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A PGD-BASED TIME SPACE DECOMPOSITION FOR THE UNSTEADY NAVIER-STOKES EQUATIONS APPLIED TO INCOMPRESSIBLE FLOWS

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Abstract. *This article addresses the theoretical and numerical formulation of an algorithm based on the Proper Generalized Decomposition (PGD) applied to the solution of unsteady incompressible viscous flows. By using a separated functional description for the space and time variables, one can formulate an algorithm which may replace the traditional incremental approach and, consequently, may reduce drastically the computational time needed for the simulation of complex unsteady flows. The choice of spatial and temporal modes for velocity and pressure is discussed and several academic applications are provided, illustrating the benefits and challenges associated with this new computational approach.*

1 INTRODUCTION

The Proper Generalized Decomposition (PGD) proposes a new theoretical framework to build the continuous partial differential equations which govern the modes defining the modal decomposition of the physical quantities of interest. Separated representations were first introduced by P. Ladevèze ([1], [2]). This remarkable idea was widely extended by F. Chinesta and his co-workers ([3], [4]). The separated representation combined with the Reduced Basis concept built with a greedy algorithm is therefore a very attractive formulation which can be used to iteratively build by successive enrichments an *a priori* separated modal description of the physical fields involved in the mathematical model of interest. In this article, one describes the use and adaptation of the PGD paradigm to build a solution of the unsteady Navier-Stokes Equations for incompressible flows. Since one does not want to reduce the framework of applications to simplified Cartesian-like or large aspect ratio geometries, the separated modal decomposition concerns only the time and spatial dimensions where the space is considered as a non-separated three-dimensional entity. From that point of view, the objective of this work is to build a constructive continuous model providing an *a priori* decomposition which is very similar to the well-known *a posteriori* time-space SVD reconstruction. No further hypothesis except the existence of a separated time-space decomposition has to be made, which means that the proposed methodology is valid for unsteady flows around geometries of arbitrary complexity. The general computational framework is the one used for the code ISIS-CFD ([5]), i.e. a generalized unstructured finite volume discretization in space and a second order accurate discretisation in time.

The existence of a separated time-space solution for the unsteady incompressible Navier-Stokes equations can not be theoretically established but one feels that the parabolicity in time and the mixed parabolic/elliptic character in space for the unsteady incompressible Navier-Stokes equations provide a mathematical background which is relatively well suited to this separated modal decomposition. In a traditional incremental approach, the unsteady solution is obtained by solving n time-steps quasi-steady three-dimensional problems (where n is the number of time steps), which becomes prohibitively expensive when the number of time steps or the global duration of the simulation are large. Replacing this traditional incremental approach by a modal strategy may result in huge gains in terms of computational efficiency if the number of modes does not grow with the number of time steps. As a matter of fact, in a separated modal representation, to compute n time steps, one needs to compute $Q * N$ three-dimensional fields, where Q is the number of enrichments (or modes) and N is the number of (fixed-point) iterations needed to determine each modal group. It is clear that if $n >> N * Q$, the PGD paradigm will become extremely attractive. Moreover, since the continuous equations used to determine the temporal modes are simple ODEs, one can use a time step as small as needed without any significant penalty in terms of computational time. By combining the stored spatial and temporal modes, one can restore the full unsteady flow evolution at any time and any point with strongly reduced storage requirements, which makes extremely easy the *a-posteriori* detailed analysis of the unsteady flow. Finally, a very interesting by-product of this formulation is a reduced order flow model which could be used in situations where a high-fidelity solution is not necessarily needed like in active flow control, for instance.

In this article, one will describe the choices of decomposition for the variables involved in the unsteady Navier-Stokes decomposition, the proposed general continuous formulations, the treatment of the unsteady boundary conditions and several illustrations on simple unsteady laminar flows.

2 PGD FORMULATION OF THE UNSTEADY NAVIER-STOKES EQUATIONS FOR INCOMPRESSIBLE FLOWS

2.1 The partial differential equations

To build the continuous PGD formulation, one starts from the following formulation of the unsteady incompressible Navier-Stokes equations:

$$\begin{array}{lcl} \frac{\partial u}{\partial t} + \hat{u} + \frac{\partial p}{\partial x} & = & f^u(\overline{x}, t) \\ \frac{\partial v}{\partial t} + \hat{v} + \frac{\partial p}{\partial y} & = & f^v(\overline{x}, t) \\ \hat{u} & = & Div(u\overrightarrow{U^{cv}}) - \frac{\partial}{\partial x}(\nu \frac{\partial u}{\partial x}) - \frac{\partial}{\partial y}(\nu \frac{\partial u}{\partial y}) \\ \hat{v} & = & Div(v\overrightarrow{U^{cv}}) - \frac{\partial}{\partial x}(\nu \frac{\partial v}{\partial x}) - \frac{\partial}{\partial y}(\nu \frac{\partial v}{\partial y}) \\ Div(\overrightarrow{U}) & = & 0 \end{array} \quad (1)$$

where $\overrightarrow{f} = (f^u, f^v)$ represents momentum source term, $\overrightarrow{U^{cv}}(u, v)$ is the convective velocity and $\overrightarrow{\hat{U}} = (\hat{u}, \hat{v})$ is an intermediate pseudo-acceleration field.

2.2 The time space PGD decomposition

Let us suppose that the velocity, pressure, pseudo-acceleration and velocity flux field can be expressed according to the following separated modal expansions :

$$\begin{array}{lcl} u(\overline{x}, t) & \approx & \sum_{k=1}^{i} X_k^u(\overline{x}) T_k^u(t) \\ v(\overline{x}, t) & \approx & \sum_{k=1}^{i} X_k^v(\overline{x}) T_k^u(t) \\ \hat{u}(\overline{x}, t) & \approx & \sum_{k=1}^{i} \hat{X}_k^u(\overline{x}) \hat{T}_k^u(t) \\ \hat{v}(\overline{x}, t) & \approx & \sum_{k=1}^{i} \hat{X}_k^v(\overline{x}) \hat{T}_k^u(t) \\ p(\overline{x}, t) & \approx & \sum_{k=1}^{i} X_k^p(\overline{x}) T_k^p(t) \end{array} \quad (2)$$

The only hypothesis made here is that the temporal modes are identical for each component of the vectorial fields. Otherwise, one supposes that the other dependent variables introduced in this separated decomposition have their own temporal modes which will have to be determined by independent differential equations.

When injected in the formulation (1), one gets the following non-linear coupled equations:

$$\sum_{k=1}^{i} X_k^u \frac{dT_k^u}{dt} + \sum_{k=1}^{i} \hat{X}_k^u \hat{T}_k^u + \sum_{k=1}^{i} \frac{\partial X_k^p}{\partial x} T_k^p = f^u(\overline{x}, t)$$

$$\sum_{k=1}^{i} X_k^v \frac{dT_k^u}{dt} + \sum_{k=1}^{i} \hat{X}_k^v \hat{T}_k^u + \sum_{k=1}^{i} \frac{\partial X_k^p}{\partial y} T_k^p = f^v(\overline{x}, t)$$

$$\sum_{k=1}^{i} \hat{X}_k^u \hat{T}_k^u - \sum_{k=1}^{i} \sum_{l=1}^{i} Div(X_k^u \overrightarrow{X_l}) T_k^u T_l^u + \sum_{k=1}^{i} T_k^u [\frac{\partial}{\partial x}(\nu \frac{\partial X_k^u}{\partial x}) + \frac{\partial}{\partial y}(\nu \frac{\partial X_k^u}{\partial y})] = 0$$

$$\sum_{k=1}^{i} \hat{X}_k^v \hat{T}_k^u - \sum_{k=1}^{i} \sum_{l=1}^{i} Div(X_k^v \overrightarrow{X_l}) T_k^u T_l^u + \sum_{k=1}^{i} T_k^u [\frac{\partial}{\partial x}(\nu \frac{\partial X_k^v}{\partial x}) + \frac{\partial}{\partial y}(\nu \frac{\partial X_k^v}{\partial y})] = 0$$

$$\sum_{k=1}^{i} Div(\overrightarrow{X_k}) T_k^u = 0 \quad (3)$$

One can write now the weak formulation in the space-time domain of these equations:

$$\int_0^{T_{max}} \int_\Omega (X_i^u T_i^{u*} + X_i^{u*} T_i^u) \left[\sum_{k=1}^{i} X_k^u \frac{dT_k^u}{dt} + \sum_{k=1}^{i} \hat{X}_k^u \hat{T}_k^u + \sum_{k=1}^{i} \frac{\partial X_k^p}{\partial x} T_k^p\right] d\overline{x}dt = \int_0^{T_{max}} \int_\Omega (X_i^u T_i^{u*} + X_i^{u*} T_i^u) f^u(\overline{x}, t) d\overline{x}dt$$

$$\int_0^{T_{max}} \int_\Omega (X_i^v T^{u*_i} + X_i^{v*} T_i^u) \left[\sum_{k=1}^{i} X_k^v \frac{dT_k^u}{dt} + \sum_{k=1}^{i} \hat{X}_k^v \hat{T}_k^u + \sum_{k=1}^{i} \frac{\partial X_k^p}{\partial y} T_k^p\right] d\overline{x}dt = \int_0^{T_{max}} \int_\Omega (X_i^v T_i^{u*} + X_i^{v*} T_i^u) f^v(\overline{x}, t) d\overline{x}dt$$

