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SEVEN DIFFERENT WAYS TO MODEL VISCOELASTICITY IN A GEOMETRICALLY EXACT SETTING

Alexey V. Shutov[1 2]

[1]Novosibirsk State University
Pirogova 2, 630090, Novosibirsk, Russia
e-mail: alexey.v.shutov@gmail.com

[2] Lavrentyev Institute of Hydrodynamics
pr. Lavrentyeva 15, 630090, Novosibirsk, Russia
e-mail: alexey.v.shutov@gmail.com

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Abstract. *Seven different approaches to the phenomenological finite-strain viscoelasticity are analyzed. The focus of the analysis is put on the Maxwell fluid, due to its paramount importance for phenomenological material modeling. The first modeling approach to the Maxwell fluid is based on the multiplicative decomposition of the deformation gradient, combined with hyperelastic relations between stresses and elastic strains. The second approach employs the additive decomposition of the strain rate tensor in combination with hypoelastic relations. The third approach makes use of the additive decomposition of the material Hencky strain. The fourth and fifth models are the covariant and contravaraint models of the Maxwell fluid, proposed by Haupt and Lion (2002). Finally, the sixth and the seventh approaches are the modifications of these two models, introduced here to enforce a pure volumetric-isochoric split. The approaches are analyzed with respect to their objectivity, thermodynamic consistency, fading memory, weak invariance under isochoric change of the reference configuration, and a possibility of a pure volumetric-iscohoric split.*

1 INTRODUCTION

Various phenomenological approaches to large strain viscoelasticity are analyzed in this talk. We put the main focus on the constitutive relations of the Maxwell body, since these relations act as the backbone for many advanced models of technical materials. Apart from viscoelastic applications, the generalization of the discussed phenomenological ideas to viscoplasticity is in many cases obvious and straightforward. Nowadays, the small-strain formulation of the Maxwell body which is based on the classical Prandtl-Reuss decomposition of the infinitesimal strain tensor is widely accepted by the modeling community [1]. Within this model, the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ is decomposed into the inelastic part $\boldsymbol{\varepsilon}_{\mathrm{i}}$ and the elastic part $\boldsymbol{\varepsilon}_{\mathrm{e}}$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\mathrm{i}} + \boldsymbol{\varepsilon}_{\mathrm{e}}. \tag{1}$$

The infinitesimal stress tensor $\boldsymbol{\sigma}$ is computed using the isotropic Hooke law

$$\boldsymbol{\sigma} = k \, \mathrm{tr}(\boldsymbol{\varepsilon}_{\mathrm{e}}) \, \mathbf{1} + 2 \, \mu (\boldsymbol{\varepsilon}_{\mathrm{e}})^{\mathrm{D}}, \tag{2}$$

where $k \geq 0$ and $\mu > 0$ are the bulk and shear moduli of the material, respectively; $\mathbf{1}$ stands for the identity tensor; $(\cdot)^{\mathrm{D}}$ is the deviatoric part of a tensor.. The evolution of the inelastic strain is given by the flow rule with an initial condition at $t = t_0$

$$\dot{\boldsymbol{\varepsilon}}_{\mathrm{i}} = \frac{1}{\eta} \, \boldsymbol{\sigma}^{\mathrm{D}}, \quad \boldsymbol{\varepsilon}_{\mathrm{i}}|_{t=t_0} = \boldsymbol{\varepsilon}_{\mathrm{i}}^{0}, \quad \mathrm{tr}(\boldsymbol{\varepsilon}_{\mathrm{i}}^{0}) = 0. \tag{3}$$

Here, $\mathrm{tr}(\cdot)$ denotes the trace of a tensor, $\eta > 0$ is the viscosity of the material.

At the same time, there is a big variety of different generalizations of this model to the geometrically exact setting of large deformations. Seven different generalizations of these equations are considered in the paper. The generalizations differ in the way how the total deformation is decomposed into the elastic and inelastic parts (multiplicative decomposition of the deformation gradient vs. additive decomposition of a generalized strain or strain rate), how the elastic properties are modeled (hyperelasticity vs. hypoelasticity), which stress tensors determine the evolution of the inelastic deformations (Mandel stress vs. weighted Cauchy stress or others). Other generalizations of the Maxwell body employ material or objective time derivatives.

Interestingly, a number of approaches was discovered and then rediscovered by other researchers. The situation is even more complicated since for several large strain approaches there is a number of equivalent formulations, and different authors were not aware of that equivalence. Thus, it is very important to analyze the links between these formulations, since for some of the formulations, simple, robust, and efficient numerical algorithms are already available. After the links are established, these algorithms can be easily adjusted to cover the equivalent formulations (as was carried out in [2] for the multiplicative inelasticity). In this paper, the seven different approaches are classified with respect to the following criteria: objectivity, thermodynamic consistency, w-invariance [3], pure decomposition of the stress response in the volumetric and the isochoric parts. New modifications of covariant and contravariant Maxwell models are proposed in order to enforce the desired volumetric-isochoric split.

2 SOME GENERAL CLASSIFICATION CRITERIA

Apart from the well-known properties like objectivity and thermodynamic consistency, some other classification criteria will be considered in this paper.

