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APPLICATION OF POLAR ELASTICITY TO THE PROBLEM OF PURE BENDING OF A THICK PLATE

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Abstract. *Polar elasticity theory is employed to derive exact solution for the pure bending problem of the transversely isotropic plate. It is then compared to the conventional elasticity solution and the corresponding composite plate solution. The material parameters of the polar model are chosen so that it represents the effective behaviour of the composite plate. The influence of the additional material constant related to the fibre size effect and affecting bending stiffness of the plate is illustrated.*

1 INTRODUCTION

The recently developed polar elasticity theory [1, 2] deals with fibre-reinforced materials containing fibres that possess bending stiffness. The contribution of the fibre bending stiffness is accounted for by employing couple stresses and introducing additional elastic parameters to the constitutive equations. To gain a better understanding of the additional constant and the role of couple stress in polar theory [2] we focus on exact analytical (polar elasticity) solutions for the pure bending of thick infinite plates.

The paper is organized in the following manner: in Section 2 the displacement solution is derived for the transversely isotropic thick plate under pure bending (and the conventional – Cauchy theory of elasticity); in Section 3 the solution is derived for the case of longitudinal modulus being a periodic function of position; in Section 4 the first problem is analysed from the standpoint of couple stress theory. In Section 5 solutions of all the three models, taken with the matching effective properties and the same effective loading, are compared.

2 TRANSVERSELY ISOTROPIC MATERIAL (CONVENTIONAL THEORY OF ELASTICITY)

Let us consider a rectangular plate subjected to a known bending moment M_3 [Fig. 1]. The material is transversely isotropic and characterized by the following constants: E_1, E_2, ν_{12}, $\nu_{23}=\nu_{32}$, G_{21}.

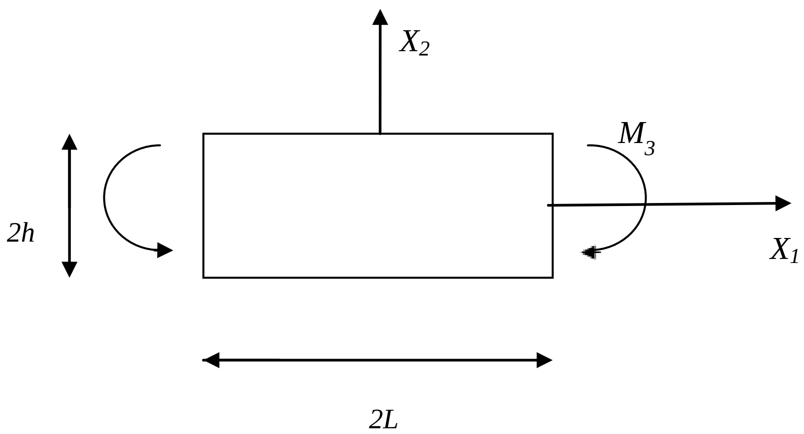

Fig.1: A thick plate with infinite length in X_3 direction.

The following boundary conditions have to hold for the tractions t_1 and t_2 on the specified faces:

$$X_2=\pm h \Rightarrow t_2=0 \tag{1a}$$

$$X_1=\pm L \Rightarrow \begin{array}{l}\int_{-h}^{h} t_1 dx_2 = 0,\\ \int_{-h}^{h} t_1 x_2 dx_2 = M_3.\end{array} \quad (1b)$$

Equilibrium equations (with negligible volume loads) must be satisfied as well:

$$\frac{\partial \sigma_{ji}}{\partial x_j} = 0 \quad , \quad (2)$$

The following stress distribution satisfies both (1) and (2):

$$\sigma_{22} = 0, \quad \sigma_{11} = \alpha x_2, \quad \sigma_{12} = 0 . \quad (3)$$

The stress distribution throughout the plate is the same at any cross-section (Fig.2).

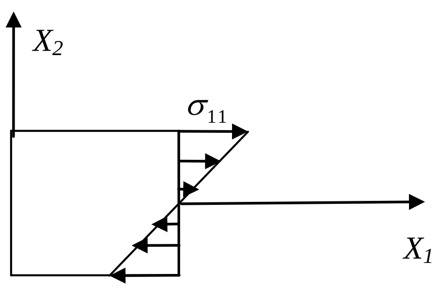

Fig.2: The stress distribution at any cross-section.

