Bmatrix} k_o u_g \cos \bar{\omega} t \\ 0 \end{Bmatrix}}_{f} \tag{1}
$$

where, $[m], [k]$ are respectively mass and stiffness matrix of the system. Therefore, the natural frequencies of the two degree of freedom system are:

$$
\omega_{1,2} = \frac{k_i}{2m_i} + \left( \frac{k_o + k_i}{2m_o} \right) \pm \sqrt{ -\frac{k_i}{m_i}\frac{k_o}{m_o} + \frac{1}{4}\left( \frac{k_o + k_i}{m_0} + \frac{k_i}{m_i} \right)^2 } \tag{2}
$$

and the modal matrix, which contains the mode shapes for each natural frequency can be expressed as:

$$\tilde{\Phi} = \begin{bmatrix} a_1 & a_2 \\ 1 & 1 \end{bmatrix}; \quad a_n = -\frac{m_i}{k_i}\omega_n^2 + 1 \tag{3}$$

Now, the Eq. (1) is a coupled second order differential equation. To uncouple this Eq.(1), the system $\{x\}$ is transferred to the principal coordinate $\{q\}$, by coordinate transformation.

$$\underbrace{\begin{Bmatrix} u_o \\ u_i \end{Bmatrix}}_{x} = \underbrace{\begin{bmatrix} \dfrac{a_1}{m_i a_1^2 + m_o} & \dfrac{a_2}{m_i a_2^2 + m_o} \\ \dfrac{1}{m_i a_1^2 + m_o} & \dfrac{1}{m_i a_2^2 + m_o} \end{bmatrix}}_{\Phi} \underbrace{\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}}_{q} \tag{4}$$

where, $\Phi$ is the mass normalized modal matrix. After shifting the coordinate system to principal coordinate, the mass, stiffness and force matrix changes to:

$$M = \Phi^T m \Phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; K = \Phi^T k \Phi = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}; P = \Phi^T f \tag{5}$$

Therefore, Eq.(1) can be solved in different modes as a single degree of freedom system and the equation of motion of the metamaterial unit in the principal coordinate system is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \cos \bar{\omega} t \tag{6}$$

$$\rightarrow \ddot{q}_i + \omega_i^2 q_i = P_i \cos \bar{\omega} t \quad i = 1, 2$$

The solution of the $i^{th}$ mode Eq.(6) is:

$$q_i = \sin(\omega_i t) A + \cos(\omega_i t) B + Z_i \cos \bar{\omega} t \tag{7}$$

where, A and B are two unknowns based on the initial condition and $Z_i = P_i / (\omega_i^2 - \bar{\omega}^2)$.

## 2.2 An example of linear system

Now, as an example, a linear system, having the inner mass is 1kg, outer mass is 2 kg, inner stiffness 3N/m and outer stiffness is 6N/m, subjected to a base excitation, having amplitude of 1m displacement and frequency 10 Hz = 0.628 rad/s is solved. The natural frequencies are $(\omega_1^2, \omega_2^2) = (1.5, 6)$ and the reduced force vector is $(P_1, P_2) = (2.4495, 3.4641)$. Now, to estimate the values of the unknowns, A and B, initial condition need to be substituted. At $t = 0$, the displacement and velocity of inner and outer mass are $(x_1, x_2) = (0, 0)$ and $(\dot{x}_1, \dot{x}_2) = (0, 0)$, which implies that $(q_1, q_2, \dot{q}_1, \dot{q}_2) = (0, 0, 0, 0)$; thus solution of Eq.(7) is $q_i = Z_i \cos \bar{\omega} t$. The displacement of inner and outer mass with its both side, assuming sides are of 0.5 m apart, is plotted in Figure 2.
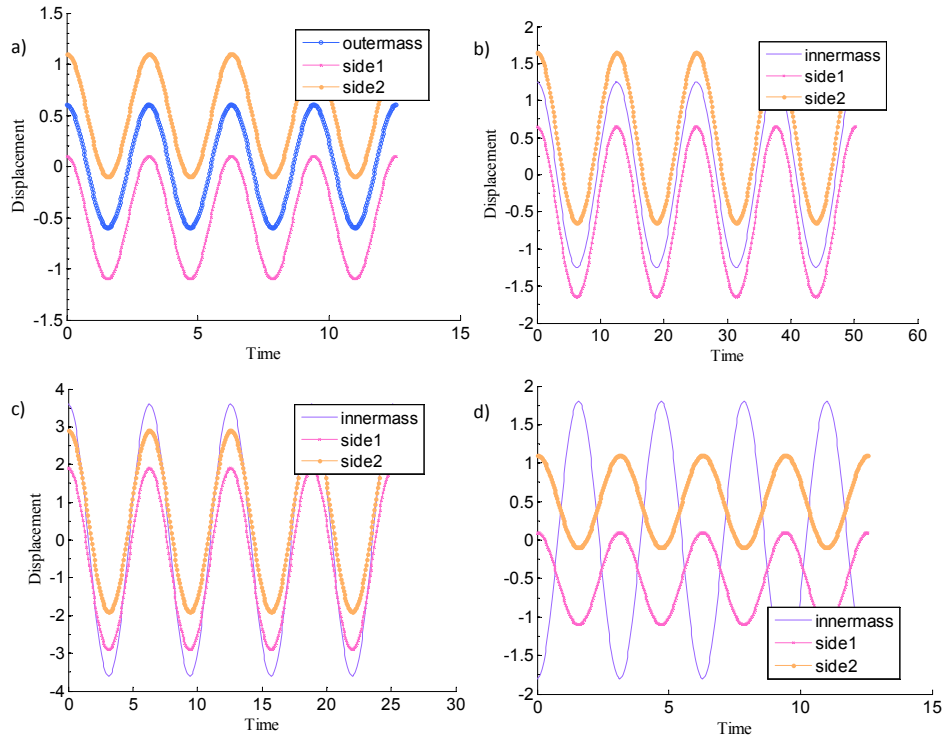
**Figure 2: Steady-state displacement response of inner and outer mass showing the side1 and side2 for various frequency b) 0.5 rad/s, c) 1.0 rad/s, d) 2.0 rad/s**