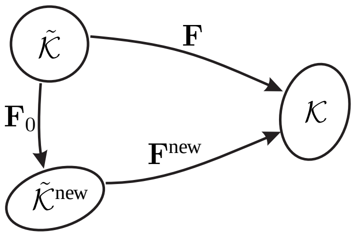

Figure 1: Commutative diagram, illustrating a change of the local reference configuration: $\tilde{\mathcal{K}}$ is replaced by a new reference $\tilde{\mathcal{K}}^{\text{new}}$.

2.1 Weak invariance under isochoric reference change

Let us consider a material model in the following form: The current value of the Kirchhoff stress tensor $\mathbf{S}$ depends on the local history of the deformation gradient $\mathbf{F}$ and a set of initial conditions $\mathcal{Z}_0$

$$\mathbf{S}(t) = \underset{t_0 \leq t' \leq t}{\mathfrak{S}} \big(\mathbf{F}(t'), \mathcal{Z}_0\big). \tag{4}$$

Here, a general response functional of a simple material appears on the right-hand side; t_0 is the initial time instance.

Next, we consider a second-rank tensor $\mathbf{F}_0$, such that $\det(\mathbf{F}_0) = 1$. Let $\widetilde{\mathcal{K}}$ be the original reference configuration and $\widetilde{\mathcal{K}}^{\text{new}} := \mathbf{F}_0 \widetilde{\mathcal{K}}$ be a new reference (cf. Figure 1). The corresponding new deformation gradient (also known as the relative deformation gradient) is obtained by the following push-forward

$$\mathbf{F}^{\text{new}}(t) := \mathbf{F}(t)\ \mathbf{F}_0^{-1}. \tag{5}$$

We say that the material model (4) is weakly invariant under the transformation (5) if there is a new set of initial data

$$\mathcal{Z}_0^{\text{new}} = \mathcal{Z}_0^{\text{new}}(\mathcal{Z}_0, \mathbf{F}_0) \tag{6}$$

such that the same stress response is predicted:

$$\underset{t_0 \leq t' \leq t}{\mathfrak{S}} \big(\mathbf{F}(t'), \mathcal{Z}_0\big) = \underset{t_0 \leq t' \leq t}{\mathfrak{S}} \big(\mathbf{F}^{\text{new}}(t'), \mathcal{Z}_0^{\text{new}}\big). \tag{7}$$

If a certain model is invariant under arbitrary isochoric changes of the reference configuration, we say that the model is *weakly invariant* or, shortly, *w-invariant*. From the mathematical standpoint, this w-invariance constitutes a certain (generalized) symmetry of the governing equations. As any other symmetry property, w-invariance provides additional insights into the structure of the constitutive equation.

2.2 Pure volumetric-iscohoric split

Suppose that (4) holds true. Let $\overline{\mathbf{F}}(t) := \det(\mathbf{F}(t))^{-1/3}\ \mathbf{F}(t)$ be the isochoric part of the deformation history $\mathbf{F}(t)$. We say that the material model (4) exhibits a pure volumetric-isochoric split (v-i split) with elastic volume changes, if the following three conditions are satisfied:

i:

$$\text{tr}\Big(\underset{t_0 \leq t' \leq t}{\mathfrak{S}} \big(\overline{\mathbf{F}}(t'), \mathcal{Z}_0\big)\Big) \equiv 0, \ \text{whenever} \ \text{tr}\, \mathbf{S}|_{t=t_0} = 0;$$

ii: there is $\mathcal{Z}_0^{dev} = \mathcal{Z}_0^{dev}(\mathcal{Z}_0)$ such that

$$\Big(\underset{t_0 \leq t' \leq t}{\mathfrak{S}} \big(\mathbf{F}(t'), \mathcal{Z}_0\big)\Big)^{\text{D}} \equiv \underset{t_0 \leq t' \leq t}{\mathfrak{S}} \big(\overline{\mathbf{F}}(t'), \mathcal{Z}_0^{dev}\big);$$

iii: $\mathrm{tr}\big(\mathbf{S}(t)\big)$ is a function of the instant value $\det\big(\mathbf{F}(t)\big)$.

Remark 1: The essential part in obtaining a pure volumetric-iscohoric split is the condition i. Indeed, consider, for example, a model where the initial conditions are formulated with respect to the Kirchhoff stresses: $\mathcal{Z}_0 = \{\mathbf{S}|_{t=t_0}\}$. If the property i is satisfied for a certain model, then the properties ii and iii can be enforced by putting

$$\underset{t_0 \leq t' \leq t}{\mathbf{S}} \big(\mathbf{F}(t'), \mathbf{S}|_{t=t_0}\big) := \underset{t_0 \leq t' \leq t}{\mathbf{S}} \big(\overline{\mathbf{F}}(t'), (\mathbf{S}|_{t=t_0})^{\mathrm{D}}\big) + p\big(\det(\mathbf{F}(t))\big)\mathbf{1}, \tag{8}$$

where $p = \mathrm{tr}\,\mathbf{S}$ is a suitable function of the current $\det \mathbf{F}$.