$$\int_0^{T_{max}} \int_\Omega (X_i^u T^{u*_i} + X_i^{u*} T_i^u) \left[\sum_{k=1}^{i} \hat{X}_k^u \hat{T}_k^u - \sum_{k=1}^{i} \sum_{l=1}^{i} Div(X_k^u \overrightarrow{X_l}) T_k^u T_l^u + \sum_{k=1}^{i} T_k^u [\frac{\partial}{\partial x}(\nu \frac{\partial X_k^u}{\partial x}) + \frac{\partial}{\partial y}(\nu \frac{\partial X_k^u}{\partial y})]\right] d\overline{x}dt = 0 \quad (4)$$

$$\int_0^{T_{max}} \int_\Omega (X_i^v T^{u*_i} + X_i^{u*} T_i^u) \left[\sum_{k=1}^{i} \hat{X}_k^v \hat{T}_k^u - \sum_{k=1}^{i} \sum_{l=1}^{i} Div(X_k^v \overrightarrow{X_l}) T_k^u T_l^u + \sum_{k=1}^{i} T_k^u [\frac{\partial}{\partial x}(\nu \frac{\partial X_k^v}{\partial x}) + \frac{\partial}{\partial y}(\nu \frac{\partial X_k^v}{\partial y})]\right] d\overline{x}dt = 0$$

$$\int_0^{T_{max}} \int_\Omega (X_i^p T_i^{p*} + X_i^{p*} T_i^p) \left[\sum_{k=1}^{i} Div(\overrightarrow{X_k}) T_k^u\right] = 0$$

where $X_i^{u*}, X_i^{v*}, T_i^{u*}$, (resp. X_i^{p*}, T_i^{p*}) are velocity (resp. pressure) test functions, $[0, T_{max}]$ is the temporal interval and Ω is the spatial domain of integration.

2.3 The incompressibility condition

The incompressibility condition is a steady partial differential equation which expresses the local conservation of mass:

$$Div(\overrightarrow{U}) = 0 \quad (5)$$

Using a PGD decomposition of the velocity, one gets the following equation:

$$\sum_{k=1}^{i} Div(\overrightarrow{X}_k) T_k^u = 0 \quad (6)$$

Projected on the time and space dimensions using a Galerkin projection, one gets two equations for the spatial and temporal modes, respectively:

$$\sum_{k=1}^{i}(\int_{0}^{T_{max}} T_i^p T_k^u dt) Div(\overrightarrow{X}_k) = 0 \tag{7}$$

and

$$\sum_{k=1}^{i}(\int_{\Omega} X_i^p Div(\overrightarrow{X}_k) dV) T_k^u = 0 \tag{8}$$

The following PGD coefficients are introduced:

$$\begin{array}{lcl} \xi_{ik} & = & \int_0^{T_{max}} T_i^p T_k^u dt \\ e_k^p & = & \int_\Omega X_i^p Div(\overrightarrow{X}_k) dV \end{array} \tag{9}$$

which leads to:

$$\begin{array}{lcl} \sum_{k=1}^{i} \xi_{ik} Div(\overrightarrow{X}_k) & = & 0 \\ \sum_{k=1}^{i} e_k^p T_k^u & = & 0 \end{array} \tag{10}$$

Actually, if the first mode is built such that it satisfies the steady Navier-Stokes equations as it will be explained later on, $Div(\overrightarrow{X_1}) = 0$.
This means that the second mode satisfies the same conditions, and by recurrence, that every spatial mode is incompressible for any k. Therefore, the PGD solution will be decomposed on the basis of spatial modes which satisfy individually the incompressibility condition. This means also that the temporal equation is trivially satisfied since $e_k^p = 0$ and can not be used.

3 DETERMINING THE SPATIAL MODES

3.1 The PGD momentum equations

To build the spatial PDE's and temporal ODE's needed to determine the spatial and temporal modes, we use the standard Galerkin projection of the equations (3) alternatively on the spatial and temporal directions and gets a set of coupled non-linear equations. The spatial PGD equations are defined by:

$$\begin{array}{lcl} \sum_{k=1}^{i} \alpha_{ik} X_k^u + \sum_{k=1}^{i} \hat{\beta}_{ik} \hat{X}_k^u + \sum_{k=1}^{i} \zeta_{ik} \frac{\partial X_k^p}{\partial x} & = & \delta_i^u(\overline{x}) \\ \sum_{k=1}^{i} \alpha_{ik} X_k^v + \sum_{k=1}^{i} \hat{\beta}_{ik} \hat{X}_k^v + \sum_{k=1}^{i} \zeta_{ik} \frac{\partial X_k^p}{\partial y} & = & \delta_i^v(\overline{x}) \\ \sum_{k=1}^{i} \hat{\beta}_{ik} \hat{X}_k^u - \sum_{k=1}^{i} \sum_{l=1}^{i} \gamma_{ikl} Div(X_k^u \overrightarrow{X_l}) + \sum_{k=1}^{i} \beta_{ik} [\frac{\partial}{\partial x}(\nu \frac{\partial X_k^u}{\partial x}) + \frac{\partial}{\partial y}(\nu \frac{\partial X_k^u}{\partial y})] & = & 0 \\ \sum_{k=1}^{i} \hat{\beta}_{ik} \hat{X}_k^v - \sum_{k=1}^{i} \sum_{l=1}^{i} \gamma_{ikl} Div(X_k^v \overrightarrow{X_l}) + \sum_{k=1}^{i} \beta_{ik} [\frac{\partial}{\partial x}(\nu \frac{\partial X_k^v}{\partial x}) + \frac{\partial}{\partial y}(\nu \frac{\partial X_k^v}{\partial y})] & = & 0 \\ Div(\overrightarrow{X}^i) & = & 0 \end{array} \tag{11}$$

with the following coefficients:

$$\begin{array}{rcl}\alpha_{ik} & = & \int_0^{T_{max}} T_i^u \frac{dT_k^u}{dt} dt \\ \hat{\beta}_{ik} & = & \int_0^{T_{max}} T_i^u \hat{T}_k^u dt \\ \beta_{ik} & = & \int_0^{T_{max}} T_i^u T_k^u dt \\ \gamma_{ikl} & = & \int_0^{T_{max}} T_i^u T_k^u T_l^u dt \\ \zeta_{ik} & = & \int_0^{T_{max}} T_i^u T_k^p dt \\ \delta_i^u(\overline{x}) & = & \int_0^{T_{max}} T_i^u f^u(\overline{x},t) dt \\ \delta_i^v(\overline{x}) & = & \int_0^{T_{max}} T_i^u f^v(\overline{x},t) dt \end{array} \tag{12}$$

3.2 Transformation of the incompressibility condition into a Laplace pressure equation - Formulation 1

Let us now see how it is possible to transform the incompressibility condition into an operational partial differential equation enabling the determination of the pressure spatial modes.
As indicated before, each spatial velocity mode satisfies:

$$Div(\overrightarrow{X}_i) = 0 \tag{13}$$

The modal momentum equations read:

$$\begin{array}{rcl}\sum_{k=1}^{i} \alpha_{ik} X_k^u + \sum_{k=1}^{i} \hat{\beta}_{ik} \hat{X}_k^u + \sum_{k=1}^{i} \zeta_{ik} \frac{\partial X_k^p}{\partial x} & = & \delta_i^u(\overline{x}) \\ \sum_{k=1}^{i} \alpha_{ik} X_k^v + \sum_{k=1}^{i} \hat{\beta}_{ik} \hat{X}_k^v + \sum_{k=1}^{i} \zeta_{ik} \frac{\partial X_k^p}{\partial y} & = & \delta_i^v(\overline{x}) \end{array} \tag{14}$$

This means that :

$$\begin{array}{rcl}\alpha_{ii} X_i^u + \hat{\beta}_{ii} \hat{X}_i^u + \zeta_{ii} \frac{\partial X_i^p}{\partial x} & = & \delta_i^u(\overline{x}) \\ & - & \sum_{k=1}^{i-1} \alpha_{ik} X_k^u - \sum_{k=1}^{i-1} \hat{\beta}_{ik} \hat{X}_k^u - \sum_{k=1}^{i-1} \zeta_{ik} \frac{\partial X_k^p}{\partial x} \\ \alpha_{ii} X_i^v + \hat{\beta}_{ii} \hat{X}_i^v + \zeta_{ii} \frac{\partial X_i^p}{\partial y} & = & \delta_i^v(\overline{x}) \\ & - & \sum_{k=1}^{i-1} \alpha_{ik} X_k^v - \sum_{k=1}^{i-1} \hat{\beta}_{ik} \hat{X}_k^v - \sum_{k=1}^{i-1} \zeta_{ik} \frac{\partial X_k^p}{\partial y} \end{array} \tag{15}$$

Using the incompressibility of every spatial modal velcity yields the so-called “Laplace pressure equation” which couples together the spatial pressure and pseudo-acceleration modes:

$$-\hat{\beta}_{ii} Div(\overrightarrow{\hat{X}}_i) - \zeta_{ii} \Delta X_i^p = -\delta_i^{p*} \tag{16}$$

with:

$$\delta_i^{p*} = \delta_i^p - \sum_{k=1}^{i-1} \hat{\beta}_{ik} Div(\overrightarrow{\hat{X}}_k) - \sum_{k=1}^{i-1} \zeta_{ik} \Delta X_k^p \tag{17}$$

It also provides the relation which should be used to update the modal velocity flux at the control volume interface ∂V_c:

$$\begin{array}{rcl}\alpha_{ii} \overrightarrow{X}_i \cdot \overrightarrow{n} & = & \overrightarrow{\delta_i(\overline{x})} \cdot \overrightarrow{n} - \hat{\beta}_{ii} \overrightarrow{\hat{X}}_i \cdot \overrightarrow{n} - \zeta_{ii} \overrightarrow{\nabla X_i^p} \cdot \overrightarrow{n} \\ & - & \sum_{k=1}^{i-1} \alpha_{ik} \overrightarrow{X}_k \cdot \overrightarrow{n} - \sum_{k=1}^{i-1} \hat{\beta}_{ik} \overrightarrow{\hat{X}}_k \cdot \overrightarrow{n} - \sum_{k=1}^{i-1} \zeta_{ik} \overrightarrow{\nabla X_k^p} \cdot \overrightarrow{n} \end{array} \tag{18}$$

where every term is reconstructed at the interface and not interpolated from cell-centered values in agreement with the Rhie and Chow procedure to avoid any checkerboard oscillations.