Figure 2 depicts that at low frequency, for example $\bar{\omega} = 0.5$ rad/s, inner mass vibrates in-phase with the outer block and it does not touch the either side of the outer block. On the other hand, inner mass touches the side1 and side2 of the outer mass although both the responses are in same phase. A further increment of excitation frequency makes the vibration of inner mass out of phase with the outer mass and therefore it touches both the side of outer mass more frequently. In both the cases, when the displacement response of inner mass crosses the boundary of the outer block then impact occurs and it needs a special analysis.

## 2.3  Impacting system

From the discussion of the linear system, it can be easily found that the inner mass can comes in contact several times with the two side walls of the outer mass not only during the out-of-phase response but also for the in phase response. Therefore, impacting system demands a thorough investigation on the responses of the system. In this paper, the frequency response of an impacting system is computed by mostly extending the method described by Nigm and Shabana [39].
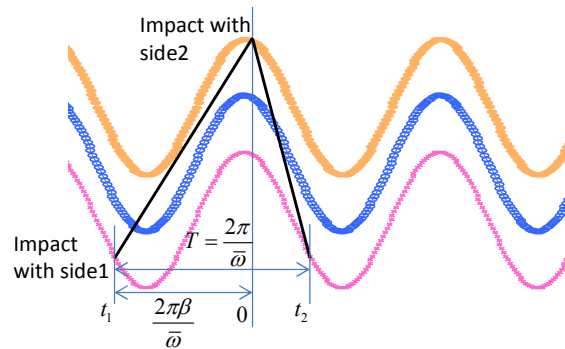


**Figure 3: conceptualization of the impacting events**

Now, as the vibration of the blocks is steady state in nature, after a constant interval, time period $T = 2\pi/\overline{\omega}$, inner mass hits the side1 of the outer block. It can be assumed that inner mass hits the side 1 of outer block at $t=t_1$ and $t=t_2$. Assuming that only double impact takes place in each period, the solution of the Eq. (6) can be written as:

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} \sin\omega_1 t & 0 \\ 0 & \sin\omega_2 t \end{bmatrix}}_{s} \underbrace{\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}}_{A} + \underbrace{\begin{bmatrix} \cos\omega_1 t & 0 \\ 0 & \cos\omega_2 t \end{bmatrix}}_{c} \underbrace{\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix}}_{B} + \begin{Bmatrix} Z_1\cos(\overline{\omega}t+\alpha) \\ Z_2\cos(\overline{\omega}t+\alpha) \end{Bmatrix}, \quad t_1 \le t \le 0^-$$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} \sin\omega_1 t & 0 \\ 0 & \sin\omega_2 t \end{bmatrix}}_{s} \underbrace{\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix}}_{C} + \underbrace{\begin{bmatrix} \cos\omega_1 t & 0 \\ 0 & \cos\omega_2 t \end{bmatrix}}_{c} \underbrace{\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix}}_{D} + \begin{Bmatrix} Z_1\cos(\overline{\omega}t+\alpha) \\ Z_2\cos(\overline{\omega}t+\alpha) \end{Bmatrix}, \quad 0^+ \le t \le t_2$$