Remark 2: Condition ii takes the simple form thanks to the appropriate choice of the stress measure. If the Kirchhoff stress $\mathbf{S}$ is replaced by the Cauchy (true) stress $\mathbf{T}$, condition ii can be recast in the equivalent form:

$$\Big(\underset{t_0 \leq t' \leq t}{\mathbf{T}} \big(\mathbf{F}(t'), \mathcal{Z}_0\big)\Big)^{\mathrm{D}} \equiv \frac{1}{\det(\mathbf{F}(t))} \underset{t_0 \leq t' \leq t}{\mathbf{T}} \big(\overline{\mathbf{F}}(t'), \mathcal{Z}_0^{dev}\big). \tag{9}$$

3 MULTIPLICATIVE DECOMPOSITION OF THE DEFORMATION GRADIENT COMBINED WITH HYPERELASTICITY

In this section we discuss the Maxwell model which is covered by general constitutive equations of Simo and Miehe [4]. The model is based on the multiplicative decomposition of the deformation gradient $\mathbf{F}$ into the viscous (dissipative) part $\mathbf{F}_{\mathrm{i}}$ and the elastic (conservative or energetic) part $\mathbf{F}_{\mathrm{e}}$

$$\mathbf{F} = \mathbf{F}_{\mathrm{e}}\,\mathbf{F}_{\mathrm{i}}. \tag{10}$$

This constitutive assumption generalizes (1) to the case of finite strains. Another important ingredient of the model is that the elastic properties are modeled using hyperelastic relations between stresses and elastic strains.

3.1 Formulation on the reference configuration

Let us summarize the constitutive equations of this model, formulated on the reference configuration (here we follow the presentation of Lion [5]). The total strain can be described with the right Cauchy-Green tensor $\mathbf{C} := \mathbf{F}^{\mathrm{T}}\mathbf{F}$. Its inelastic part is captured using the inelastic right Cauchy-Green tensor

$$\mathbf{C}_{\mathrm{i}} := \mathbf{F}_{\mathrm{i}}^{\mathrm{T}}\mathbf{F}_{\mathrm{i}}. \tag{11}$$

Let $\tilde{\mathbf{T}}$ be the second Piola-Kirchhoff stress tensor; ρ_{R} stands for the mass density in the reference configuration; $\psi(\mathbf{C}\,\mathbf{C}_{\mathrm{i}}^{-1})$ stands for the free energy per unit mass. The function $\psi(\cdot)$ is assumed to be an isotropic function of its argument. Moreover, we assume the following volumetric-isochoric split

$$\psi(\mathbf{C}\,\mathbf{C}_{\mathrm{i}}) = \psi_{\mathrm{iso}}(\overline{\mathbf{C}\,\mathbf{C}_{\mathrm{i}}^{-1}}) + \psi_{\mathrm{vol}}(\det(\mathbf{C}\,\mathbf{C}_{\mathrm{i}}^{-1})). \tag{12}$$

The closed system of constitutive equations is given by

$$\tilde{\mathbf{T}} = 2\rho_{\mathrm{R}} \frac{\partial \psi(\mathbf{C}\mathbf{C}_{\mathrm{i}}{}^{-1})}{\partial \mathbf{C}}\Big|_{\mathbf{C}_{\mathrm{i}}=\mathrm{const}}, \quad \dot{\mathbf{C}}_{\mathrm{i}} = \frac{2}{\eta}\big(\mathbf{C}\tilde{\mathbf{T}}\big)^{\mathrm{D}}\mathbf{C}_{\mathrm{i}}, \quad \mathbf{C}_{\mathrm{i}}|_{t=t_0} = \mathbf{C}_{\mathrm{i}}^{0}. \tag{13}$$

Here, the superimposed dot denotes the material time derivative.

3.2 Formulation on the current configuration

The elastic strains can be characterized by the elastic left Cauchy-Green tensor

$$\mathbf{B}_{\rm e} := \mathbf{F}_{\rm e}\, \mathbf{F}_{\rm e}^{\rm T}. \tag{14}$$

The velocity gradient tensor is defined as

$$\mathbf{L} := \dot{\mathbf{F}}\, \mathbf{F}^{-1}. \tag{15}$$

Using this operator, the following objective derivative, known as Lee derivative or contravariant Oldroyd rate, is introduced

$$\mathscr{L}_v(\mathbf{A}) := \mathfrak{O}_{contravar}(\mathbf{A}) := \mathbf{F}\frac{d}{dt}(\mathbf{F}^{-1}\mathbf{A}\mathbf{F}^{-\rm T})\mathbf{F}^{\rm T} = \dot{\mathbf{A}} - \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^{\rm T}. \tag{16}$$

The system of constitutive equations on the current configuration takes the following form

$$\mathbf{S} = 2\rho_{\rm R}\frac{\partial\psi(\mathbf{B}_{\rm e})}{\partial\mathbf{B}_{\rm e}}\mathbf{B}_{\rm e}, \quad -\mathscr{L}_v(\mathbf{B}_{\rm e})\mathbf{B}_{\rm e}^{-1} = \frac{2}{\eta}\mathbf{S}^{\rm D}, \quad \mathbf{B}_{\rm e}|_{t=0} = \mathbf{B}_{\rm e}^{0}. \tag{17}$$