3.3 Transformation of the incompressibility condition into a Poisson pressure equation - Formulation 2

If one wishes to build a pressure equation which resembles the pressure equation used in the classic incremental formulation of the unsteady Navier Stokes equations, we should proceed in a slightly different way which is explained now. The momentum equations are first rebuilt by replacing the pseudo-acceleration field by its expression, which leads to :

$$\begin{array}{rl} & \sum_{k=1}^{i} \alpha_{ik} X_k^u + \sum_{k=1}^{i} \sum_{l=1}^{i} \gamma_{ikl} Div(X_k^u \overrightarrow{X_l}) \\ - & \sum_{k=1}^{i} \beta_{ik} [\frac{\partial}{\partial x}(\nu \frac{\partial X_k^u}{\partial x}) + \frac{\partial}{\partial y}(\nu \frac{\partial X_k^u}{\partial y})] + \sum_{k=1}^{i} \zeta_{ik} \frac{\partial X_k^p}{\partial x} = \delta_i^u(\overline{x}) \\ & \sum_{k=1}^{i} \alpha_{ik} X_k^v + \sum_{k=1}^{i} \sum_{l=1}^{i} \gamma_{ikl} Div(X_k^v \overrightarrow{X_l}) \\ - & \sum_{k=1}^{i} \beta_{ik} [\frac{\partial}{\partial x}(\nu \frac{\partial X_k^v}{\partial x}) + \frac{\partial}{\partial y}(\nu \frac{\partial X_k^v}{\partial y})] + \sum_{k=1}^{i} \zeta_{ik} \frac{\partial X_k^p}{\partial y} = \delta_i^v(\overline{x}) \end{array} \quad (19)$$

If we introduce the following source terms δ_i^{u*} and δ_i^{v*}:

$$\begin{array}{rcl} \delta_i^{u*} & = & \delta_i^u - \sum_{k=1}^{i-1} \{ \alpha_{ik} X_k^v + \sum_{l=1}^{i} \gamma_{ikl} Div(X_k^u \overrightarrow{X_l}) - \beta_{ik} \frac{\partial}{\partial x}(\nu \frac{\partial X_k^u}{\partial x}) - \frac{\partial}{\partial y}(\nu \frac{\partial X_k^u}{\partial y})] \\ & + & \zeta_{ik} \frac{\partial X_k^p}{\partial x} \} \\ \delta_i^{v*} & = & \delta_i^v - \sum_{k=1}^{i-1} \{ \alpha_{ik} X_k^v + \sum_{l=1}^{i} \gamma_{ikl} Div(X_k^u \overrightarrow{X_l}) - \beta_{ik} \frac{\partial}{\partial x}(\nu \frac{\partial X_k^v}{\partial x}) - \frac{\partial}{\partial y}(\nu \frac{\partial X_k^v}{\partial y})] \\ & + & \zeta_{ik} \frac{\partial X_k^p}{\partial y} \} \end{array} \quad (20)$$

the PGD momentum equations for the current i mode read:

$$\begin{array}{l} \alpha_{ii} X_i^u + \sum_{l=1}^{i} \gamma_{iil} Div(X_i^u \overrightarrow{X_l}) - \beta_{ii} [\frac{\partial}{\partial x}(\nu \frac{\partial X_i^u}{\partial x}) + \frac{\partial}{\partial y}(\nu \frac{\partial X_i^u}{\partial y})] + \zeta_{ii} \frac{\partial X_i^p}{\partial x} = \delta_i^{u*} \\ \alpha_{ii} X_i^v + \sum_{l=1}^{i} \gamma_{iil} Div(X_i^v \overrightarrow{X_l}) - \beta_{ii} [\frac{\partial}{\partial x}(\nu \frac{\partial X_i^v}{\partial x}) + \frac{\partial}{\partial y}(\nu \frac{\partial X_i^v}{\partial y})] + \zeta_{ii} \frac{\partial X_i^p}{\partial y} = \delta_i^{v*} \end{array} \quad (21)$$

Once discretised with a finite volume discretization, the momentum equations read for each control volume V_c:

$$\begin{array}{rcl} Vol_c \frac{\alpha_{ii}}{\beta_{ii}} X_c^u + \sum C_{nb} X_{nb}^u + C_d X_c^u + Vol_c \frac{\zeta_{ii}}{\beta_{ii}} Discr\left[\frac{\partial}{\partial x} X^p\right] + Src_U & = & 0 \\ Vol_c \frac{\alpha_{ii}}{\beta_{ii}} X_c^v + \sum C_{nb} X_{nb}^v + C_d X_c^v + Vol_c \frac{\zeta_{ii}}{\beta_{ii}} Discr\left[\frac{\partial}{\partial y} X^p\right] + Src_V & = & 0 \end{array} \quad (22)$$

where:

$$\begin{array}{rcl} Src_U & = & -Vol_c \frac{\delta_i^{u*}}{\beta_{ii}} \\ Src_V & = & -Vol_c \frac{\delta_i^{v*}}{\beta_{ii}} \end{array} \quad (23)$$

The indices i of the mode i are no more mentioned for the sake of simplicity. The operator $Discr$ stands for "Discretization of". Vol_c is the volume of the cell of integration, C_{nb} and C_d are the discretisation coefficients of the convective-diffusive common operator present in both momentum equations.

In order to build a pressure equation, a new discrete pseudo-velocity field $(\widetilde{X}_c^u, \widetilde{X}_c^v)$ is introduced:

$$\begin{array}{rcl} \widetilde{X}_c^u - \frac{1}{Vol_c} \sum C_{nb} X_{nb}^u & = & \frac{SrcU}{Vol_c} = -\frac{\delta_i^{u*}}{\beta_{ii}} \\ \widetilde{X}_c^v - \frac{1}{Vol_c} \sum C_{nb} X_{nb}^v & = & \frac{SrcV}{Vol_c} = -\frac{\delta_i^{v*}}{\beta_{ii}} \end{array} \quad (24)$$

which leads to:

$$\begin{array}{rcl} X_c^u & = & -C_p \left(\widetilde{X}_c^u + Vol_c \frac{\zeta_{ii}}{\beta_{ii}} Discr\left[\frac{\partial}{\partial x} X^p\right] \right) \\ X_c^v & = & -C_p \left(\widetilde{X}_c^v + Vol_c \frac{\zeta_{ii}}{\beta_{ii}} Discr\left[\frac{\partial}{\partial y} X^p\right] \right) \end{array} \quad (25)$$

with $C_{Diag} = C_d + Vol_c \frac{\alpha_{ii}}{\beta_{ii}}$ and $C_p = \frac{Vol_c}{C_{Diag}}$.
By using the solenoidality condition, one gets and alternate pressure equation which can be written with the following semi-discretized formulation:

$$-Div(C_p \vec{\tilde{X}}) - Div(C_p \frac{\zeta_{ii}}{\beta_{ii}} \overrightarrow{\nabla X^p}_i) \quad = \quad 0 \tag{26}$$

3.4 Comments on the spatial formulation

In this section, we have built the spatial PGD equations which have to be solved to determine the spatial velocity and pressure modes. In order to have spatial velocity modes satisfying individually the incompressibility condition, we have proposed two possibilities to transform the solenoidality condition into a pressure equation. The first one is built from the continuous formulation of the momentum equations in which an intermediate pseudo-acceleration vectorial field has been introduced. Taking the divergence of these equations leads to a partial differential equation combining a Laplace pressure equation and the divergence of the pseudo-acceleration field. The second option which corresponds to the usual procedure for the finite volume approach for incompressible flows consists in discretizing first the momentum equations and reconstructing the fluxes of the velocity components and pressure gradient, while all the contributions of the neighboring data and source terms are gathered into the so-called pseudo-velocity fluxes. Proceeding in this way transforms the solenoidality condition into a Poisson pressure equation and a divergence of the pseudo velocity vectorial field. In that case, the discretization coefficients of the momentum equations enter the formulation of the spatial pressure equation through the central coefficient C_p and the definition of the pseudo-velocity fields.