(8)

where, $0^+$ and $0^-$ are represent the condition just after and just before t=0. The velocities are:

$$\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} \omega_1\cos\omega_1 t & 0 \\ 0 & \omega_2\cos\omega_2 t \end{bmatrix}}_{\omega c} \underbrace{\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}}_{A} - \underbrace{\begin{bmatrix} \omega_1\sin\omega_1 t & 0 \\ 0 & \omega_2\sin\omega_2 t \end{bmatrix}}_{\omega s} \underbrace{\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix}}_{B} - \begin{Bmatrix} Z_1\overline{\omega}\sin(\overline{\omega}t+\alpha) \\ Z_2\overline{\omega}\sin(\overline{\omega}t+\alpha) \end{Bmatrix}, t_1 \le t \le 0^-$$

$$\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} \omega_1\cos\omega_1 t & 0 \\ 0 & \omega_2\cos\omega_2 t \end{bmatrix}}_{\omega c} \underbrace{\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix}}_{C} - \underbrace{\begin{bmatrix} \omega_1\sin\omega_1 t & 0 \\ 0 & \omega_2\sin\omega_2 t \end{bmatrix}}_{\omega s} \underbrace{\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix}}_{D} - \begin{Bmatrix} Z_1\overline{\omega}\sin(\overline{\omega}t+\alpha) \\ Z_2\overline{\omega}\sin(\overline{\omega}t+\alpha) \end{Bmatrix}, 0^+ \le t \le t_2$$

(9)

where, A, B, $A_1$, $B_1$ are unknown initial condition and $Z_n = \dfrac{P_n}{\left(\omega_n^2 - \overline{\omega}^2\right)}$ .

Due to the periodicity,

$$x(0^-) = x(0^+); \quad x(t_2) = x(t_1) \tag{10}$$

After converting Eq.(10)a) to the principal coordinate, we can get:

$$x(0^-) = \Phi q = \Phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} + \begin{Bmatrix} Z_1\cos\alpha \\ Z_2\cos\alpha \end{Bmatrix}\right)$$

$$x(0^+) = \Phi q = \Phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} + \begin{Bmatrix} Z_1\cos\alpha \\ Z_2\cos\alpha \end{Bmatrix}\right)$$

$$\rightarrow B_1 = D_1; \quad B_2 = D_2$$

(11)

The converted equation of the Eq.(10) b) in principal coordinate is:

$$x(t_2) = \Phi q = \Phi \left( s_2 C + c_2 B + \begin{Bmatrix} Z_1 \cos \bar{\omega} t_2 \\ Z_2 \cos \bar{\omega} t_2 \end{Bmatrix} \cos \alpha - \begin{Bmatrix} Z_1 \sin \bar{\omega} t_2 \\ Z_2 \sin \bar{\omega} t_2 \end{Bmatrix} \sin \alpha \right)$$

$$x(t_1) = \Phi q = \Phi \left( s_1 A + c_1 B + \begin{Bmatrix} Z_1 \cos \bar{\omega} t_1 \\ Z_2 \cos \bar{\omega} t_1 \end{Bmatrix} \cos \alpha - \begin{Bmatrix} Z_1 \sin \bar{\omega} t_1 \\ Z_2 \sin \bar{\omega} t_1 \end{Bmatrix} \sin \alpha \right) \tag{12}$$

$$\rightarrow s_2 C - s_1 A + (c_2 - c_1) B + \underbrace{\begin{Bmatrix} Z_1 (\cos \bar{\omega} t_2 - \cos \bar{\omega} t_1) \\ Z_2 (\cos \bar{\omega} t_2 - \cos \bar{\omega} t_1) \end{Bmatrix}}_{Zc} \cos \alpha - \underbrace{\begin{Bmatrix} Z_1 (\sin \bar{\omega} t_2 - \sin \bar{\omega} t_1) \\ Z_2 (\sin \bar{\omega} t_2 - \sin \bar{\omega} t_1) \end{Bmatrix}}_{Zs} \sin \alpha = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assuming the impact as an instantaneous event, the change of velocity can be estimated by:

$$\{\dot{x}(0^+)\} = G\{\dot{x}(0^-)\}; \{\dot{x}(t_2)\} = G\{\dot{x}(t_1)\}; G = \frac{1}{m_i + m_o} \begin{bmatrix} m_o - m_i \varepsilon & m_i (1 + \varepsilon) \\ m_o (1 + \varepsilon) & m_i - m_o \varepsilon \end{bmatrix} \tag{13}$$

where, $\varepsilon$ is the coefficient of restitution between the inner and the outer mass. Eq.(13) a) can be converted in principal coordinate system as:

$$\dot{x}(0^+) = \Phi \dot{q}(0^+) = \Phi \left( \underbrace{\begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix}}_{\omega} \underbrace{\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix}}_{C} - \underbrace{\begin{Bmatrix} Z_1 \bar{\omega} \\ Z_2 \bar{\omega} \end{Bmatrix}}_{Z\bar{\omega}} \sin \alpha \right)$$

$$\dot{x}(0^-) = \Phi \dot{q}(0^-) = \Phi \left( \underbrace{\begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix}}_{\omega} \underbrace{\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}}_{A} - \begin{Bmatrix} Z_1 \bar{\omega} \\ Z_2 \bar{\omega} \end{Bmatrix} \sin \alpha \right) \tag{14}$$

$$\rightarrow -G\Phi \omega A + \Phi \omega C - (\Phi Z\bar{\omega} - G\Phi Z\bar{\omega}) \sin \alpha = \{0\}_{2 \times 1}$$