3.3 Properties of the model

Constitutive equations (13) and (17) are equivalent (cf. [2]). The material model is objective and thermodynamically consistent [5]. The inelastic flow is incompressible: $\det(\mathbf{C}_{\rm i}) \equiv 1$ and $\det(\mathbf{B}_{\rm e}) \equiv (\det(\mathbf{F}))^2$. The model exhibits a pure volumetric-iscohoric split in the sense of properties i, ii, iii. Moreover, as shown in [6], this model is w-invariant under arbitrary isochoric changes of the reference configuration. In particular, the w-invariance implies that the form of the constitutive equations $(13)_1$ and $(13)_2$ does not change if a new reference is considered; only the initial conditions $(13)_3$ have to be adjusted to match the new reference. In case of a very fast loading (as viscosity $\eta \rightarrow \infty$) the inelastic flow becomes frozen and the material response converges to pure hyperelasticity. Another important property of the model is the fading memory. More precisely: the model exhibits the so-called exponential stability of the solution with respect to small perturbations of the initial values [6]. The model is free from spurious oscillations of shear stresses under simple shear. Concerning the numerical implementation of the model, this exponential stability allows one to suppress the accumulation of the integration error [6], both for implicit and explicit time-stepping procedures. A simple explicit update formula for implicit time integration is available for neo-Hookean potentials [2]. Its generalization for Yeoh potential is presented in [7]. Some generalizations of this multiplicative Maxwell model are possible to capture a nonlinear kinematic hardening [8, 9, 10] and even a nonlinear distortional hardening [11, 12].

4 ADDITIVE DECOMPOSITION OF THE STRAIN RATE WITH HYPOELASTICITY

4.1 Constitutive equations

Another class of Maxwell models is based on the additive decomposition of the strain rate tensor $\mathbf{D} := \frac{1}{2}(\mathbf{L} + \mathbf{L}^{\rm T})$ into the inelastic part $\mathbf{D}_{\rm i}$ and the elastic part $\mathbf{D}_{\rm e}$ (cf. [13, 14]).

$$\mathbf{D} = \mathbf{D}_{\rm i} + \mathbf{D}_{\rm e}. \tag{18}$$

The elastic strain rate $\mathbf{D}_\mathrm{e}$ is related to a certain objective rate of the stress tensor $\mathbf{S}$ via grade-zero hypoelasticity

$$\mathbf{D}_\mathrm{e} = \mathbb{H} : \overset{o}{\mathbf{S}}, \tag{19}$$

where $\mathbb{H}$ is a constant fourth rank tensor (compliance tensor). Here we suppose that elastic properties are isotropic. Therefore, for the inverse $\mathbb{H}^{-1}$ (stiffness tensor) we have

$$\mathbb{H}^{-1} : \mathbf{X} = \frac{E}{3(1-2\nu)} \mathrm{tr}(\mathbf{X})\, \mathbf{1} + \frac{E}{1+\nu} \mathbf{X}^\mathrm{D}, \text{ for all } \mathbf{X} \in \mathrm{Sym}, \tag{20}$$

where E and ν are elastic constants. Note that (20) naturally implies the volumetric-isochoric split. In this section we consider corotational rates only:

$$\overset{o}{\mathbf{S}} = \dot{\mathbf{S}} + \mathbf{S}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{S}, \quad \boldsymbol{\Omega} \in \mathrm{Skew}, \tag{21}$$

where Skew is the set of skew-symmetric second-rank tensors, the tensor $\boldsymbol{\Omega}$ is the so-called spin tensor, which is a kinematic quantity (it depends on the deformation process only). The inelastic flow rule takes the form

$$\mathbf{D}_\mathrm{i} = \frac{1}{\eta}\, \mathbf{S}^\mathrm{D}. \tag{22}$$

Combining (19) with (22) and introducing appropriate initial condition for the Kirchhoff stress, we obtain the following local initial value problem

$$\overset{o}{\mathbf{S}} = \mathbb{H}^{-1} : (\mathbf{D} - \frac{1}{\eta}\, \mathbf{S}^\mathrm{D}), \quad \mathbf{S}|_{t=t_0} = \mathbf{S}_0. \tag{23}$$

To close the system of constitutive equations, it remains to define the spin tensor $\boldsymbol{\Omega}$. In case of the classical Zaremba-Jaumann rate, we have

$$\boldsymbol{\Omega}^\mathrm{ZJ} := \mathbf{W} := \frac{1}{2}(\mathbf{L} - \mathbf{L}^\mathrm{T}). \tag{24}$$

Another popular choice is the Green-Naghdi rate, also known as Green-Naghdi-Dienes rate, Green-McInnis rate or polar rate, defined by the spin

$$\boldsymbol{\Omega}^\mathrm{GN} := \dot{\mathbf{R}}\mathbf{R}^\mathrm{T}, \quad \mathbf{R} := (\mathbf{F}\mathbf{F}^\mathrm{T})^{-1/2}\, \mathbf{F} = \mathbf{F}(\mathbf{F}^\mathrm{T}\mathbf{F})^{-1/2}. \tag{25}$$

A common drawback of these corotational rates is that the constitutive equations are not integrable even for a frozen inelastic flow (which corresponds to $\eta = \infty$): Even for $\mathbf{D} \equiv \mathbf{D}_\mathrm{e}$ the stress state depends on the loading path, which contradicts the notion of path-independent elasticity.