3.5 Final spatial formulation

To summarize, one gets the final coupled non-linear formulation for the spatial PGD modes:

$$\begin{array}{lcl}
\sum_{k=1}^{i} \alpha_{ik} X_k^u + \sum_{k=1}^{i} \hat{\beta}_{ik} \hat{X}_k^u + \sum_{k=1}^{i} \zeta_{ik} \frac{\partial X_k^p}{\partial x} & = & \delta_i^u(\overline{x}) \\
\sum_{k=1}^{i} \alpha_{ik} X_k^v + \sum_{k=1}^{i} \hat{\beta}_{ik} \hat{X}_k^v + \sum_{k=1}^{i} \zeta_{ik} \frac{\partial X_k^p}{\partial y} & = & \delta_i^v(\overline{x}) \\
\sum_{k=1}^{i} \hat{\beta}_{ik} \hat{X}_k^u - \sum_{k=1}^{i} \sum_{l=1}^{i} \gamma_{ikl} Div(X_k^u \vec{X_l}) + \sum_{k=1}^{i} \beta_{ik} [\frac{\partial}{\partial x}(\nu \frac{\partial X_k^u}{\partial x}) + \frac{\partial}{\partial y}(\nu \frac{\partial X_k^u}{\partial y})] & = & 0 \\
\sum_{k=1}^{i} \hat{\beta}_{ik} \hat{X}_k^v - \sum_{k=1}^{i} \sum_{l=1}^{i} \gamma_{ikl} Div(X_k^v \vec{X_l}) + \sum_{k=1}^{i} \beta_{ik} [\frac{\partial}{\partial x}(\nu \frac{\partial X_k^v}{\partial x}) + \frac{\partial}{\partial y}(\nu \frac{\partial X_k^v}{\partial y})] & = & 0 \\
- \sum_{k=1}^{i} \hat{\beta}_{ik} Div(\vec{\hat{X}}_k) - \sum_{k=1}^{i} \zeta_{ik} \Delta X_k^p & = & -\delta_i^p(\overline{x}) \\
or & & \\
-\beta_{ii} Div(C_p \vec{\tilde{X}}_i) - \zeta_{ii} Div(C_p \nabla X_i^p) & = & 0
\end{array} \tag{27}$$

with:

$$\begin{array}{lcl}\delta_i^u(\overline{x}) & = & \int_0^{T_{max}} T_i^u f^u(\overline{x},t)dt \\ \delta_i^v(\overline{x}) & = & \int_0^{T_{max}} T_i^u f^v(\overline{x},t)dt \\ \delta_i^p(\overline{x}) & = & \int_0^{T_{max}} T_i^u Div(\overrightarrow{f}(\overline{x},t))dt \\ \alpha_{ik} & = & \int_0^{T_{max}} T_i^u \frac{dT_k^u}{dt}dt \\ \hat{\beta}_{ik} & = & \int_0^{T_{max}} T_i^u \hat{T}_k^u dt \\ \beta_{ik} & = & \int_0^{T_{max}} T_i^u T_k^u dt \\ \gamma_{ikl} & = & \int_0^{T_{max}} T_i^u T_k^u T_l^u dt \\ \zeta_{ik} & = & \int_0^{T_{max}} T_i^u T_k^p dt \end{array} \quad (28)$$

4 DETERMINING THE TEMPORAL MODES

4.1 Comments on the pressure temporal modes

It has been noticed previously that it is impossible to build an equation to determine the temporal modes of the pressure T_i^p if one starts from the original incompressibility condition $Div(\overrightarrow{U}) = 0$. Moreover, if one starts from the classic semi-discretized Poisson equation for the pressure (formulation 2), it is also impossible to build a consistent temporal differential equation since this semi-discretized equation comprises terms (C_p and $\overrightarrow{\overrightarrow{X_i}}$) which come from the discretisation of the momentum equations. It appears therefore that the only valid starting point to determine the pressure temporal modes is the differential Laplace pressure equation introduced in 3.2.

4.2 Temporal PGD ordinary differential equations

Based on the set of equations (3), the temporal PGD equations consist in a set of five coupled non-linear algebraic differential equations:

$$\begin{array}{lcl}\sum_{k=1}^i a_{ik}^u \frac{dT_k^u}{dt} + \sum_{k=1}^i \hat{b}_{ik}^u \hat{T}_k^u + \sum_{k=1}^i e_{ik}^u T_k^p & = & d_i^u(t) \\ \sum_{k=1}^i a_{ik}^v \frac{dT_k^u}{dt} + \sum_{k=1}^i \hat{b}_{ik}^v \hat{T}_k^u + \sum_{k=1}^i e_{ik}^v T_k^p & = & d_i^v(t) \\ \sum_{k=1}^i \hat{b}_{ik}^u \hat{T}_k^u - \sum_{k=1}^i \sum_{l=1}^i c_{ikl}^u T_k^u T_l^u - \sum_{k=1}^i b_{ik}^u T_k^u & = & 0 \\ \sum_{k=1}^i \hat{b}_{ik}^v \hat{T}_k^u - \sum_{k=1}^i \sum_{l=1}^i c_{ikl}^v T_k^u T_l^u - \sum_{k=1}^i b_{ik}^v T_k^u & = & 0 \\ \sum_{k=1}^i a_{ik}^p T_k^p + \sum_{k=1}^i \hat{b}_{ik}^p \hat{T}_k^u & = & d_i^p(t) \end{array} \quad (29)$$

where the following coefficients have been introduced:

$$
\begin{array}{lcl}
a^u_{ik} & = & \int_\Omega X^u_i X^u_k d\overline{x} \\
a^v_{ik} & = & \int_\Omega X^v_i X^v_k d\overline{x} \\
\hat{b}^u_{ik} & = & \int_\Omega X^u_i \hat{X}^u_k d\overline{x} \\
\hat{b}^v_{ik} & = & \int_\Omega X^v_i \hat{X}^v_k d\overline{x} \\
c^u_{ikl} & = & \int_\Omega X^u_i Div(X^u_k \overrightarrow{X_l}) d\overline{x} \\
c^v_{ikl} & = & \int_\Omega X^v_i Div(X^v_k \overrightarrow{X_l}) d\overline{x} \\
b^u_{ik} & = & \int_\Omega X^u_i \left[-\frac{\partial}{\partial x}(\nu \frac{\partial X^u_k}{\partial x}) - \frac{\partial}{\partial y}(\nu \frac{\partial X^u_k}{\partial y}) \right] d\overline{x} \\
b^v_{ik} & = & \int_\Omega X^v_i \left[-\frac{\partial}{\partial x}(\nu \frac{\partial X^v_k}{\partial x}) - \frac{\partial}{\partial y}(\nu \frac{\partial X^v_k}{\partial y}) \right] d\overline{x} \\
e^u_{ik} & = & \int_\Omega X^u_i \frac{\partial X^p_k}{\partial x} d\overline{x} \\
e^v_{ik} & = & \int_\Omega X^v_i \frac{\partial X^p_k}{\partial y} d\overline{x} \\
d^u_i & = & \int_\Omega X^u_i f^u(\overline{x}, t) d\overline{x} \\
d^v_i & = & \int_\Omega X^v_i f^v(\overline{x}, t) d\overline{x} \\
a^p_{ik} & = & \int_\Omega X^p_i (-Div \left[\overrightarrow{\nabla X^p_k} \right]) d\overline{x} \\
\hat{b}^p_{ik} & = & \int_\Omega X^p_i (-Div \left[\overrightarrow{\hat{X}_k} \right]) d\overline{x} \\
d^p_i & = & \int_\Omega X^p_i (-Div[\overrightarrow{f}]) d\overline{x}
\end{array} \tag{30}
$$

We have now five ordinary differential equations to determine three unknowns T^u_i, $\hat{T}^u_i$ and T^p_i. We will use the sum of the first and second equations and the sum of the third and fourth equations to provide a system coupling together T^u_i, $\hat{T}^u_i$ and T^p_i:

$$
\begin{array}{lcl}
\sum_{k=1}^i (a^u_{ik} + a^v_{ik}) \frac{dT^u_k}{dt} + \sum_{k=1}^i (\hat{b}^u_{ik} + \hat{b}^v_{ik}) \hat{T}^u_k + \sum_{k=1}^i (e^u_{ik} + e^v_{ik}) T^p_k & = & d^u_i(t) + d^v_i(t) \\
\sum_{k=1}^i (\hat{b}^u_{ik} + \hat{b}^v_{ik}) \hat{T}^u_k - \sum_{k=1}^i \sum_{l=1}^i (\hat{c}^u_{ikl} + \hat{c}^v_{ikl}) T^u_k T^u_l - \sum_{k=1}^i (\hat{f}^u_{ik} + \hat{f}^v_{ik}) T^u_k & = & 0 \\
\sum_{k=1}^i a^p_{ik} T^p_k + \sum_{k=1}^i \hat{b}^p_{ik} \hat{T}^u_k & = & d^p_i(t)
\end{array} \tag{31}
$$

and these coupled non-linear differential equations will be iteratively solved with an appropriate linearisation strategy. It is worthwhile to underline that the last algebraic equation provides a non-linear (actually quadratic) relation between the pressure and the velocity temporal modes which is the temporal counterpart of the Bernoulli equation.
At this stage, it should be noticed that it is not necessary easy to provide accurate evaluations of these temporal coefficients with a formal 2nd order accurate finite volume solver. This means that specific discretisation schemes have to be built to provide an accurate enough discretisation of these operators on a fully unstrctured grid. For the sake of conciseness, we will not elaborate more on this topic which is however felt crucial for the modal convergence of the PGD decomposition.

5 SPACE-TIME CONSISTENCY RELATIONS

It might be useful to notice that there are additional equations linking together the spatial and temporal PGD coefficients. These relations are obtained, for instance, by integrating over the whole flow domain the spatial modal equations. This yields the following relations:

$$
\begin{array}{lcl}
\sum_{l=1}^{k}\alpha_{kl}a_{il}^{u}+\sum_{l}^{k}\hat{\beta}_{kl}b_{il}^{u}+\sum_{l}^{k}\zeta_{kl}e_{il}^{u} & = & \int_{\Omega}X_{i}^{u}\delta_{k}^{u}d\overline{x} \\
\sum_{l=1}^{k}\alpha_{kl}a_{il}^{v}+\sum_{l}^{k}\hat{\beta}_{kl}b_{il}^{v}+\sum_{l}^{k}\zeta_{kl}e_{il}^{v} & = & \int_{\Omega}X_{i}^{v}\delta_{k}^{v}d\overline{x} \\
\sum_{l=1}^{k}\hat{\beta}_{kl}\hat{b}_{ik}^{u}-\sum_{l=1}^{k}\sum_{m=1}^{k}\gamma_{klm}c_{ikm}^{u}-\sum_{l=1}^{k}\beta_{kl}b_{ik}^{u} & = & 0 \\
\sum_{l=1}^{k}\hat{\beta}_{kl}\hat{b}_{ik}^{v}-\sum_{l=1}^{k}\sum_{m=1}^{k}\gamma_{klm}c_{ikm}^{v}-\sum_{l=1}^{k}\beta_{kl}b_{ik}^{v} & = & 0 \\
\sum_{l=1}^{k}\zeta_{kl}a_{ik}^{p}T_{k}^{p}+\sum_{l=1}^{k}\hat{\beta}_{kl}\hat{b}_{ik}^{p} & = & \int_{\Omega}X_{i}^{p}\delta_{k}^{p}d\overline{x}
\end{array}
\qquad (32)
$$

which are valid for any indices i and k. These relations establish a kind of global time-space consistency of the PGD coefficients and can be used to ensure structural relations which might not be verified at a discrete level, due to the influence of the spatial discretisation errors on the determination of the temporal PGD coefficients.

