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# THE FEATURES OF ANALYTICAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS OF THE ELASTICITY THEORY FOR FINITE DOMAINS WITH ANGULAR POINTS OF A BOUNDARY

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**Abstract.** The paper is dedicated to the classical problem of elasticity theory – to the solution of the biharmonic equation in a semi-strip (rectangle) and to some conclusions that follow from the analysis of exact solutions of the biharmonic problem. The formulas describing the distribution of stresses and displacements in the semi-strip (rectangle) with free longitudinal sides and stresses set on the end face, as well as the mathematical apparatus used in this case can be found in the works of authors such as [1, 2]. The solutions are represented as the expansions in the Fadle–Papkovich functions. The coefficients of the expansions are determined as the simple Fourier integrals from the known boundary functions.

The exact solutions in the rectangle have some unusual properties that do not exist in any of the known exact solutions of elasticity theory. For example, they are not unique and, therefore, they may describe the residual stresses. Thus, solving the biharmonic problem for the rectangle, thus we constructed the theory of residual stresses – one of the fundamental problems of elasticity theory. The obtained solutions allow us to understand what it means to set boundary conditions on the rectilinear boundary of a domain; because she is already not rectilinear after application of the load. What is the angular point of an infinite wedge and how it differs from the angular point of the rectangle? Has got the angular point in the rectangle the singularity always, if the type of boundary conditions is changed at the angular point? And so on.

Based on the developed mathematical apparatus, it is possible to obtain the exact solutions of various mixed boundary value problems in the rectangle. For example, when there is a crack in the rectangle, when the part of the boundary of the rectangle is rigidly fixed, and the part is free, etc. The developed methods can be transferred to other coordinate systems: to polar coordinate system (cambered beam), to oblique (triangle, trapezium). They can be generalized to three-dimensional problems (rectangular parallelepiped).

Interest in solutions of boundary-value problems of elasticity theory in canonical domains with corner points of a boundary, in particular, in a rectangle, did not cease ever, having reached a peak years in 1940–1980, primarily thanks to the Soviet school of mathematics and mechanics. Several thousands of papers were published throughout these years. The last review (2003) V. V. Meleshko contains more than 700 references to the most important research in almost 200 years [3]. There are several schools that have been established in these years in the Soviet Union and were represented by the greatest mathematicians and mechanics of those years: the Leningrad school (P. F. Papkovich, A. I. Lurie, G. A. Grinberg, G. I. Dzhanelidze, V. K. Prokopov, B. M. Nuller and others), the Moscow school (M. I. Gusein-Zade, S. A. Lurie, V. V. Vasiliev, V. I. Maly, E. M. Zveryaev and many others), the Rostov-on-Don school under the leadership of I. I. Vorovich, very strong and numerous the Ukrainian school (V. T. Grinchenko, A. F. Ulitko, A. M. Gomilko, V. V. Meleshko). Strong and bright works have been published in the Proceedings of the Armenian, Azerbaijani, Georgian Academy of Sciences. After the 1980s there were no notable publications. Western papers were sketchy and much weaker than the Soviet.

The heart of the problem is simple. Let us explain it on the example of the first fundamental boundary-value problem for the semi-strip in the case of symmetric deformation. We can find the solution of the biharmonic equation in the semi-strip $\{x \ge 0, |y| \le 1\}$, that has free longitudinal sides, i.e.

$$\sigma_y(x,\pm 1) = \tau_{xy}(x,\pm 1) = 0, \tag{1}$$

and normal $\sigma_x(0,y) = \sigma(y)$ and tangential $\tau_{xy}(0,y) = \tau(y)$ stresses are set on the end face

$$\sigma_x(0,y) = \sigma(y),\ \tau_{xy}(0,y) = \tau(y). \tag{2}$$

Without loss of generality we can assume that the stresses $\sigma(y)$ are self-balanced. Solving the problem by the method of separation of variables, we come to the problem of determining the coefficients $a_k$ of expansions of the two functions $\sigma(y), \tau(y)$ set on the semi-strip's end face in a series in the two systems of boundary value problem's eigenfunctions – the so-called Fadle–Papkovich functions:

$$\sigma(y) = \sum_{k=1}^{\infty} a_k s_x(\lambda_k, y) + \overline{a_k} s_x(\overline{\lambda_k}, y),\ \tau(y) = \sum_{k=1}^{\infty} a_k t_{xy}(\lambda_k, y) + \overline{a_k} t_{xy}(\overline{\lambda_k}, y). \tag{3}$$