Under plane strain conditions (ε_3=0) and for stresses in the form (3) the constitutive equations yield:

$$\begin{array}{l}\varepsilon_1 = \dfrac{1-\nu_{21}\nu_{12}}{E_1}\sigma_1\\ \varepsilon_2 = \dfrac{-\nu_{12}(1+\nu_{23})}{E_1}\sigma_1\end{array} \quad (4)$$

By integrating (4) we obtain

$$\begin{array}{l}u_1 = \int \varepsilon_1 dx_1 + g_1(x_2)\\ u_2 = \int \varepsilon_2 dx_2 + g_2(x_1)\end{array} \quad (5)$$

$$\begin{cases} u_1 = \dfrac{1-\nu_{21}\nu_{12}}{E_1}\alpha x_1 x_2 + g_1(x_2)\\ u_2 = \dfrac{-\nu_{12}(1+\nu_{23})}{2E_1}\alpha x_2{}^2 + g_2(x_1)\end{cases} \quad (6)$$

With the additional condition of zero shear strain

$$\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}=0 \tag{7}$$

we find that

$$\frac{1-\nu_{21}\nu_{12}}{E_1}\alpha x_1+g_1{}'(x_2)+g_2{}'(x_1)=0 \tag{8}$$

and

$$\begin{cases} g_2{}'(x_1)=-\dfrac{1-\nu_{21}\nu_{12}}{E_1}\alpha x_1+c, \\ g_1{}'(x_2)=-c, \end{cases} \tag{9}$$

$$\begin{cases} g_2(x_1)=-\dfrac{1-\nu_{21}\nu_{12}}{2E_1}\alpha x_1{}^2+cx_1+c_1, \\ g_1(x_2)=-cx_2+c_2, \end{cases} \tag{10}$$

where the constants c_1, c_2, c are set to zero (c_1, c_2 refers to rigid body translation, c to the rigid body rotation). The final solution in displacements then acquires the form

$$\begin{cases} u_1=\dfrac{1-\nu_{21}\nu_{12}}{E_1}\alpha x_1 x_2, \\ u_2=\dfrac{-\nu_{12}(1+\nu_{23})}{2E_1}\alpha x_2{}^2-\dfrac{1-\nu_{21}\nu_{12}}{2E_1}\alpha x_1{}^2. \end{cases} \tag{11}$$

3 HETEROGENEOUS MATERIAL WITH PERIODIC PROPERTIES

We still consider a plate subjected to a known bending moment M_3 [Fig. 1]. The material is now characterized by the following parameters: $E_1=E_0+\tilde{E}\cos(\frac{\pi N}{h}x_2)$, $E_2=\left(\frac{N}{2h}\int_0^{2N/h}\frac{dx_2}{E_1(x_2)}\right)^{-1}$, ν_{12}, ν_{23}, G_{21}.

The same boundary conditions (1) and equilibrium equations (2) must be satisfied. In order to ensure pure bending, the applied normal stress at the ends of the plate must accommodate to the strain distribution $\varepsilon_1=kx_2$, where the curvature is introduced as $k=-u_{2,11}$. The curvature can be defined in relation to the load as follows:

$$M_3=\frac{k}{1-\nu_{1T}\nu_{T1}}\int_{-h}^{h}E_1x_2^2dx_2 \tag{12}$$

Thus the stresses are

$$\sigma_{22}=0,\quad \sigma_{11}=\frac{E_0+\tilde{E}\cos(\frac{\pi N}{h}x_2)}{1-\nu_{1T}\nu_{T1}}kx_2,\quad \sigma_{12}=0. \tag{13}$$

Following the logics of the previous section and the Hooke's law relations for plane strain (4) we arrive to the displacement solution

$$\begin{cases} u_1 = kx_1x_2, \\ u_2 = \int_0^x \varepsilon_2 dx_2 - \frac{1}{2}kx_1{}^2. \end{cases} \tag{14}$$

4 TRANSVERSELY ISOTROPIC MATERIAL (POLAR THEORY OF ELASTICITY)

Now let us consider the pure bending of the plate with the additional elastic constant $\boldsymbol{d_{31}}$ indicative of the bending stiffness of the substructure. The loading conditions stay the same as on the Fig. (1).
The boundary conditions on the faces in this case are as follows:

$$x_2 = \pm h: \quad t_2 = 0 \tag{15a}$$

$$x_1 = \pm L: \quad \begin{matrix} \int_{-h}^{h} t_1 dx_2 = 0, \\ \int_{-h}^{h} t_1 x_2 dx_2 + \int_{-h}^{h} m_{13} dx_2 = M_3 \end{matrix} \tag{15b}$$

where the couple stress [1] $\boldsymbol{m_{13}}$ is introduced.
The stress components are divided into the symmetric and antisymmetric parts

$$\sigma_{ij} = \sigma_{(ij)} + \sigma_{[ij]}. \tag{16}$$

Equilibrium equations in the polar theory for the present case are augmented by the equation connecting the antisymmetric shear stress and the couple stress:

$$\begin{matrix} \frac{\partial \sigma_{ji}}{\partial x_j} = 0 \;, \\ 2\sigma_{[21]} = \frac{\partial m_{13}}{\partial x_1}. \end{matrix} \tag{17}$$