Considereing the models of type (23) with $\mathbb{H} = const$, $\boldsymbol{\Omega} \in \mathrm{Skew}$ and $\eta = \infty$, only the so-called logarithmic spin $\boldsymbol{\Omega}^\mathrm{log}$ allows one to obtain path-independent stress-strain relations (cf. [15]). According to (21), the corotational logarithmic rate is given by

$$\overset{o}{\mathbf{S}}{}^\mathrm{log} := \dot{\mathbf{S}} + \mathbf{S}\boldsymbol{\Omega}^\mathrm{log} - \boldsymbol{\Omega}^\mathrm{log}\mathbf{S}, \tag{26}$$

where $\boldsymbol{\Omega}^\mathrm{log}$ stands for the so-called log-spin [16]. The logarithmic spin is defined implicitly through the following property

$$\mathbf{D} = (\ln \overset{o}{\mathbf{V}})^\mathrm{log}, \quad \text{where} \quad \mathbf{V} := (\mathbf{F}\mathbf{F}^\mathrm{T})^{1/2}. \tag{27}$$

An explicit expression for $\boldsymbol{\Omega}^\mathrm{log}$ can be found in [16].

4.2 Properties of the models

Let us summarize the basic properties of the models described by (23). The models are objective and the inelastic flow is incompressible. The material response exhibits a pure volumetric-iscohoric split in the sense of conditions i, ii, iii. In particular, $\mathrm{tr}(\mathbf{S}) = k \ln(\det(\mathbf{F}))$. Among the considered spins $\mathbf{\Omega}^{\mathrm{ZJ}}$, $\mathbf{\Omega}^{\mathrm{GN}}$, and $\mathbf{\Omega}^{\log}$, only the logarithmic spin $\mathbf{\Omega}^{\log}$ allows one to obtain a thermodynamically consistent stress response. For this spin, in case of a very fast loading (as $\eta \rightarrow \infty$), the stress response converges to a certain hyperelastic response (with a quadratic logarithmic strain energy function). The model (23) is w-invariant iff the corresponding spin is independent of the choice of the reference configuration [3]. The Zaremba-Jaumann spin $\mathbf{\Omega}^{\mathrm{ZJ}}$ yields a weakly invariant model, whereby $\mathbf{\Omega}^{\mathrm{GN}}$ and $\mathbf{\Omega}^{\log}$ depend on the choice of the reference [3]. Thus, dealing with relations of type (23), it is impossible to combine thermodynamic consistency and the w-invariance in a single model. Finally, it is well known that the spin $\mathbf{\Omega}^{\mathrm{ZJ}}$ yields unphysical oscillatory stress response under simple shear.

5 MODEL BASED ON THE ADDITIVE DECOMPOSITION OF THE LOGARITHMIC STRAIN

5.1 Constitutive equations

Following [17, 18, 19], we consider an approach which employs the structure of the the small-strain relations (1) – (3). In order to use these relations in the large strain case, the linearized strain tensor $\boldsymbol{\varepsilon}(t)$ is replaced by the referential logarithmic strain (Hencky strain)

$$\mathbf{H}(t) := \frac{1}{2} \ln(\mathbf{C}(t)). \tag{28}$$

The stress response $\boldsymbol{\sigma}$, resulting from (1) – (3) with $\boldsymbol{\varepsilon}(t) = \mathbf{H}(t)$, is a stress measure operating on the reference configuration which has to be power conjugate to $\mathbf{H}$

$$\boldsymbol{\sigma} : \dot{\mathbf{H}} = \widetilde{\mathbf{T}} : \Big(\frac{1}{2}\dot{\mathbf{C}}\Big) \text{ for all } \dot{\mathbf{C}} \in Sym. \tag{29}$$

In order to obtain the relation between $\boldsymbol{\sigma}$ and the second Piola-Kirchhoff stress, we recust (29) in the form

$$\boldsymbol{\sigma} : \Big(\frac{1}{2}\frac{\partial \ln(\mathbf{C})}{\partial \mathbf{C}} : \dot{\mathbf{C}}\Big) = \widetilde{\mathbf{T}} : \Big(\frac{1}{2}\dot{\mathbf{C}}\Big) \text{ for all } \dot{\mathbf{C}} \in Sym. \tag{30}$$

Thus, we arrive at

$$\Big(\frac{\partial \ln(\mathbf{C})}{\partial \mathbf{C}} : \boldsymbol{\sigma}\Big) : \dot{\mathbf{C}} = \widetilde{\mathbf{T}} : \dot{\mathbf{C}} \text{ for all } \dot{\mathbf{C}} \in Sym. \tag{31}$$

Therefore,

$$\widetilde{\mathbf{T}} = \frac{\partial \ln(\mathbf{C})}{\partial \mathbf{C}} : \boldsymbol{\sigma}, \quad \mathbf{T} = \frac{1}{\det \mathbf{F}} \mathbf{F} \Big(\frac{\partial \ln(\mathbf{C})}{\partial \mathbf{C}} : \boldsymbol{\sigma}\Big) \mathbf{F}^{\mathrm{T}}. \tag{32}$$