6 COMMENTS ON THE GENERAL COMPUTATIONAL STRATEGY

This PGD algorithm is therefore comprised of four embedded loops:

1. an enrichment loop which adds new temporal and spatial modes to enrich the modal decomposition until modal convergence,

2. a fixed point loop which ensures the mutual convergence of the couple of spatial and temporal modes which are solved sequentially. It should be noted here that we have chosen to converge on the non-linearity at each fixed point iteration, either for the spatial or temporal modes. Although such a procedure is likely to be very expensive, we preferred to use it to ensure a safe overall convergence.

3. a non-linear loop which accounts for the non-linearities present in the spatial PDE's and the temporal ODE's,

4. a linear coupling loop which takes into account the linear coupling existing between the linearized spatial momentum and pressure equations on one hand, and on the other hand, the linear coupling between the linearized temporal equations for the temporal velocity and pressure modes.

In this version of the **ISIS-CFD** code, the linear coupling between the spatial equations is treated with a fully coupled formulation for which the fully-coupled saddle-point linear system corresponding to the linearised momentum and pressure equations is directly solved by *ad-hoc* preconditioned Krylov solvers in the spirit of previous developments performed in our team ([6]).

7 TAKING INTO ACCOUNT THE BOUNDARY CONDITIONS AND RELATED SIMPLIFICATIONS

7.1 Satisfying unsteady boundary conditions

The PGD formulation is well posed when the boundary conditions are homogeneous. When the unsteadiness of the flow can be reached naturally, thanks to the development of internal instabilities in the flow like for the unsteady behind a cylinder, this peculiarity of the PGD methodology does not pose any problem. However, when the unsteadiness of the flow is imposed by moving boundaries, one should build a first set of modes which

satisfy the unsteady boundary conditions. This means that the velocity boundary condition should be written in a time space separated form in order to comply with the present choice. For simple academic flows like the unsteady driven cavity, or the flow between two concentric cyinders, it is easy to design a procedure to take into account the unsteadiness of the boundary conditions.
Let us take for instance the case of a driven square cavity where the horizontal component of the velocity is imposed at the upper wall by:

$$U_{wall} = f(t) \tag{33}$$

We select the steady flow solution associated with $U_{wall} = 1$ to provide the initial spatial modes and one multiplies this steady solution by the temporal function involved at the upper wall of the driven cavity to build a peculiar flow solution satisfying the unsteady boundary conditions but, of course, not the unsteady Navier-Stokes equations. The steady flow solution $X_1^u(\overline{x}), X_1^v(\overline{x}), X_1^p(\overline{x})$ satisfies the steady Navier-Stokes equations and we can use it as first spatial modes, the first temporal velocity modes being provided by $T_1^u(t) = f(t)$ while the first pressure temporal mode has still to be specified. The first modes are therefore designed in such a way that the unsteady boundary conditions are satisfied, which means that the next modes just need to satisfy homogeneous Dirichlet (velocity) or Neumann (pressure) boundary conditions.

$$\begin{array}{rcl} u(\overline{x},t) & = & u^{(2)}(\overline{x},t) + X_1^u(\overline{x})T_1^u(t) \\ v(\overline{x},t) & = & v^{(2)}(\overline{x},t) + X_1^v(\overline{x})T_1^u(t) \\ p(\overline{x},t) & = & p^{(2)}(\overline{x},t) + X_1^p(\overline{x})T_1^p(t) \end{array} \tag{34}$$

However, we also need to prescribe the temporal and spatial first modes for the pressure and the pseudo-acceleration in the framework of the proposed formulation. For the pseudo-acceleration, we can simply use its definition:

$$\begin{array}{rcl} \hat{b}_{11}^u \hat{T}_{11}^u & = & c_{111}^u (T_1^u)^2 + b_{11}^u T_1^u \\ \hat{b}_{11}^v \hat{T}_{11}^u & = & c_{111}^v (T_1^u)^2 + b_{11}^v T_1^u \end{array} \tag{35}$$

Moreover, the fact that the first spatial mode is solution of the steady Navier-Stokes equations provides the following relations:

$$\begin{array}{rcl} \hat{b}_{11}^u & = & c_{111}^u + b_{11}^u \\ \hat{b}_{11}^v & = & c_{111}^v + b_{11}^v \end{array} \tag{36}$$

which leads us to the formula used to initialize the first pseudo-acceleration temporal mode:

$$\hat{T}_1^u = \frac{(c_{111}^u + c_{111}^v)(T_1^u)^2 + (b_{11}^u + b_{11}^v)T_1^u}{c_{111}^u + c_{111}^v + b_{11}^u + b_{11}^v} \tag{37}$$

The optimal initialisation of the first pressure temporal mode is still an open problem. To determine the initial temporal pressure mode, we might suppose that the flow solution based on the first PGD mode satisfies the incompressibility condition:

$$a_{11}^p T_1^p + \hat{b}_{11}^p \hat{T}_1^u = 0 \tag{38}$$

If one supposes that the first spatial pressure mode is solution of the steady Navier-Stokes equations:

$$a_{11}^p + \hat{b}_{11}^p = 0 \tag{39}$$

this leads to:

$$T_1^p \quad = \quad \hat{T}_1^u \tag{40}$$

But, there is no justification of the above-mentioned hypothesis and the determination of the first temporal mode can be considered as an open question.
When the unsteady boundary conditions are more complex, one can build a time-space Singular Value Decomposition of the unsteady boundary conditions, which leads to the following expression:

$$\begin{array}{rcl} u^{(bnd)} & = & \sum_{k=1}^{N} X_k^{u(bnd)}(\overline{x}) T_k^u(t) \\ v^{(bnd)} & = & \sum_{k=1}^{N} X_k^{v(bnd)}(\overline{x}) T_k^u(t) \\ p^{(bnd)} & = & \sum_{k=1}^{N} X_k^{p(bnd)}(\overline{x}) T_k^p(t) \end{array} \tag{41}$$

We can then solve N steady Navier-Stokes equations with the boundary conditions provided by $X_k^{u(bnd)}$ and $X_k^{v(bnd)}$ to initialize the spatial modes in the flow domain. Once these modes have been determined, the remaining PGD modes will satisfy homogeneous velocity conditions. This procedure can be tedious in case of very complex unsteady boundary conditions but this is a known limitation of the PGD paradigm.

7.2 Simplifications related with the choice of the initial temporal and spatial modes

As indicated before, when the boundary conditions are not homogeneous, it is necessary to select the k first temporal and spatial modes such that the unsteady boundary conditions are satisfied by their combination. It is interesting to notice that if, these first k spatial modes satisfy also the steady incompressible Navier-Stokes equations and if $\hat{T}_k^u = T_k^p$ for all the k modes used to ensure the boundary conditions, then for every next $i > k$ mode, the following relations hold:

$$\begin{array}{rcl} -Div[\overrightarrow{\nabla X_i^p}] - Div[\overrightarrow{\hat{X}_i}] & = & 0 \\ \hat{T}_i^u = T_i^p & & \end{array} \tag{42}$$

which greatly simplifies the PGD formulation and reduces the number of independent PGD coefficients.

8 SIMPLIFICATIONS FOR THE STOKES EQUATIONS

For the Stokes equations, the non-linear convective terms are absent. If one chooses the first spatial mode such that it satisfies the steady Stokes equations, we have the following relations:

$$\begin{array}{rcl} \hat{X}_1^u & = & -\frac{\partial X_1^p}{\partial x} \\ \hat{X}_1^v & = & -\frac{\partial X_1^p}{\partial y} \\ \hat{X}_1^u & = & \frac{\partial}{\partial x}(\nu \frac{\partial X_1^u}{\partial x}) + \frac{\partial}{\partial y}(\nu \frac{\partial X_1^u}{\partial y}) \\ \hat{X}_1^v & = & \frac{\partial}{\partial x}(\nu \frac{\partial X_1^v}{\partial x}) + \frac{\partial}{\partial y}(\nu \frac{\partial X_1^v}{\partial y}) \end{array} \tag{43}$$

and then, for every i:

$$\begin{array}{rcl} \hat{b}_{i1}^u & = & b_{i1}^u \\ \hat{b}_{i1}^v & = & b_{i1}^v \\ \hat{b}_{i1}^u & = & -e_{i1}^u \\ \hat{b}_{i1}^v & = & -e_{i1}^v \\ a_{i1}^p & = & -\hat{b}_{i1}^p \end{array} \tag{44}$$