The Fadle–Papkovich functions (in the case of symmetric deformation of the semi-strip) have the form:

$$\begin{aligned} s_x(\lambda_k, y) &= (1+\mu)\lambda_k \{(\sin\lambda_k - \lambda_k\cos\lambda_k)\cos\lambda_k y - \lambda_k y \sin\lambda_k \sin\lambda_k y\}, \\ t_{xy}(\lambda_k, y) &= (1+\mu)\lambda_k^2 \{\cos\lambda_k \sin\lambda_k y - y\sin\lambda_k\cos\lambda_k y\}. \end{aligned} \tag{4}$$

$\mu$ is a Poisson's ratio. The numbers $\lambda_k$ are the set $\{\pm\lambda_k, \pm\overline{\lambda_k}\}_{k=1}^{\infty} = \Lambda$ of all complex zeros of the exponential type's entire function:

$$L(\lambda) = \lambda + \sin\lambda\cos\lambda. \tag{5}$$

If the expansion coefficients $a_k$ are found, the final solution of the boundary value problem in the semi-strip will have the form (below only expressions for stresses are shown)

$$
\begin{aligned}
\sigma_x(x,y) &= \sum_{k=1}^{\infty} a_k s_x(\lambda_k, y) e^{\lambda_k x} + \overline{a_k} s_x(\overline{\lambda_k}, y) e^{\overline{\lambda_k} x}, \\
\sigma_y(x,y) &= \sum_{k=1}^{\infty} a_k s_y(\lambda_k, y) e^{\lambda_k x} + \overline{a_k} s_y(\overline{\lambda_k}, y) e^{\overline{\lambda_k} x}, \\
\tau_{xy}(x,y) &= \sum_{k=1}^{\infty} a_k t_{xy}(\lambda_k, y) e^{\lambda_k x} + \overline{a_k} t_{xy}(\overline{\lambda_k}, y) e^{\overline{\lambda_k} x} \quad (\mathrm{Re}\,\lambda_k < 0),
\end{aligned} \tag{6}
$$

where

$$
s_y(\lambda_k, y) = (1+\mu)\lambda_k \{(\sin\lambda_k + \lambda_k \cos\lambda_k)\cos\lambda_k y + \lambda_k y \sin\lambda_k \sin\lambda_k y\}.
$$

The boundary conditions on the longitudinal sides of the semi-strip (1) will be satisfied exactly, because the Fadle–Papkovich functions are such, that

$$
t_{xy}(\lambda_k, \pm 1) = s_y(\lambda_k, \pm 1) = 0. \tag{7}
$$

If the periodicity conditions were set on the longitudinal sides of the semi-strip, eigenfunctions would be the usual trigonometric set of functions, and then we would obtain instead of expansions (3), for example,

$$
\begin{aligned}
\sigma(y) &= -\sum_{k=1}^{\infty} \left( A_k (1+\mu) k\pi + 2B_k \right) \cos k\pi y, \\
\tau(y) &= \sum_{k=1}^{\infty} \left( A_k (1+\mu) k\pi + B_k (3+\mu) \right) \sin k\pi y.
\end{aligned} \tag{8}
$$

The trigonometric sets of functions included in (8) are orthogonal and form a basis on the semi-strip's end face. Therefore, there aren't problems with determining of the coefficients $A_k, B_k$ of expansions. Thus the well-known Filon–Ribiere solutions are obtained.

The Fadle–Papkovich functions are much more difficult than trigonometric series: they are complex-valued and non-orthogonal. But their main feature is that they do not form basis on the segment on which the expanded functions (end faces of the rectangle or the semi-strip) are set. Therefore, it is impossible to obtain explicit formulas for the coefficients of expansions in terms of the Fadle–Papkovich functions on the basis of the classical methods of the theory of the basis of functions. Numerous methods have been proposed for the determination of the coefficients of expansions in terms non-orthogonal sets of the Fadle–Papkovich functions. However, almost all of them, anyway, were reduced to an approximate determination of the unknown coefficients from the infinite, not reducible system of algebraic equations.