Similarly, Eq.(13) b) can be converted in principal coordinate as:

$$\dot{x}(t_2) = G\dot{x}(t_1)$$

$$\Phi \left( \omega c_2 C - \omega s_2 B - \begin{Bmatrix} Z_1 \bar{\omega} \sin \bar{\omega} t_2 \\ Z_2 \bar{\omega} \sin \bar{\omega} t_2 \end{Bmatrix} \cos \alpha - \begin{Bmatrix} Z_1 \bar{\omega} \cos \bar{\omega} t_2 \\ Z_2 \bar{\omega} \cos \bar{\omega} t_2 \end{Bmatrix} \sin \alpha \right)$$

$$= G\Phi \left( \omega c_1 A - \omega s_1 B - \begin{Bmatrix} Z_1 \bar{\omega} \sin \bar{\omega} t_1 \\ Z_2 \bar{\omega} \sin \bar{\omega} t_1 \end{Bmatrix} \cos \alpha - \begin{Bmatrix} Z_1 \bar{\omega} \cos \bar{\omega} t_1 \\ Z_2 \bar{\omega} \cos \bar{\omega} t_1 \end{Bmatrix} \sin \alpha \right) \tag{15}$$

$$\rightarrow -G\Phi \omega c_1 A + (G\Phi \omega s_1 - \Phi \omega s_2) B + \Phi \omega c_2 C + \left( G\Phi \begin{Bmatrix} Z_1 \bar{\omega} \sin \bar{\omega} t_1 \\ Z_2 \bar{\omega} \sin \bar{\omega} t_1 \end{Bmatrix} - \Phi \begin{Bmatrix} Z_1 \bar{\omega} \sin \bar{\omega} t_2 \\ Z_2 \bar{\omega} \sin \bar{\omega} t_2 \end{Bmatrix} \right) \cos \alpha$$

$$+ \left( G\Phi \begin{Bmatrix} Z_1 \bar{\omega} \cos \bar{\omega} t_1 \\ Z_2 \bar{\omega} \cos \bar{\omega} t_1 \end{Bmatrix} - \Phi \begin{Bmatrix} Z_1 \bar{\omega} \cos \bar{\omega} t_2 \\ Z_2 \bar{\omega} \cos \bar{\omega} t_2 \end{Bmatrix} \right) \sin \alpha = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The impact occurs at the side 1 at t=0 and at side 2 at $t_1$ and $t_2$; therefore,

$$u_i(0^-) - u_o(0^-) = \Delta; \quad u_i(t_1) - u_o(t_1) = -\Delta \tag{16}$$

The transformation from Eq.(16)a) to principal coordinate system can be expressed as:

$$\Phi_2 \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} + \begin{Bmatrix} Z_1 \\ Z_2 \end{Bmatrix} \cos \alpha \right) - \Phi_1 \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} + \begin{Bmatrix} Z_1 \\ Z_2 \end{Bmatrix} \cos \alpha \right) = \Delta \qquad (17)$$

The converted equation of Eq.(16)b) in principal coordinate is:

$$\Phi_2 \left( s_1 A + c_1 B + \begin{Bmatrix} Z_1 \cos \bar\omega t_1 \\ Z_2 \cos \bar\omega t_1 \end{Bmatrix} \cos \alpha - \begin{Bmatrix} Z_1 \sin \bar\omega t_1 \\ Z_2 \sin \bar\omega t_1 \end{Bmatrix} \sin \alpha \right)$$

$$-\Phi_1 \left( s_1 A + c_1 B + \begin{Bmatrix} Z_1 \cos \bar\omega t_1 \\ Z_2 \cos \bar\omega t_1 \end{Bmatrix} \cos \alpha - \begin{Bmatrix} Z_1 \sin \bar\omega t_1 \\ Z_2 \sin \bar\omega t_1 \end{Bmatrix} \sin \alpha \right) = -\Delta \qquad (18)$$

where, $\Phi_i$ is the i$^{th}$ row of the modal matrix $\Phi$.

The unknown values of initial conditions, namely A, B, $A_1$, $B_1$, and phase angle $\alpha$ can be estimated by writing Eq.(12),(14),(15),(17) and (18) in a matrix form:

$$\begin{bmatrix} -s_1 & c_2 - c_1 & s_2 & -\overline{Zs} & \overline{Zc} \\ -G\Phi\omega & 0 & \Phi\omega & T_{24} & 0 \\ -G\Phi\omega c_1 & T_{32} & \Phi\omega c_2 & T_{34} & T_{35} \\ 0 & \Phi_2 I - \Phi_1 I & 0 & 0 & \Phi_2 Z - \Phi_1 Z \\ \Phi_2 s_1 - \Phi_1 s_1 & \Phi_2 c_1 - \Phi_1 c_1 & 0 & T_{54} & T_{55} \end{bmatrix} \begin{Bmatrix} A_{2\times1} \\ B_{2\times1} \\ C_{2\times1} \\ \sin\alpha \\ \cos\alpha \end{Bmatrix} = \begin{Bmatrix} \{0\}_{2\times1} \\ \{0\}_{2\times1} \\ \{0\}_{2\times1} \\ \Delta \\ -\Delta \end{Bmatrix} \qquad (19)$$