Since $\hat{T}_1^u = T_1^u$, if one chooses $T_1^p = T_1^u$, this leads to the following coupled equations for the second PGD spatial and temporal modes:

$$\begin{array}{lcl}
\alpha_{22}X_2^u + \hat{\beta}_{22}\hat{X}_2^u + \zeta_{22}\frac{\partial X_2^p}{\partial x} & = & -\alpha_{21}X_1^u \\
\alpha_{22}X_2^v + \hat{\beta}_{22}\hat{X}_2^v + \zeta_{22}\frac{\partial X_2^p}{\partial y} & = & -\alpha_{21}X_1^v \\
\hat{\beta}_{22}\hat{X}_2^u + \beta_{22}[\frac{\partial}{\partial x}(\nu\frac{\partial X_2^u}{\partial x}) + \frac{\partial}{\partial y}(\nu\frac{\partial X_2^u}{\partial y})] & = & 0 \\
\hat{\beta}_{22}\hat{X}_2^v + \beta_{22}[\frac{\partial}{\partial x}(\nu\frac{\partial X_2^v}{\partial x}) + \frac{\partial}{\partial y}(\nu\frac{\partial X_2^v}{\partial y})] & = & 0 \\
-\zeta_{22}Div[\overrightarrow{\nabla X_2^p}] - \hat{\beta}_{22}Div[\overrightarrow{\hat{X}_2}] & = & 0 \\
(a_{22}^u + a_{22}^v)\frac{dT_2^u}{dt} + (\hat{b}_{22}^u + \hat{b}_{22}^v)\hat{T}_2^u + (e_{22}^u + e_{22}^v)T_2^p & = & -(a_{21}^u + a_{21}^v)\frac{dT_1^u}{dt} \\
(\hat{b}_{22}^u + \hat{b}_{22}^v)\hat{T}_2^u - (b_{22}^u + b_{22}^v)T_2^u & = & 0 \\
a_{22}^p T_2^p + \hat{b}_{22}^p\hat{T}_2^u & = & 0
\end{array} \tag{45}$$

The solutions of this system are :

$$\begin{array}{lcl}
T_2^p & = & T_2^u \\
\hat{T}_2^u & = & T_2^u \\
(a_{22}^u + a_{22}^v)\frac{dT_2^u}{dt} + (\hat{a}_{22}^u + e_{22}^u + \hat{a}_{22}^v + e_{22}^v)T_2^u & = & -(a_{21}^u + a_{21}^v)\frac{dT_1^u}{dt} \\
-Div[\overrightarrow{\nabla X_2^p}] - Div[\overrightarrow{\hat{X}_2}] & = & 0 \\
\hat{X}_2^u + [\frac{\partial}{\partial x}(\nu\frac{\partial X_2^u}{\partial x}) + \frac{\partial}{\partial y}(\nu\frac{\partial X_2^u}{\partial y})] & = & 0 \\
\hat{X}_2^v + [\frac{\partial}{\partial x}(\nu\frac{\partial X_2^v}{\partial x}) + \frac{\partial}{\partial y}(\nu\frac{\partial X_2^v}{\partial y})] & = & 0 \\
\alpha_{22}\hat{X}_2^u + \hat{\beta}_{22}(\hat{X}_2^u + \frac{\partial X_2^p}{\partial x}) & = & -\alpha_{21}\hat{X}_1^u \\
\alpha_{22}\hat{X}_2^v + \hat{\beta}_{22}(\hat{X}_2^v + \frac{\partial X_2^p}{\partial y}) & = & -\alpha_{21}\hat{X}_1^v
\end{array} \tag{46}$$

It is trivial to demonstrate by recurrence for the next temporal modes that $\hat{T}_i^u = T_i^u$ and $\hat{T}_i^p = T_i^u$, which proves that this formulation naturally provides a decomposition for which every temporal pressure mode is equal to the velocity temporal mode, thanks to the linearity of the Stokes equations.

9 APPLICATIONS

9.1 Analytic solutions

9.1.1 Stokes equations

A. Dumont ([7]) used in his PhD thesis the following 2D analytic solutions for the Stokes equations:

$$\begin{array}{lcl}
\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - \frac{\partial}{\partial x}(\nu\frac{\partial u}{\partial x}) - \frac{\partial}{\partial y}(\nu\frac{\partial u}{\partial y}) + f^u(\overline{x}, t) & = & 0 \\
\frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} - \frac{\partial}{\partial x}(\nu\frac{\partial v}{\partial x}) - \frac{\partial}{\partial y}(\nu\frac{\partial v}{\partial y}) + f^v(\overline{x}, t) & = & 0 \\
-Div\left[\overrightarrow{\nabla p}\right] - Div\left[\overrightarrow{\hat{U}}\right] + f^p(\overline{x}, t) & = & 0
\end{array} \tag{47}$$

with:

$$\begin{array}{lcl} u(x,y,t) & = & cos(\omega x)sin(\omega y)e^{-t} \\ v(x,y,t) & = & -sin(\omega x)cos(\omega y)e^{-t} \\ p(x,y,t) & = & (1-2\omega^2\nu)sin(\omega x)sin(\omega y)e^{-t} \\ f^u(x,y,t) & = & -(1-\omega)(2\omega^2\nu-1)cos(\omega x)sin(\omega y)e^{-t} \\ f^v(x,y,t) & = & -(1+\omega)(1-2\omega^2\nu)sin(\omega x)cos(\omega y)e^{-t} \\ f^p(x,y,t) & = & -(\frac{\partial f^u}{\partial x}+\frac{\partial f^v}{\partial y}) \\ & = & -2\omega^2(2\omega^2\nu-1)sin(\omega x)sin(\omega y)e^{-t} \end{array} \tag{48}$$

with $\omega = 2\pi$. This manufactured solution is imposed in a squared cavity $[0.;1.]\times[0.;1.]$ with the following boundary conditions and the PGD decomposition converges in one mode analytically and numerically.

9.1.2 Navier-Stokes equations

The unsteady Taylor-Green vortex is an analytic solution of the unsteady Navier-Stokes equations. It is given by:

$$\begin{array}{lcl} u(x,y,t) & = & sin(\omega x)cos(\omega y)e^{-2\omega^2\nu t} \\ v(x,y,t) & = & -cos(\omega x)sin(\omega y)e^{-2\omega^2\nu t} \\ p(x,y,t) & = & \frac{1}{4}(cos(2\omega x)+cos(2\omega y))e^{-4\omega^2\nu t} \end{array} \tag{49}$$

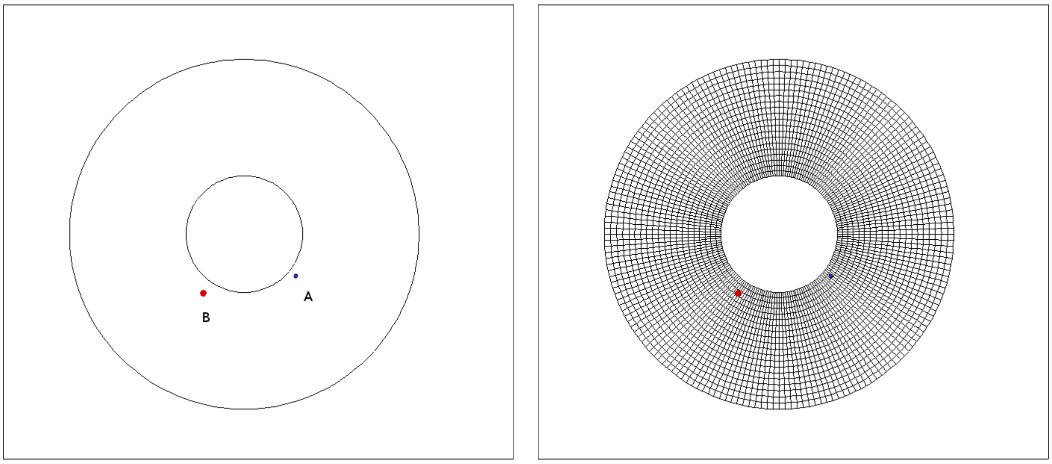
with $\omega = 2\pi$. In that case, the spatial and temporal modes are provided by:

$$\begin{array}{lcl} X_1^u & = & sin(\omega x)cos(\omega y) \\ X_1^v & = & -cos(\omega x)sin(\omega y) \\ X_1^p & = & \frac{1}{4}(cos(2\omega x)+cos(2\omega y)) \\ T_1^u & = & e^{-2\omega^2\nu t} \\ T_1^p & = & e^{-4\omega^2\nu t} \end{array} \tag{50}$$

As previously, the PGD decomposition converges in one mode analytically and numerically. This is the first example where the temporal mode for the pressure is the square of the temporal velocity mode, justifying the choice of independent pressure and velocity temporal modes.