The entire model is subdivided into three modules (or computation steps) [18]:

- equation (28) as a geometric preprocessor ($\mathbf{C} \mapsto \mathbf{H}$)
- equations (1) – (3) as a constitutive model ($\mathbf{H} \equiv \boldsymbol{\varepsilon} \mapsto \boldsymbol{\sigma}$)
- equation $(32)_1$ as a geometric postprocessor ($\boldsymbol{\sigma} \mapsto \widetilde{\mathbf{T}}$)

5.2 Properties of the model

The considered material model is objective and thermodynamically consistent. For a given deformation process, the model exhibits exponential stability of the solution with respect to small perturbations of initial data. The inelastic flow is incompressible. Through some computations, it can be shown that the model exhibits a pure volumetric-iscohoric split in the sense of restrictions i, ii, iii. In particular, $\mathrm{tr}(\mathbf{C}\widetilde{\mathbf{T}}) = \mathrm{tr}\ \mathbf{S} = \mathrm{tr}\ \boldsymbol{\sigma} = k \ln(\det(\mathbf{F}))$. The model is *not w-invariant*: Even for $\eta = \infty$, the elastic response depends on the choice of the reference configuration [3]. The model is free from spurious oscillations of shear stresses under simple shear. Some well-established numerical procedures of small-strain theory can be applied for the numerical integration of the evolution equations.

6 COVARIANT MAXWELL MODEL

6.1 Constitutive equations

Let us denote the covaraint Oldroyd derivative by $\mathfrak{O}_{covar}$:

$$\mathfrak{O}_{covar}(\mathbf{S}) := \dot{\mathbf{S}} + \mathbf{L}^{\mathrm{T}}\ \mathbf{S} + \mathbf{S}\ \mathbf{L}. \tag{33}$$

The so-called covariant Maxwell fluid, as proposed by Haupt and Lion to model incompressible materials [20], is given by the following local initial value problem

$$\mathfrak{O}_{covar}(\mathbf{S}) + \frac{2\mu}{\eta}\mathbf{S} = 2\mu\mathbf{D}, \quad \mathbf{S}|_{t=t_0} = \mathbf{S}_0. \tag{34}$$

Contravariant pull-back of (34) to the reference configuration yields its equivalent formulation in terms of the convected stress tensor $\widetilde{\mathbf{t}} := \mathbf{F}^{\mathrm{T}}\ \mathbf{S}\ \mathbf{F}$ and Green's strain tensor $\mathbf{E} := \frac{1}{2}(\mathbf{C} - \mathbf{1})$

$$\dot{\widetilde{\mathbf{t}}} + \frac{2\mu}{\eta}\widetilde{\mathbf{t}} = 2\mu\dot{\mathbf{E}}, \quad \widetilde{\mathbf{t}}|_{t=t_0} = \widetilde{\mathbf{t}}_0. \tag{35}$$

Since the model (33) was originally developed for incompressible materials, the bulk modulus k does not enter this constitutive formulation.

In order to incorporate the volumetric response in case of a general compressible material, one may consider the following generalization: The stresses predicted by (34) are now interpreted as auxiliary stresses $\mathbf{S}_{\text{aux}}$, used to determine the deviatoric part of the overall stress response

$$\mathbf{S}(t) = (\mathbf{S}_{\text{aux}}(t))^{\mathrm{D}} + p(\det(\mathbf{F}(t)))\mathbf{1}, \tag{36}$$

$$\dot{\mathbf{S}}_{\text{aux}} + (\mathbf{L}^{\mathrm{D}})^{\mathrm{T}}\ \mathbf{S}_{\text{aux}} + \mathbf{S}_{\text{aux}}\ \mathbf{L}^{\mathrm{D}} + \frac{2\mu}{\eta}\mathbf{S}_{\text{aux}} = 2\mu\mathbf{D}^{\mathrm{D}}, \quad \mathbf{S}_{\text{aux}}|_{t=t_0} = \mathbf{S}_0. \tag{37}$$

Here, $p(\det(\mathbf{F}(t)))$ is a suitable function. In analogy to the previous models, discussed in Sections 4 and 5, one may put $p(\det(\mathbf{F}(t))) = k \ln(\det(\mathbf{F}(t)))$.

6.2 Properties of the model

The model (34) is objective and thermodynamically consistent (cf. [20]). It follows immediately from (35) that for a given (local) deformation process, the solution exhibits exponential stability with respect to small perturbations of the initial data. Equations (34) are w-invariant, since all quantities which appear in (34) are invariant under the change of the reference configuration. For that reason, equations (35) are w-invariant as well. In case of a very fast loading (as $\eta \to \infty$), the stress response converges to a pure hyperelasticity. Thanks to the very simple structure of equations (35), efficient and robust time-stepping algorithms are available. Unfortunately, the covariant model (34) *does not exhibit* a pure volumetric-iscohoric split: the condition i is violated. Indeed, for arbitrary incompressible process (with $\det(\mathbf{F}) \equiv 1$), taking the trace of (33), we arrive at

$$\frac{d}{dt}(\text{tr}\mathbf{S}) + 2\,\mathbf{S} : \mathbf{D} + \frac{2\mu}{\eta}\text{tr}\mathbf{S} = 0. \tag{38}$$

In general, the stress power is not equal to zero: $\mathbf{S} : \mathbf{D} \neq 0$. Thus, according to (38), tr$\mathbf{S} \neq 0$ even if tr$\mathbf{S}|_{t=t_0} = 0$. According to (38), even for incompressible processes, the deviatoric part of the stress influences the evolution of the volumetric part.