where, $T_{24} = G\Phi Z\bar\omega - \Phi Z\bar\omega$ , $T_{32} = G\Phi\omega s_1 - \Phi\omega s_2$ , $T_{35} = G\Phi Z\bar\omega \sin\bar\omega t_1 - \Phi Z\bar\omega \sin\bar\omega t_2$ , $T_{34} = G\Phi Z\bar\omega \cos\bar\omega t_1 - \Phi Z\bar\omega \cos\bar\omega t_2$, $T_{54} = (\Phi_1 Z - \Phi_2 Z)\sin\bar\omega t_1$, $T_{55} = (\Phi_1 Z - \Phi_2 Z)\cos\bar\omega t_1$

## 2.4 Extension of the previous example with impact modification

Nigm and Shabana [39] shows that solution becomes non-viable for $0 \le \beta \le 0.18$. In this case a symmetric impact can be considered because the gaps on both the sides are same; therefore, $\beta = 0.5$ is assumed to conduct the analysis. In case of the previous example, solved in section 2.2, shows that at excitation frequency 1 rad/s and 2 rad/s, response of inner mass crosses the boundary of outer mass, which is impractical. This issue is resolved and shown in . after consideration of impact.
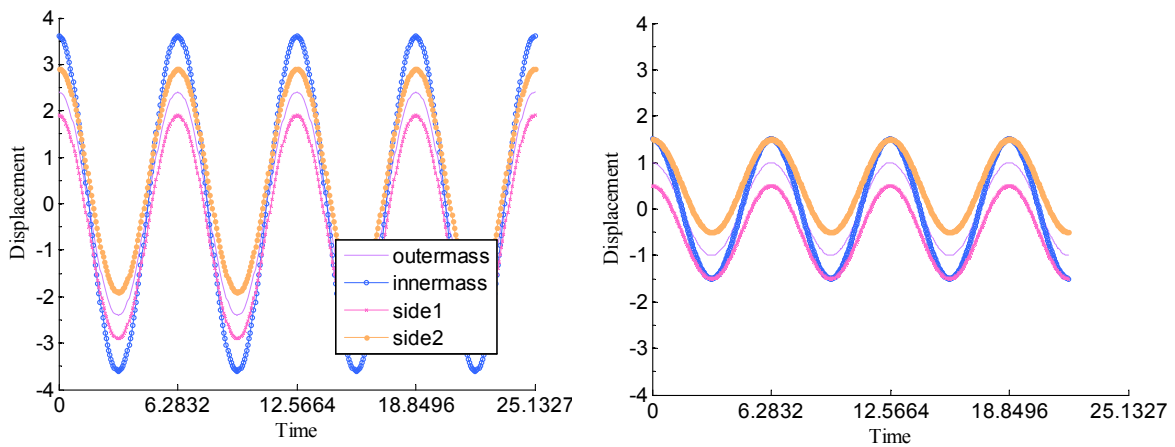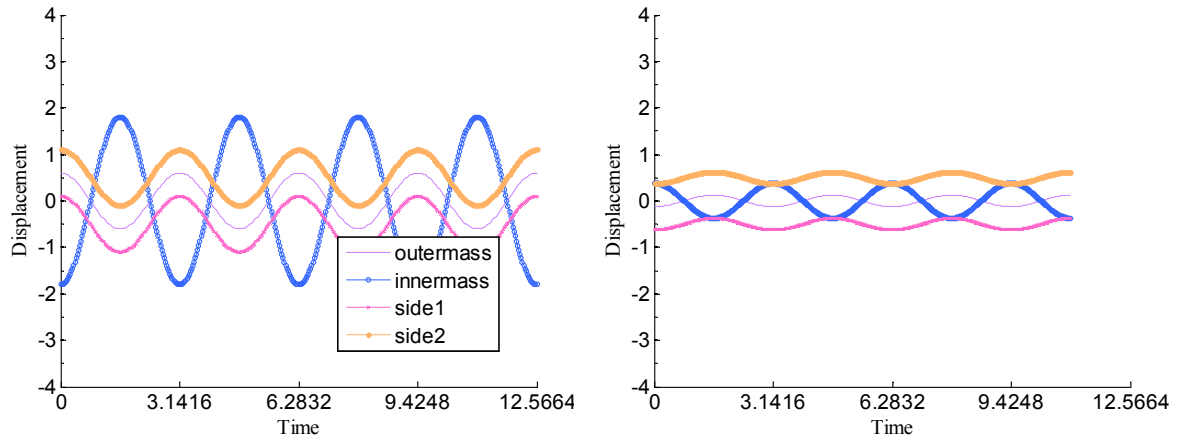
**Figure 4: Time domain steady-state response of the linear system without considering impact (in left column) and then modified impacting solution (right column) for excitation frequency 1 rad/s and 2 rad/s**