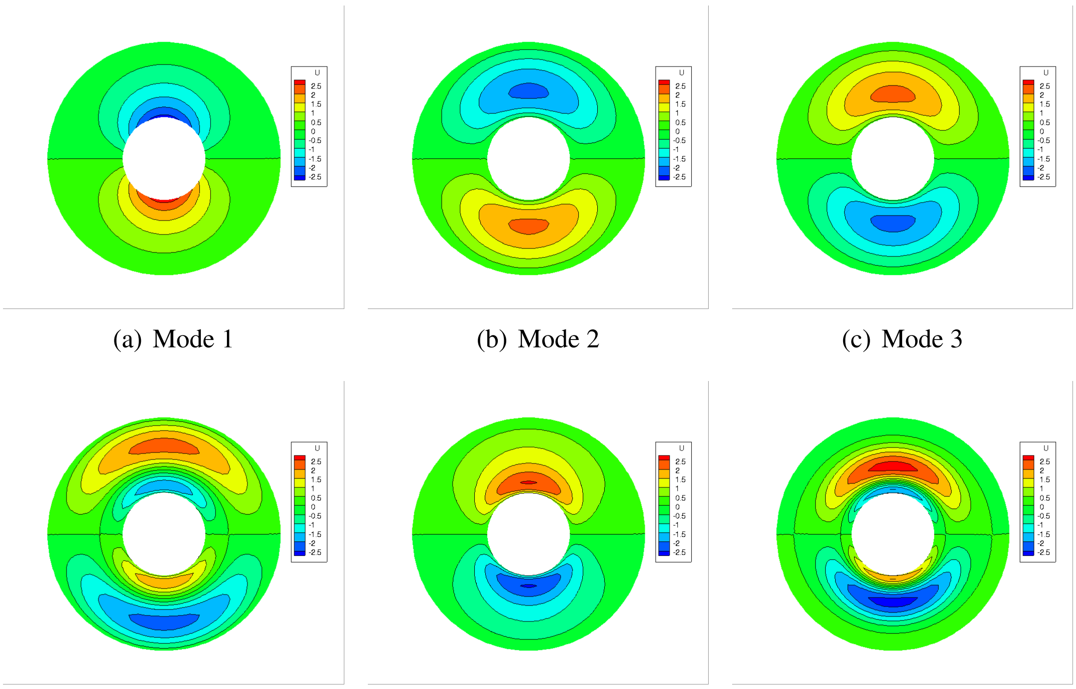
9.2 UNSTEADY FLOW BETWEEN TWO CONCENTRIC CYLINDERS

The overall procedure is validated on an academic flow configuration, the computation of the unsteady viscous flow between two concentric cylinders (see figure1(a)). The flow domain is limited by two cylinders and (R_{in}=1., R_{out}=3.0) and the inner cylinder is rotating with a rotation velocity defined by $\Omega(t) = \Omega_0 sin(2\pi t)$. The Reynolds number of this flow is $Re = 20$. A structured grid composed of 180 (resp. 21) points distributed circumferentially (resp. radially) is shown in figure 1(b) where two specific blue (A) and red (B) points are introduced. The point A is located close to the inner rotating wall while the point B is positioned further away in a region where the convective effects are less prominent.
Figures 2, 3, 4 show the six first normalized spatial modes for the two components of the velocity and the pressure.



(a) Computational domain (b) Grid

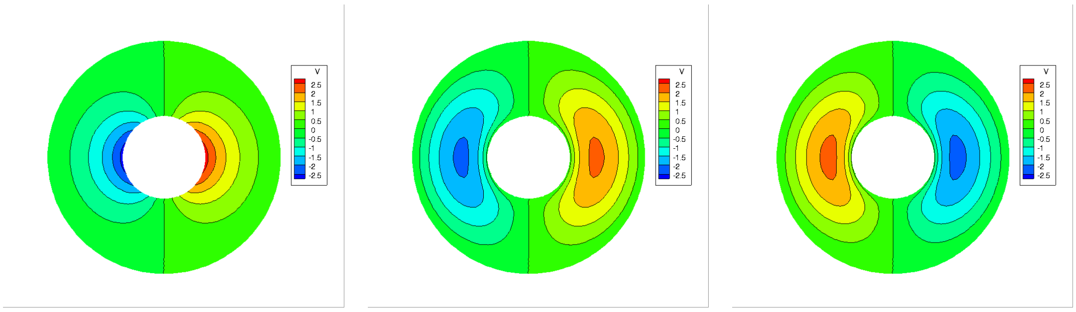
Figure 1: Unsteady viscous flow between two concentric cylinders - Computational domain



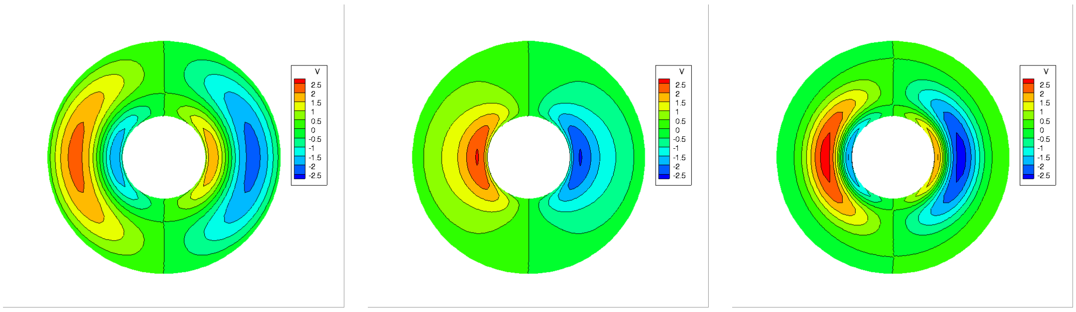
(a) Mode 1 (b) Mode 2 (c) Mode 3

(d) Mode 4 (e) Mode 5 (f) Mode 6

Figure 2: The six first normalized spatial PGD modes for U

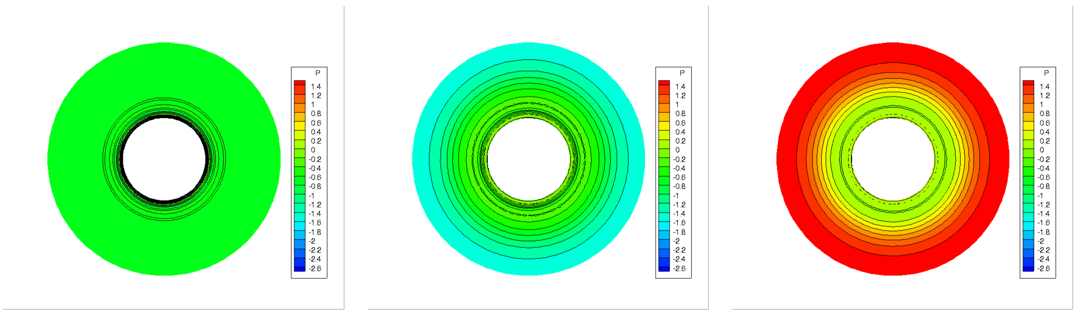


(a) Mode 1 (b) Mode 2 (c) Mode 3

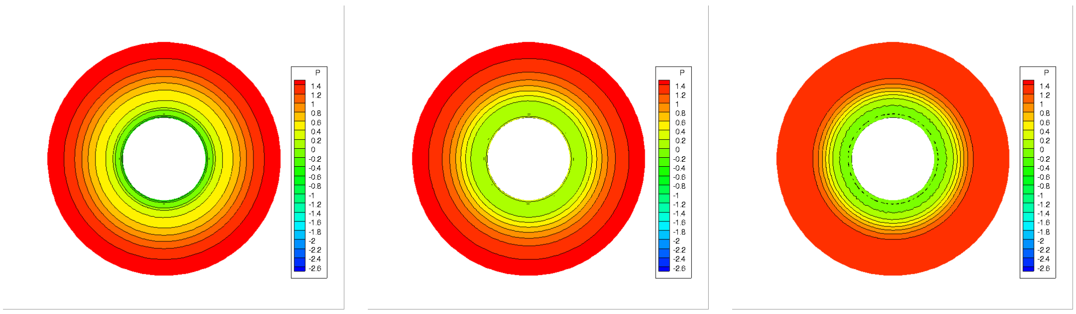


(d) Mode 4 (e) Mode 5 (f) Mode 6

Figure 3: The six first normalized spatial PGD modes for V

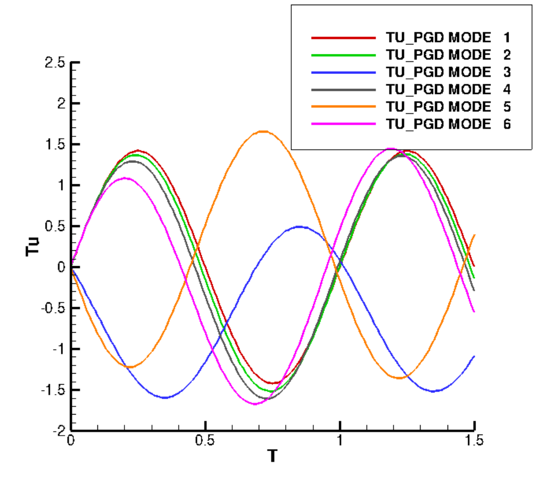


(a) Mode 1 (b) Mode 2 (c) Mode 3

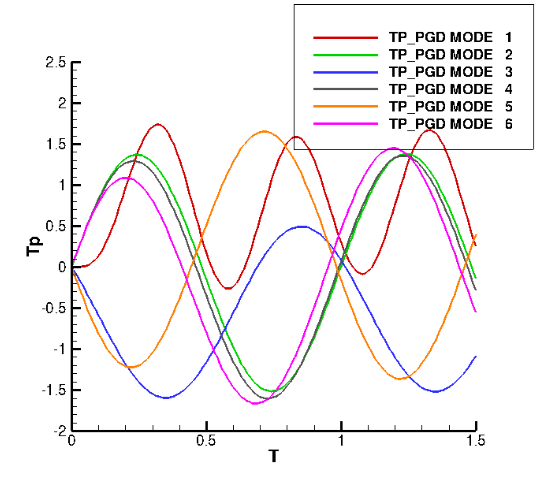


(d) Mode 4 (e) Mode 5 (f) Mode 6

Figure 4: The six first normalized spatial PGD modes for P



(a) Temporal velocity modes



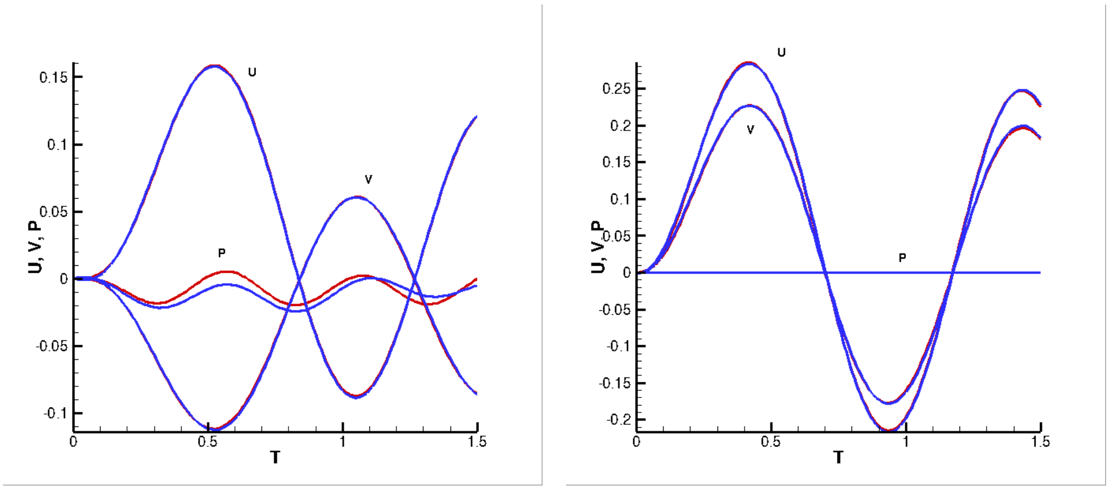
(b) Temporal pressure modes

Figure 5: The six first normalized temporal PGD modes for the velocity and pressure

The six first normalized temporal modes for the velocity and pressure are also shown in the figures 5.