7 MODIFIED COVARIANT MODEL WITH A PURE V-I SPLIT

7.1 Constitutive equations

In this paper we modify the model (34) in order to enforce the property i: For arbitrary incompressible deformations we consider

$$\mathfrak{O}_{covar}(\mathbf{S}) + \frac{2\mu}{\eta}\mathbf{S} = 2\mu\mathbf{D} + \frac{2}{3}(\mathbf{S} : \mathbf{D})\mathbf{1}, \quad \mathbf{S}|_{t=t_0} = \mathbf{S}_0. \tag{39}$$

The covariant pull-back of these relations yields

$$\dot{\widetilde{\mathbf{t}}} + \frac{2\mu}{\eta}\widetilde{\mathbf{t}} = 2\mu\dot{\mathbf{E}} - \frac{1}{3}\Big(\widetilde{\mathbf{t}} : \frac{d}{dt}(\mathbf{C}^{-1})\Big)\,\mathbf{C}, \quad \widetilde{\mathbf{t}}|_{t=t_0} = \widetilde{\mathbf{t}}_0. \tag{40}$$

In order to model the stress response of a compressible Maxwell fluid, in analogy to (36)-(37), we may consider the following generalization

$$\mathbf{S}(t) = \mathbf{S}_{\text{aux}}(t) + p(\det(\mathbf{F}(t)))\mathbf{1}, \tag{41}$$

$$\dot{\mathbf{S}}_{\text{aux}} + (\mathbf{L}^{\text{D}})^{\text{T}}\,\mathbf{S}_{\text{aux}} + \mathbf{S}_{\text{aux}}\,\mathbf{L}^{\text{D}} + \frac{2\mu}{\eta}\mathbf{S}_{\text{aux}} = 2\mu\mathbf{D}^{\text{D}} + \frac{2}{3}(\mathbf{S}_{\text{aux}} : \mathbf{D}^{\text{D}})\mathbf{1}, \quad \mathbf{S}_{\text{aux}}|_{t=t_0} = \mathbf{S}_0. \tag{42}$$

7.2 Properties of the model

The suggested model is objective and w-invariant. The condition i is satisfied. Indeed, let $\det(\mathbf{F}) \equiv 1$. Then, taking the trace of (39), we obtain

$$\frac{d}{dt}(\text{tr}\mathbf{S}) + \frac{2\mu}{\eta}\text{tr}\mathbf{S} = 0, \quad \text{tr}\mathbf{S}(t) = \exp\Big(-\frac{2\mu}{\eta}(t-t_0)\Big)\,\text{tr}\mathbf{S}|_{t=t_0}. \tag{43}$$

If tr$\mathbf{S}|_{t=t_0} = 0$, then tr$\mathbf{S} \equiv 0$, which is exactly the required property i. Next, for the model (41)-(42), the properties i, ii, and iii are satisfied.

8 CONTRAVARIANT MAXWELL MODEL

8.1 Constitutive equations

Recall the contravaraint Oldroyd derivative $\mathfrak{O}_{contravar}$:

$$\mathfrak{O}_{contravar}(\mathbf{S}) := \dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^{\text{T}}. \tag{44}$$

The so-called contravariant Maxwell fluid was proposed by Haupt and Lion in the incompressible case [20]:

$$\mathfrak{O}_{contravar}(\mathbf{S}) + \frac{2\mu}{\eta}\mathbf{S} = 2\mu\mathbf{D}, \quad \mathbf{S}|_{t=t_0} = \mathbf{S}_0. \tag{45}$$

Contravariant pull-back of (45) to the reference configuration yields the following equivalent formulation in terms of the second Piola-Kirchhoff stress

$$\dot{\widetilde{\mathbf{T}}} + \frac{2\mu}{\eta}\widetilde{\mathbf{T}} = -2\mu\frac{d}{dt}\mathbf{C}^{-1}, \quad \widetilde{\mathbf{T}}|_{t=t_0} = \widetilde{\mathbf{T}}_0. \tag{46}$$

Analogously to the previous section, we generalize the model to cover arbitrary deformations: The stresses predicted by (45) are, again, auxiliary stresses $\mathbf{S}_{\text{aux}}$, used to determine the deviatoric part of the overall stress response

$$\mathbf{S}(t) = (\mathbf{S}_{\text{aux}}(t))^{\text{D}} + p(\det(\mathbf{F}(t)))\mathbf{1}, \tag{47}$$

$$\dot{\mathbf{S}}_{\text{aux}} - \mathbf{L}^{\text{D}}\,\mathbf{S}_{\text{aux}} - \mathbf{S}_{\text{aux}}\,(\mathbf{L}^{\text{D}})^{\text{T}} + \frac{2\mu}{\eta}\mathbf{S}_{\text{aux}} = 2\mu\mathbf{D}^{\text{D}}, \quad \mathbf{S}_{\text{aux}}|_{t=t_0} = \mathbf{S}_0. \tag{48}$$