Figure 4 depicts the linear response of the two degree of freedom systems where the vibration of inner mass exceeds the boundary of the outer mass. In modified steady state solution the vibration of the inner mass is limited between the two side wall of the outer mass. The interaction between the side wall and the inner mass damped the vibration of the outer mass, even for the fully elastic collision; therefore, vibro-impacting system is potentially very useful for vibration attenuation.

## 3    FREQUENCY-RESPONSE ANALYSIS

Frequency-response curve of impacting system is computed to determine the attenuation bandwidth of a single cell of an impacting metamaterial due to the impact between the walls of the outer mass and the inner resonating unit. The total frequency dependent steady-state response can be classified into two main parts: in-phase motion, where the inner and outer masses are in same phase and out-of-phase motion, where both of them are in opposite phase. Different state of vibration for various excitation frequencies is summarized in Table 1.

**Table 1: Different state of response for the system given in section 2.2**

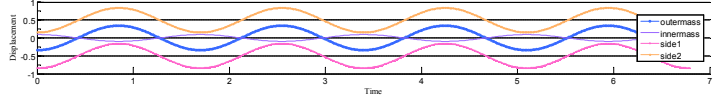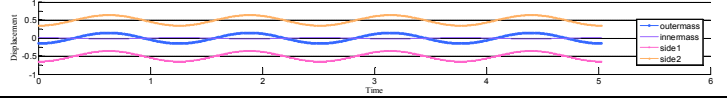| Phase | Condition | Plot (linear system) | Remarks |
|-------|-----------|----------------------|---------|
| In-phase | $X_i < X_o + \Delta$ |  | No impact |
|  | $X_i > X_o + \Delta$ |  | Impact |
| Out of phase | $X_i > X_o + \Delta$ |  | Impact |
|  | $X_i < X_o - \Delta$ |  | Impact |
|  | $X_i > \Delta - X_o$ |  | Impact |

| | | |
|---|---|---|
| $X_i < \lvert \Delta - X_o \rvert$ |  | No impact |
| $X_i < X_o$ |  | No impact |

Table 1 classifies the conditions for in-phase and out-of-phase impacts. These conditions are employed in computing the frequency response curve. Simple linear solution is adopted and maximum of the displacement is considered as amplitude in those frequency ranges, where no impact takes place, otherwise modified solution procedure discussed in section 2.3 is adopted. In Table 1, $X_i$ and $X_o$ are stands for the amplitude of the inner mass and outer mass. The transmittance of vibration, which is described as $T(dB) = 20 \log\left(u_0/u_g\right)$, is plotted in the frequency domain in Figure 5.
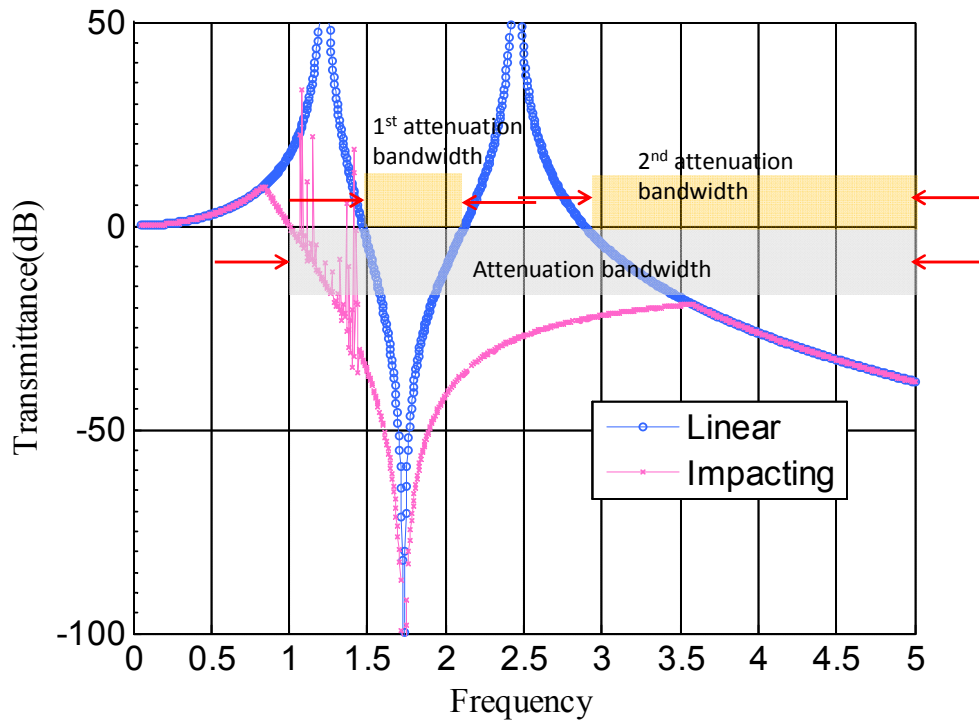