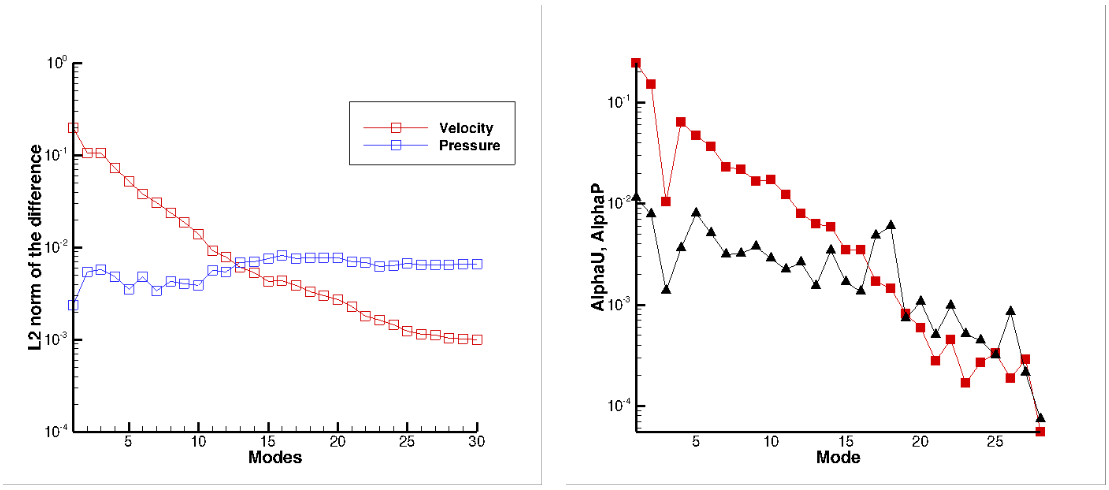
The figures 6 show a comparison of the temporal evolution of the velocity components and pressure at the two points mentioned before. 30 modes are used to build this PGD decomposition. We can notice a very good agreement on the velocity components and a less perfect agreement on the pressure, especially at points (here point A) which are located in a region where the convective effects are significant (and therefore $T_i^u \neq T_i^p$). It should be underlined here that the discretization errors committed in the discrete evaluation of the temporal PGD coefficients preclude any perfect agreement between the incremental and the modal approaches for a given time and space discretization. To draw conclusions on the solution convergence, it would be necessary to perform a systematic grid and time convergence study, which has not yet been carried out in this preliminary study.

The figure 7(a) shows the L2 norm in space and time of the difference between the solutions computed with the incremental and modal approaches. It is clear that the difference on the velocity fields is reduced as long as the modal content is enriched before reaching a minimum value of 10^{-3} which is representative of the different discretization errors characterizing each methodology. However, such a convergence is not observed on the pressure field which is still affected by a systematic (but small) defect. It is surprising to observe that the convergence on the velocity is not affected by the lack of convergence on the pressure.
The figure 7(b) shows the evolution of the norms of the velocity (in red) and pressure (in black) PGD modes. The modal convergence is smoother on the velocity than on the pressure. We can notice that it is difficult to reach more than five orders of magnitude for the Navier-Stokes equations, although a zero-machine convergence is reached without any problem for the Stokes equations (not shown here for the sake of conciseness). It is however interesting to assess the quality of the PGD decomposition with a reduced number of modes. Figure 8 show the temporal signal at the same points A and B for only 15 modes. One can notice that the agreement on the velocity and pressure signals is already quite satisfactory.



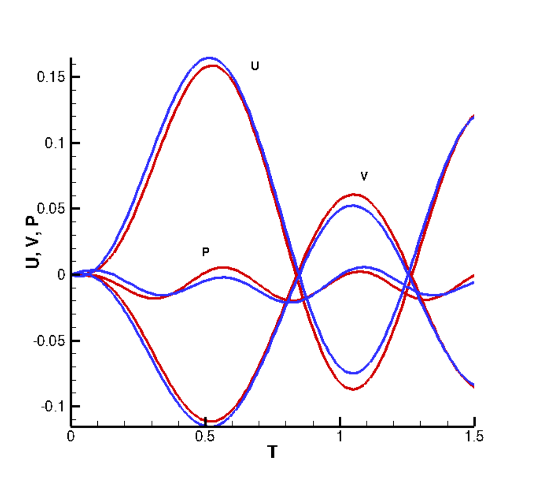
(a) Point A (b) Point B

Figure 6: Comparison between the incremental (red) and modal (blue) temporal signals for the two components of the velocity and the pressure (30 modes)

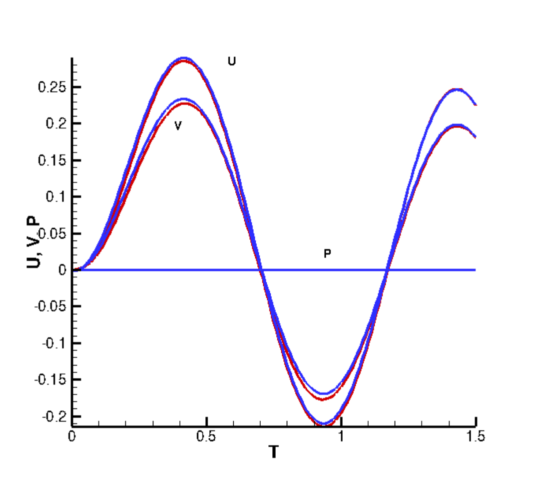


(a) L2 Norm of the difference (b) Modal convergence

Figure 7: Modal convergence of the velocity and pressure



(a) Point A



(b) Point B

Figure 8: Comparison between the incremental (red) and modal (blue) temporal signals for the two components of the velocity and the pressure (15 modes)

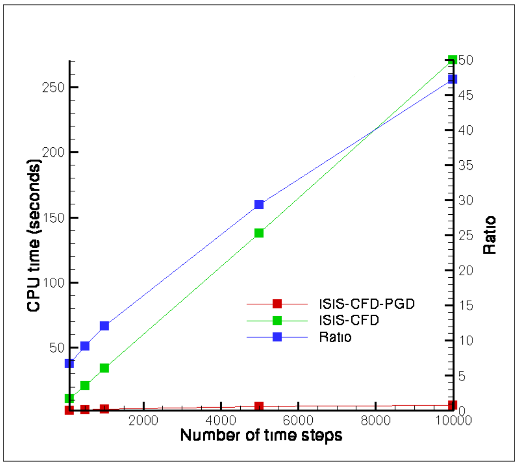


Figure 9: Evolution of the computational time to solve the unsteady Navier-Sokes equations - Comparison between incremental (ISIS-CFD) and modal (ISIS-CFD-PGD) approaches

Finally, the figure 9 shows a comparison of the evolution of computational time with the number of time steps, both for a classical incremental approach (ISIS-CFD) and for the present modal decomposition (ISIS-CFD-PGD). This comparison is made for a very modest grid and already illustrates the impressive speed-up which can be obtained with the modal paradigm.

10 CONCLUSIONS AND PERSPECTIVES

This article has briefly described a first formulation of a time space separated PGD decomposition applied to the unsteady Navier-Stokes equations. While the implementation of the PGD paradigm in the momentum equations is relatively straightforward for the velocity spatial and temporal modes, the determination of the temporal pressure modal decomposition poses a severe problem. This difficulty has to be related with the fact that the conservation of mass is a steady constraint linking velocity gradients. It is therefore necessary to transform this solenoidality condition into a pressure equation. Moreover, contrary to what is done in traditional finite volume segregated computational approaches, the pressure equation which is used here is not a semi-discretized Poisson equation but a Laplace pressure equation obtained by taking the divergence of the momentum equations. The first results are encouraging although the robustness of the overall procedure should be improved. The lack of machine-zero modal convergence for such a simple flow configuration might be an issue for higher Reynolds numbers which should be understood and solved. The test case used to illustrate this first formulation is very academic but the formulation is written in such a way that it can be applied to arbitrary geometries on fully unstructured grids. It is worthwhile to notice that the PGD decomposition has been implemented into an industrial fully unstructured finite volume solver **ISIS-CFD** world-wide distributed by NUMECA Int. under the name FINE^{TM}/Marine. This means that it is feasible to introduce a PGD formulation into industrial softwares without dramatic modifications of the code as long as a time-space separation is envisaged.

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