The following stress and couple stress distribution satisfies both (15) and (17) (also see Fig.3):

$$\begin{matrix} \sigma_{22} = 0, \quad \sigma_{21} = 0, \\ \sigma_1 = \alpha x_2, \quad \sigma_{12} = 0, \\ m_{13} = const = \beta. \end{matrix} \tag{18}$$

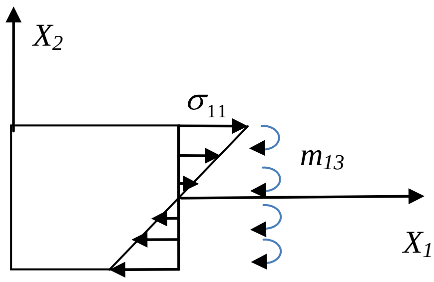

Fig.3: Distribution of the non-zero stress and couple stress components in a section of the plate.

The strain-stress relations hold in the form of (4) with the additional equation [1]:

$$m_{13} = d_{31} u_{2,11} \tag{19}$$

which can be rewritten here as

$$\frac{\partial^2 u_2}{\partial x_1^{\,2}} = \frac{\beta}{d_{31}}. \tag{20}$$

From the integration of (4) and zero shear condition (7) we obtain the displacement solution identical to (11):

$$\begin{cases} u_1 = \dfrac{1-\nu_{21}\nu_{12}}{E_1}\alpha x_1 x_2, \\ u_2 = \dfrac{-\nu_{12}(1+\nu_{23})}{2E_1}\alpha x_2^{\,2} - \dfrac{1-\nu_{21}\nu_{12}}{2E_1}\alpha x_1^{\,2}. \end{cases} \tag{21}$$

If we now insert the second equation of (21) into (20) we see that this solution is valid only if

$$-\frac{1-\nu_{21}\nu_{12}}{E_1}\alpha = \frac{\beta}{d_{31}} \ . \tag{22}$$

So (22) is the result of requirement of zero shear strain (20). It means that for the given material constant d_{31} the applied stress parameter α and the applied couple stress value $\boldsymbol{\beta}$ must relate via (22) in order to ensure pure bending.

As this section shows, it can be concluded that the displacement solution in the form of (21) for the pure bending problem is valid for

a) transversely isotropic plate (d_{31}=0) under end loading $\boldsymbol{\sigma_{11}=\alpha x_2}$ or
b) transversely isotropic polar material plate ($d_{31}\neq 0$) under the end loading $\boldsymbol{\sigma_{11}=\alpha x_2}$ and $\boldsymbol{m_{13}=const=\beta}$ under the condition that (22) holds.

It can be verified that the units of the elastic constant d_{31} correspond to Newtons [N].

5 COMPARISON AND DISCUSSION

On the basis of the previous developments we compare the solutions of the three corresponding models under the given value of the applied bending moment. The model from the Section 2 is taken as a reference model. Its periodically changing stiffness can be regarded as an approximation of the fibre reinforced composite. This model we refer to as the PS (periodical stiffness) model. If we aim to replace the PS model with a homogeneous one, we can do it in two ways. Firstly, the model from the Section 1, taken with the corresponding properties, can serve this goal. This model we refer to as the EC (effective classical) model. Secondly, the model from the Section 3, further called the EP (effective polar) model, can simulate the given problem. The goal is to compare results from two of the alternative homogeneous models (ES and EP) with the PS model and to evaluate their accuracy under varying N which can be regarded as analogous to the nominal number of fibres per height in a unidirectional fibre composite. The properties of both homogeneous models are set as effective (averaged) properties of the PS model. In addition, the bending stiffness parameter in the EP model is set as $d_{31} = \dfrac{2h^2\tilde{E}}{\pi^2 N^2 (1-\nu_{21}\nu_{12})}$ and serves as a correction parameter which ensures that the overall bending stiffness of the plate in the EP model equals that one of the PS model. All models are loaded at both ends of the plate by two equilibrated couples with magnitude (per unit width) of $M=-500\ MPa{\cdot}mm^2$ inducing the same bending moment in all sections along the plate length. The dimensions are: $2L=200\ mm$ (length of the plate), $2h=50\ mm$ (thickness).