**Figure 5: Attenuation bandwidth for linear and impacting metamaterial unit**

Figure 5 depicts that the transmission of excitation is very high at the two resonating peaks, which are at 1.25 and 2.45 rad/s. In between these two peaks, an anti-resonating valley exists at 1.72 rad/s, which symbolized that almost no vibration is transmitted to the outer block from the base, in this frequency range. This anti-resonating valley is generally termed as attenuation bandwidth because it does not allow waves to propagate within the lattice in this frequency range. After the 2nd resonating peak, another infinitely long attenuation band generates. In case of the linear metamaterial unit, the 1st attenuation bandwidth starts at 1.5 rad/s and extended upto 2.2 rad/s and 2nd attenuation bandwidth originates from 2.9 rad/s and extended upto infinity. On the other hand, the attenuation bandwidth of the impacting metamaterial is initiate from 1.0 rad/s and extended upto infinity without having any transmission band inside.

Therefore, as the low frequency wave insulator and wideband mechanical filter, this impacting metamaterial can be used very efficiently.

## 4    CONCLUSIONS

The attenuation bandwidth increment capability of a single unit of impacting metamaterial over a single unit of equivalent linear metamaterial is analytically evaluated in this paper. The amplitude of the steady state response of impacting metamaterial unit for a specific monochromatic excitation is computed based on the method suggested by Nigm and Shabana. From the transmittance plot it can be easily conclude that impacting metamaterial widen the attenuation bandwidth in lower and higher frequency side, which proves that the impacting metamaterial can potentially be used as wideband filter and low frequency sound insulator. Further parametric study on the effect of the coefficient of restitution and the gap distance on the attenuation bandwidth is need to be investigated towards the designing of a wideband metamaterial unit.

## REFERENCES

[1]    Banerjee, B., *An Introduction to Metamaterials and Waves in Composites*. 2011, Auckland: Taylor & Fransis.

[2]    Pai, P.F. and H. Peng. *Acoustic metamaterial structures based on multi-frequency vibration absorbers*. in *Proceedings of SPIE - The International Society for Optical Engineering*. 2014.

[3]    Sun, H., et al. *Acoustic metamaterial with negative parameter*. in *Proceedings of SPIE - The International Society for Optical Engineering*. 2014.

[4]    Huang, H.H. and C.T. Sun, *Anomalous wave propagation in a one- dimensional acoustic metamaterial having simultaneously negative mass density and Young's modulus*. The Journal of the Acoustical Society of America, 2012. **132**: p. 2887.

[5]    Huang, G.L. and C.T. Sun, *Band Gaps in a Multiresonator Acoustic Metamaterial*. Journal of Vibration and Acoustics, 2010. **132**(3): p. 031003-031003.

[6]    Yao, S., X. Zhou, and G. Hu, *Experimental study on negative effective mass in a 1D mass–spring system*. New Journal of Physics, 2008. **10**(4): p. 043020.

[7]    Sheng, P., et al., *Locally resonant sonic materials*. Physica B: Condensed Matter, 2003. **338**(1–4): p. 201-205.

[8]    Sun, H., X. Du, and P. Frank Pai. *Metamaterial broadband vibration absorbers by local resonance*. in *Collection of Technical Papers - AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference*. 2011.

[9]    Pope, S.A. and H. Laalej, *A multi-layer active elastic metamaterial with tuneable and simultaneously negative mass and stiffness*. Smart Materials and Structures, 2014. **23**(7).

[10]   Calius, E.P., et al., *Negative mass sound shielding structures: Early results*. Physica Status Solidi B-Basic Solid State Physics, 2009. **246**(9): p. 2089-2097.

[11]   Huang, H.H., C.T. Sun, and G.L. Huang, *On the negative effective mass density in acoustic metamaterials*. International Journal of Engineering Science, 2009. **47**(4): p. 610-617.

[12]   Lu, M.-H., L. Feng, and Y.-F. Chen, *Phononic crystals and acoustic metamaterials*. Materials Today, 2009. **12**(12): p. 34-42.

[13]   Lin, G.C., et al., *Simulation of a metamaterial beam for mechanical wave absorption.* Wuli Xuebao/Acta Physica Sinica, 2011. **60**(3).

[14]   Sun, H., et al. *Theory and experiment research of metamaterial beams for broadband vibration absorption.* in *Collection of Technical Papers - AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference.* 2011.

[15]   Sun, H., et al. *Theory and experiment research of metamaterial panel for mechanical waves absorption.* in *Collection of Technical Papers - AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference.* 2013.