The basis of this method is a generalization of the classical notion of the basis of functions on the segment. If the classical basis on the segment can be considered as a basis in the complex plane, the Fadle–Papkovich functions form a basis on the Riemann surface of the logarithm. Therefore, it is possible to construct biorthogonal sets of functions, which are determined on the Riemann surface of the logarithm and, then, to find the coefficients of expansions. The Fadle–Papkovich functions are generalization of systems of exponential functions with complex exponents. A. F. Leontiev, and then Yu. F. Korobeinik were studied the basis properties of exponential functions' systems with complex exponents and were investigated that they do not form a basis in the classical sense, and the expansions on them are not unique. Korobeinik has named such function systems as the representing function systems. The base for study of basis properties of exponential functions' systems is the theory of entire functions of exponential type and Borel transform in this class of functions. The base for study of basis properties of the Fadle–Papkovich functions is the theory of quasi-entire functions of exponential type and Borel transform in this class of functions.

The quasi-entire functions of exponential type were first introduced by A. Pfluger in [4] in 1936 and it is the only work of quasi-entire functions of exponential type. Based on the Pfluger's results, it is possible to construct systems of functions which biorthogonal to the Fadle–Papkovich functions, and then exactly determine the expansion coefficients, thereby to construct an exact solution of the problem. Such a solution was first published in [5].

The solutions that are obtained using of functions' biorthogonal sets are called exact because they are constructed on the same scheme as the Filon–Ribiere solutions, and the desired coefficients are explicitly determined (and not as the solution of the infinite system of algebraic equations). Moreover, if the complex eigenvalues in obtaining exact solutions of the boundary-value problem are aim to the numbers $k\pi$ $(k = 1, 2, ...)$, the Fadle–Papkovich functions turn into the usual trigonometric functions, and the solutions – into the Filon–Ribiere solutions.

Since the biorthogonal sets of functions are determined on a certain infinite curve lying on the Riemann surface of the logarithm, it is necessary to continue the expanded functions initially defined on the segment (end face of the semi-strip) outside the segment on this curve. Because such continuations are not unique, the solutions of the boundary-value problems in the semi-strip are not unique as well. In 1996 academician E.I. Shemyakin first drew attention to the non-uniqueness of boundary value problems' solutions of elasticity theory in finite domains with boundary's angular points and changing type's points of the boundary conditions. As the solutions are not unique, the nonzero solutions do exist despite the fact that the load on the sides of the semi-strip is equal to zero. Such solutions are called eigensolutions, and the corresponding stresses – residual or initial. Constructing exact solutions in the semi-strip or in a rectangle, we do obtained the theory describing the residual stresses (one of the most important problems of elasticity theory). The residual stresses satisfy the equilibrium equations and the boundary conditions, but do not satisfy the strain compatibility conditions and so the displacements are not uniquely determined. To understand the physics of this known statement, let us cut the rectangle $\{P : |x| \le a, |y| \le 1\}$ along the axis $y$. Replace the resulting right and left rectangles by almost rectangles that if a certain load is applied to them, they take their initial shape of rectangles. Let us apply the required load and then glue together the rectangles. As a result we will obtain the rectangle $P$, where there is a nontrivial field of residual stresses such that the relations $\sigma_x(0, y) = \sigma(y)$ and $\tau_{xy}(0, y) = \tau(y)$ are satisfied on the glue line. The fact that the displacements are not determined uniquely means the following: there are infinite numbers of profiles sides of almost rectangles, connected into the original rectangle, after gluing such profiles the normal and tangential stresses are the same on the glue line.