The following properties are set in the computations:

	PS model	**EC model**	**EP model**
E_1, *MPa*	$1000 + 999\cos(\frac{\pi N}{h} x_2)$	1000	1000
E_2, *MPa*	44,71	44,71	44,71
ν_{21}	0,3	0,3	0,3
ν_{23}	0,3	0,3	0,3
d_{31}, $MPa{\cdot}mm^2$			8690 for N=4 1390 for N=10

Table 1: Material constants set for different models.

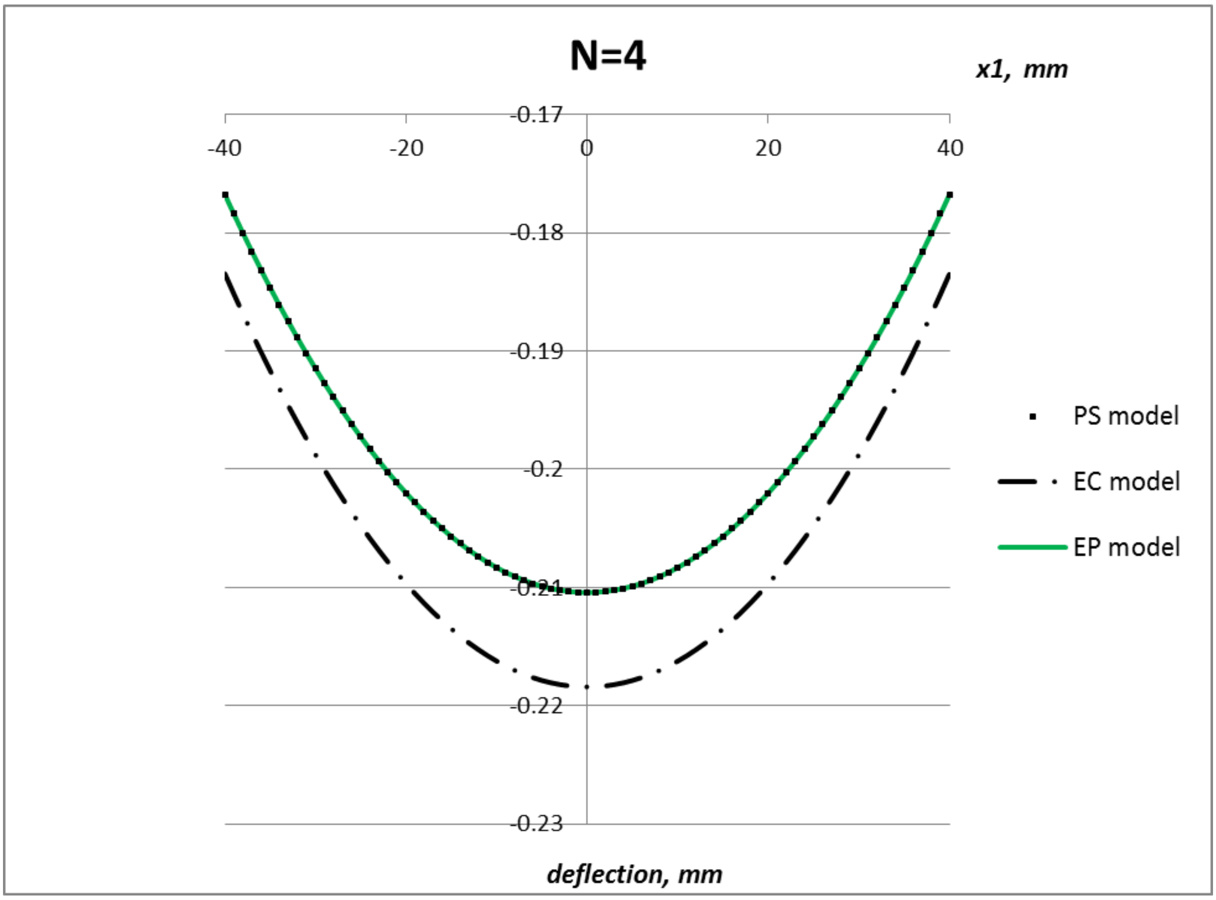

Fig.4: Plate deflection for different models, N=4.