[16]   Huang, H.H. and C.T. Sun, *Wave attenuation mechanism in an acoustic metamaterial with negative effective mass density.* New Journal of Physics, 2009. **11**(1): p. 013003.

[17]   Baughman, R.H., et al., *Negative Poisson's ratios as a common feature of cubic metals.* Nature, 1998. **392**(6674): p. 362-365.

[18]   Friis, E., R. Lakes, and J. Park, *Negative Poisson's ratio polymeric and metallic foams.* Journal of Materials Science, 1988. **23**(12): p. 4406-4414.

[19]   Kocer, C., D.R. McKenzie, and M.M. Bilek, *Elastic properties of a material composed of alternating layers of negative and positive Poisson's ratio.* Materials Science and Engineering: A, 2009. **505**(1–2): p. 111-115.

[20]   Lakes, R., *Foam structures with a negative Poisson's ratio.* Science, 1987. **235**(4792): p. 1038-1040.

[21]   Lakes, R., *Advances in negative Poisson's ratio materials.* Advanced Materials, 1993. **5**(4): p. 293-296.

[22]   Larsen, U.D., O. Sigmund, and S. Bouwstra. *Design and fabrication of compliant micromechanisms and structures with negative Poisson's ratio.* in *Micro Electro Mechanical Systems, 1996, MEMS'96, Proceedings. An Investigation of Micro Structures, Sensors, Actuators, Machines and Systems. IEEE, The Ninth Annual International Workshop on.* 1996. IEEE.

[23]   Calius, E.P., et al., *Negative mass sound shielding structures: Early results.* physica status solidi (b), 2009. **246**(9): p. 2089-2097.

[24]   Lapine, M., I.V. Shadrivov, and Y.S. Kivshar, *Colloquium: Nonlinear metamaterials.* Reviews of Modern Physics, 2014. **86**(3): p. 1093.

[25]   Khajehtourian, R. and M.I. Hussein, *Dispersion characteristics of a nonlinear elastic metamaterial.* AIP Advances, 2014. **4**(12): p. 124308.

[26]   Nadkarni, N., C. Daraio, and D.M. Kochmann, *Dynamics of periodic mechanical structures containing bistable elastic elements: From elastic to solitary wave propagation.* Physical Review E, 2014. **90**(2): p. 023204.

[27]   Klatt, T. and M.R. Haberman, *A nonlinear negative stiffness metamaterial unit cell and small-on-large multiscale material model.* Journal of Applied Physics, 2013. **114**(3).

[28]   Romeo, F. and G. Rega, *Wave propagation properties in oscillatory chains with cubic nonlinearities via nonlinear map approach.* Chaos, Solitons & Fractals, 2006. **27**(3): p. 606-617.

[29]   Lazarov, B.S. and J.S. Jensen, *Low-frequency band gaps in chains with attached non-linear oscillators.* International Journal of Non-Linear Mechanics, 2007. **42**(10): p. 1186-1193.

[30]   Georgiou, I., *On the Global Geometric Structure of the Dynamics of the Elastic Pendulum.* Nonlinear Dynamics, 1999. **18**(1): p. 51-68.

[31]    Chen, W.Z., B.B. Hu, and H. Zhang, *Interactions between impurities and nonlinear waves in a driven nonlinear pendulum chain.* Physical Review B, 2002. **65**(13): p. 134302.

[32]    Hodges, C.H. and J. Woodhouse, *Vibration isolation from irregularity in a nearly periodic structure: Theory and measurements.* The Journal of the Acoustical Society of America, 1983. **74**: p. 894.

[33]    Bonanomi, L., G. Theocharis, and C. Daraio, *Wave propagation in granular chains with local resonances.* Physical Review E, 2015. **91**(3): p. 033208.

[34]    Lydon, J., G. Theocharis, and C. Daraio, *Nonlinear resonances and energy transfer in finite granular chains.* Physical Review E, 2015. **91**(2): p. 023208.

[35]    Vandewater, L. and S. Moss, *Probability-of-existence of vibro-impact regimes in a nonlinear vibration energy harvester.* Smart Materials and Structures, 2013. **22**(9): p. 094025.

[36]    Soliman, M.S.M., et al., *A wideband vibration- based energy harvester.* Journal of Micromechanics and Microengineering, 2008. **18**(11).

[37]    Banerjee, A., A. Chanda, and R. Das, *Historical Origin and Recent Development on Normal Directional Impact Models for Rigid Body Contact Simulation: A Critical Review.* Archives of Computational Methods in Engineering, 2016: DOI: 10.1007/s11831-016-9164-5

[38]    Hsu, C.S., *Impulsive Parametric Excitation: Theory.* Journal of Applied Mechanics, 1972. **39**(2): p. 551-558.

[39]    Nigm, M.M. and A.A. Shabana, *Effect of an impact damper on a multi-degree of freedom system.* Journal of Sound and Vibration, 1983. **89**(4): p. 541-557.