The eigensolutions have the property that the boundary conditions are strictly satisfied for them on rectilinear boundaries of the rectangle. Hereby they are fundamentally different from the solutions in the classical statement, in which "boundary conditions are conventionally moved to the nondeformed surface", i.e., either the missing material is added, or the extra material is removed. In 1940 D.I. Sherman proposed a more strict understanding of boundary conditions. Let us consider the content of understanding in the following example. Suppose that the semi-strip $\{|y| \le 1, x \ge 0\}$ is given, the longitudinal sides of which are free, and normal stresses $\sigma_x(0, y) = \sigma(y)$ are set on the end face (for simplicity, we assume that the tangential stresses are equal to zero, and the function $\sigma(y)$ is even). Following Sherman understanding, we can consider an infinite plane $\{|x, y| < \infty\}$, in which let us draw two horizontal cuts $\{y=\pm 1, |x|<\infty\}$ and the vertical cut $\{x=0, |y|\le 1\}$. Let us make insertions along a cut of the same material as that of the plane material, which continuously glue with the sides of the cuts. Then the shapes of insertions can be chosen so that the normal and tangential stresses are equal to zero on axes of the horizontal insertion, and we have such relations on an axis of the vertical insertion: $\sigma_x(0,y)=\sigma(y),\ \tau_{xy}(0,y)=0$. At first sight this understanding of boundary conditions on the sides of the semi-strip seems more stricter, but actually nothing has changed since and in this case we have to "add" or "remove" material. As a result, the boundaries of the semi-strip are rectilinear before the deformation and remain rectilinear after the deformation. Due to this condition the strain compatibility conditions are satisfied, both in the classic sense of the boundary conditions, and acc. Sherman.

In solving the boundary value problems the need to continue the boundary functions set on the end faces of the rectangle $P$ outside the segment, in which they were initially determined, is physically quite understandable. In fact, within the framework of classical concepts of mechanics of solids, not only values of boundary functions, but all their derivatives, both along the axis $x$, and along the axis $y$ must be defined at each point of the rectilinear sides of the rectangle, including in its angular points (the same belongs to changing type's points of the boundary conditions). Therefore, for a correct formulation of the boundary-value problem in the rectangle it is necessary to specify: 1) how the boundary functions set on the end faces of the rectangle continues to the whole infinite straight line $|y|<\infty$ and 2) how the solution continues through the end faces of the rectangle. The necessity for the continuation of the boundary functions on the axis $y$ is dictated by the fact that sets of functions biorthogonal to the Fadle–Papkovich functions are not finite. Let us explain the second point. Consider the same semi-strip $\Pi^+$ with free longitudinal sides, when on the end face of the semi-strip

$$\sigma_x(0,y)=\sigma(y),\ \tau_{xy}(0,y)=0,\ (|y|<1). \tag{9}$$

If the solution from the right semi-strip continues to the same left semi-strip, i.e. in the semi-strip with free longitudinal sides, the stresses are finite at the angular points of the semi-strip. And if the solution continues to the left semi-strip, the longitudinal sides of which are rigidly fixed, but the conditions (9) are still satisfied, the stresses may have integrable singularities at the angular points $x=0,\ |y|=\pm 1$. The conditions, under which the singularity exists or not exist, are strictly established by the solution of the corresponding mixed boundary-value problem in a rectangle.

In conclusion, we give formulas for stresses in the semi-strip, when only the normal self-balanced stresses $\sigma_x(0,y)=\sigma(y)$ are set on its end face, and the tangential stresses are equal to zero

$$\begin{aligned}
\sigma_x(x,y) &= \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{\sigma_k \frac{s_x(\lambda_k,y)}{M_k}\frac{\mathrm{Im}(-\overline{\lambda_k} e^{\lambda_k x})}{\mathrm{Im}(\lambda_k)}\right\},\\
\sigma_y(x,y) &= \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{\sigma_k \frac{s_y(\lambda_k,y)\lambda_k\overline{\lambda_k}}{M_k\lambda_k^2}\frac{\mathrm{Im}(-\lambda_k e^{\lambda_k x})}{\mathrm{Im}(\lambda_k)}\right\},\\
\tau_{xy}(x,y) &= \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{\sigma_k \frac{t_{xy}(\lambda_k,y)}{\lambda_k M_k}\lambda_k\overline{\lambda_k}\frac{\mathrm{Im}(-e^{\lambda_k x})}{\mathrm{Im}(\lambda_k)}\right\},
\end{aligned} \tag{10}$$

where

$$\sigma_k = \int_{-1}^{1} \sigma(y) \frac{\cos \lambda_k y}{2(1+\mu)\lambda_k \sin \lambda_k} dy, \; M_k = \frac{L'(\lambda_k)}{2} = \cos^2 \lambda_k .$$

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