8.2 Properties of the model

Just as the covaraint model, the model (45) is objective and thermodynamically consistent (cf. [20]). It follows from (46) that the solution is exponentially stable with respect to small perturbations of the initial conditions. Equations (45) are w-invariant. In case of a very fast loading (as $\eta \to \infty$), the stress response converges to a pure hyperelasticity.[1] Due to the simple linear structure of equations (46), efficient and robust numerical algorithms are available. Just as the covaraint model, its contravariant counterpart (45) *does not exhibit* a pure volumetric-iscohoric split, since the condition i is again violated. To demonstrate this, consider an incompressible deformation process with $\det(\mathbf{F}) \equiv 1$. Taking the trace of (45), we have

$$\frac{d}{dt}(\text{tr}\mathbf{S}) + 2\,\mathbf{S} : \mathbf{D} + \frac{2\mu}{\eta}\text{tr}\mathbf{S} = 0. \tag{49}$$

Again, generally, the stress power $\mathbf{S} : \mathbf{D}$ is not equal to zero. Thus, according to (49), $\text{tr}\mathbf{S} \neq 0$ even if $\text{tr}\mathbf{S}|_{t=t_0} = 0$.

9 MODIFIED CONTRAVARIANT MODEL WITH A PURE V-I SPLIT

9.1 Constitutive equations

In order to enforce the property i, we modify the equations of the contravariant Maxwell fluid. For incompressible material we consider

$$\mathfrak{O}_{contravar}(\mathbf{S}) + \frac{2\mu}{\eta}\mathbf{S} = 2\mu\mathbf{D} - \frac{2}{3}(\mathbf{S} : \mathbf{D})\mathbf{1}, \quad \mathbf{S}|_{t=t_0} = \mathbf{S}_0. \tag{50}$$

An equivalent formulation is obtained by the contravariant pull-back to the reference configuration

$$\dot{\widetilde{\mathbf{T}}} + \frac{2\mu}{\eta}\widetilde{\mathbf{T}} = -2\mu\frac{d}{dt}\mathbf{C}^{-1} - \frac{1}{3}(\widetilde{\mathbf{T}} : \dot{\mathbf{C}})\,\mathbf{C}^{-1}, \quad \widetilde{\mathbf{T}}|_{t=t_0} = \widetilde{\mathbf{T}}_0. \tag{51}$$

[1] A linear combination of the covariant and contravariant models allows one to obtain a model, which generalizes Monney-Rivlin hyperelasticity to finite-strain viscoelasticity.

In a general compressible case we put

$$\mathbf{S}(t) = \mathbf{S}_{\text{aux}}(t) + p(\det(\mathbf{F}(t)))\mathbf{1}, \tag{52}$$

$$\dot{\mathbf{S}}_{\text{aux}} - \mathbf{L}^{\text{D}}\, \mathbf{S}_{\text{aux}} - \mathbf{S}_{\text{aux}}\, (\mathbf{L}^{\text{D}})^{\text{T}} + \frac{2\mu}{\eta}\mathbf{S}_{\text{aux}} = 2\mu \mathbf{D}^{\text{D}} - \frac{2}{3}(\mathbf{S}_{\text{aux}} : \mathbf{D}^{\text{D}})\mathbf{1}, \quad \mathbf{S}_{\text{aux}}|_{t=t_0} = \mathbf{S}_0. \tag{53}$$

9.2 Properties of the model

This modified contravarinat model is objective and w-invariant. The condition i is identically satisfied. To check this property, we consider an incompressible deformation with $\det(\mathbf{F}) \equiv 1$. Then, taking the trace of (50), we arrive at

$$\frac{d}{dt}(\text{tr}\mathbf{S}) + \frac{2\mu}{\eta}\text{tr}\mathbf{S} = 0, \quad \text{tr}\mathbf{S}(t) = \exp\Big(-\frac{2\mu}{\eta}(t - t_0)\Big)\, \text{tr}\mathbf{S}|_{t=t_0}. \tag{54}$$

Thus, for $\text{tr}\mathbf{S}|_{t=t_0} = 0$, we have $\text{tr}\mathbf{S} \equiv 0$ and the property i holds true. Finally, for the generalized model (52)-(53), the properties i, ii, and iii are satisfied.

10 CONCLUSIONS

Seven different approaches to finite strain Maxwell fluid are considered in this paper. The approaches are compared qualitatively, using a number of criteria. In particular, the models are analyzed with respect to the w-invariance under the isochoric change of the reference configuration and the possibility of a pure volumetric-iscohoric split. *Two new models are proposed here*, basing on the covariant and contravariant Maxwell models, previously proposed by Haupt and Lion. In contrast to the original formulations, the modified models exhibit the pure i-v split. In the follow-up paper, some other commonly used approaches to the phenomenological elasto-plasticity will be analyzed and the links between the similar frameworks will be established.

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