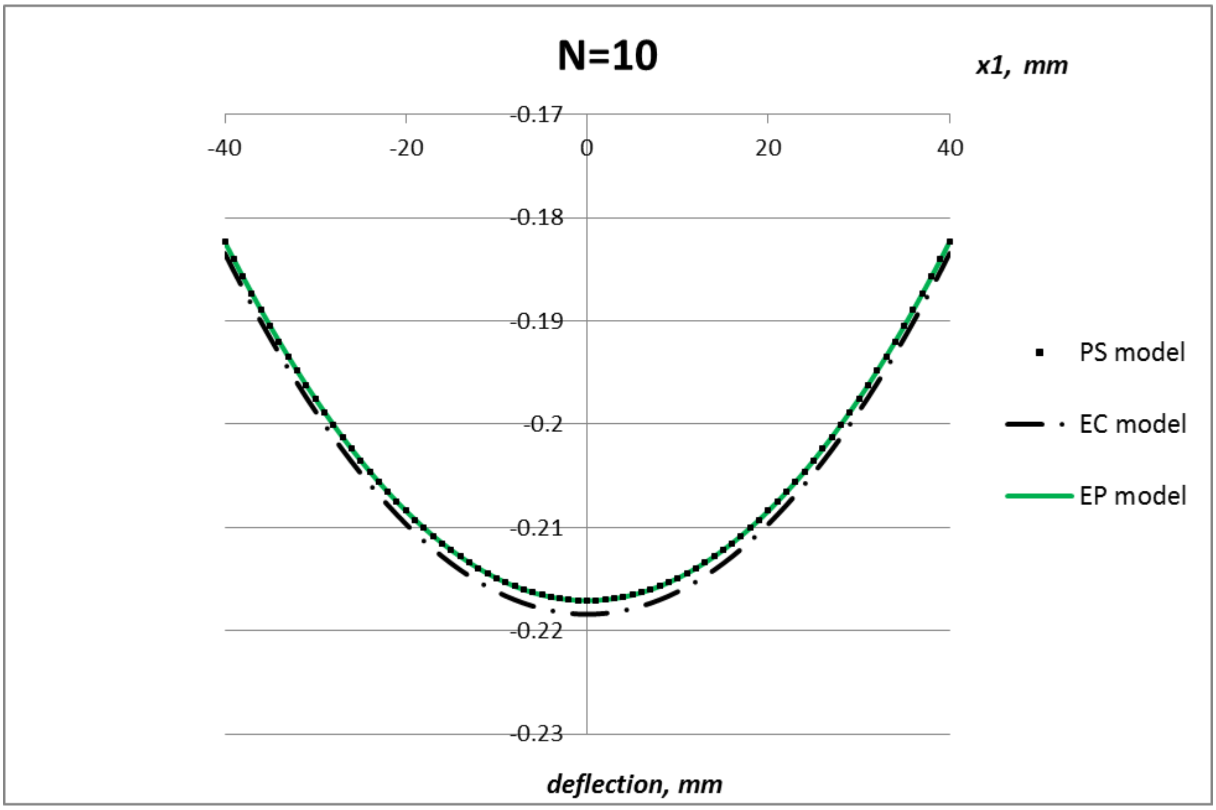

Fig.5: Plate deflection for different models, N=10.

The motivation for such comparison sprung from chapter 4 in [1] where the authors consider Euler-Bernoulli beam with the cosinusoidal Young's modulus distribution to point out the inaccuracy of the conventional theory of fibre-reinforced materials [3] in cases where the fibres are not infinitesimally thin.

In polar theory, the length scale can be introduced via the additional elastic constant d_{31}. In the present computations, the formula for d_{31} contains $(h/N)^2$ which is the (squared) dimension of a representative volume unit related to the nominal fibre thickness. (In a similar vein this constant is chosen in [4, 5].) Therefore the EP model gives correct results (equal to the PS model) for N=4 while the conventional theory of elasticity (EC model) neglects the bending stiffness of fibres and consequently underestimates the stiffness because the fibre thickness is not negligible. With increasing N the bending stiffness of fibres decreases and the correct solution (obtained by both PS and EP models) becomes closer to the EC model.

6 CONCLUSIONS

The new pure bending elasticity solution is derived for the transversely isotropic polar material. It is compared with the conventional theory solutions in order to demonstrate how the size effect can be taken into account in the homogeneous polar model. With the nominal number of fibres increasing their bending stiffness decreases and the polar elasticity model converges to the conventional elasticity model.

In the given example, the additional elastic constant has a role of a correcting parameter. It corrects the error in bending stiffness resulting from the averaging of the Young's modulus.

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