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THE VORTICITY CREATION PROCESS AT PHYSICAL SURFACES

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Abstract. *The fluids as deformable bodies without own shape, when starting from rest, experience interactions between the flowing fluid and the physical surfaces marking the bounds of flow. These interactions are a kind of impact process where there is a momentum exchange between two colliding bodies, i.e. the flow and its boundary surfaces. Within a short time of contact a post-impact shear flow occurs where two main effects are triggered off by the flow-induced collision: dramatic redistribution of the momentum and the boundary vorticity followed by the shear stress/viscosity change in the microstructure of the fluid which at the beginning behaves as linear reactive medium and latter as nonlinear dispersive medium. The disturbance of the starting flow induces the entanglement of the wall-bounded flow in the form of point-vortices or concentrated vorticity balls whence waves are emitted and propagated through flow field. The paper develops a wave mechanism for the transport of the concentrated boundary vorticity, directly related to the fascinating turbulence phenomenon, using the torsion concept of vorticity filaments associated with the hypothesis of thixotropic/nonlinear viscous fluid.*

1 INTRODUCTION

Generally, the impact/collision is a process of momentum exchange between two colliding bodies within a short time of contact. With respect to a single impacted body or structure, the loading in such a process acts with high intensity during this short period of time. As a result, the initial velocity distribution is rapidly changed (even pressure wave loadings). Such rapid loading in the contacting area is a source where waves are emitted that propagates with finite speed through the body. In the case of sufficiently small amplitudes, the linear elastic body waves propagate with the speed of sound waves; the distinction between the fast longitudinal L – wave (carrying wave) and the slower transverse T – wave complicates the pattern as it will be seen in the sequel.

When starting from rest the fluids as deformable bodies without own shape, experience interactions between the flowing fluid and the physical surfaces enclosing the flow. These interactions are a kind of impact process where there is a momentum exchange between the flow and its boundary surfaces. Within a short time of contact a post-impact shear flow occurs where two main effects are triggered off by the flow-induced collision: dramatic redistribution of the momentum and the boundary vorticity creation followed by the shear stress/viscosity change in the microstructure of the fluid which at the beginning behaves as linear reactive medium and latter as nonlinear dispersive medium. The disturbances of the starting flow ($t \to 0$) cause the entanglement of the wall-bounded flow (stream function $\Psi = 0, t \to 0$) inducing a wall torsion pressure (suction) in the form of point-vortices or concentrated vorticity balls whence waves are emitted and propagated through the flow field [1], [2]. Such vorticity concentrations and their potentially devastating rotary motions are observed in the natural case of tornados. The above phenomenology of the impact process of a starting fluid shows that at the beginning the vorticity lines are subjected to a torsion pressure [2] being concentrated at the boundaries; after that the vorticity concentrations are dispersed in a shear layer by vorticity shear waves. Thus, the description of the complicate dynamics of the concentrated boundary vorticity, in fact a kind of variable angular velocity, requests a new fluid medium able to adjust itself continuously and to respond to the flow/stress. Such a fluid playing a role of variable mass is called the thixotropic/nonlinear viscous fluid [3]. In contrast to the Newtonian fluid model, too restrictive (linear shear stress and constant viscosity), the thixotropic model is a flexible one, able to describe even intricate turbulent motions.

The boundary vorticity dynamics, (i.e. its creation, growing and finally fading under the form of the full turbulent flow) is presented in the paper through the results concerning the plate boundary layer flow and Couette flow.

2 GENERAL FORMULATION

We consider an infinite region of fluid, which is viscous and incompressible. The Navier-Stokes equations expressed in the vorticity-stream function formulation for two-dimensional motion of this fluid is recasted in the vorticity transport/transfer equation

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right), \tag{1}$$

where $\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ is the vorticity at the fluid point (rotation of fluid), t is the time, x is the coordinate in the direction of mean flow and y is the coordinate normal to that direction; x and y velocity components are $u = \frac{\partial \Psi}{\partial y}$ and $v = -\frac{\partial \Psi}{\partial x}$ and Ψ is the stream function.

The vorticity transport equation is classified as a parabolic equation with the unknown being the vorticity ω. The definition of vorticity yields

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}=-\omega, \tag{2}$$

known as the stream function equation, classified as an elliptic partial differential equation.

The unknown is the stream function Ψ, whose ω is provided from the solution of Eq. (1). Once the stream function has been computed, the velocity component may be determined from its derivatives. By introduction of new variables, namely the vorticity and the stream function, the incompressible Navier-Stokes equations are decoupled into one elliptic equation and one parabolic equation which can be solved sequentially. The main drawback of the method consists in the poor knowledge of the initial and boundary conditions that must be obtained by physical considerations. The initial condition at fluid-boundary impact ($\Psi=0, t=\frac{x}{U_e}\to 0$) given by Stuart's solution for vorticity $\omega_w=e^2$ [1] associated with the thixotropic fluid model [2] renders Eq. 1 at the boundary enclosing the flow (Ψ = 0, t) into a non-linear ordinary differential equation for the boundary vorticity, $\omega_w\left(\Psi=0, t=\frac{x}{U_e}\right)$

$$\frac{d\omega_w}{dx}=\nu\frac{d^2\omega_w}{dx^2},\ \text{- vorticity boundary conditions}, \tag{3}$$

$$\omega_w\nu=U_e^2\ \text{- shear compliance relation}, \tag{4}$$

where the $\nu(x)$ is the variable viscosity of the thixotropic viscous fluid able to respond to flow/stress and Eq. (4) represents a kind of the conservation of flow angular momentum. The initial/starting and boundary conditions (3) for vorticity are a crucial part of the mechanism by which a laminar flow becomes transient and then turbulent one.

3 ON PHYSICS OF VORTICITY

Formally, the problem of turbulence is to solve the Navier-Stokes equations subject to initial and boundary conditions. At present, it is possible to obtain fully resolved solutions at moderate Reynolds numbers via direct numerical simulations of the Navier-Stokes equations. However, the common mathematical conjecture that the turbulent flows can be correctly described in an asymptotical manner (i.e. at large Re with $\nu\to 0$) by some sort of specially selected weak solution of Euler equations has little to do with real physics at any large Reynolds number. But, beside the difficulties of a formal/technical nature, there is another difficulty of a more general nature. It is lack of knowledge about the basic physical processes of turbulence and its generation and origin, and poor understanding of the processes which are already known.

Here, in contrast with the common view that the origin of turbulence lies in the instability of some basic laminar flows, the turbulence is the result of the boundary vorticity dynamics of the flow-induced collision and its consequences where shear waves are emitted and propagated in the flow field from the vibrating concentrated vorticity at the boundaries.

This is understood in the sense that any flow starts from rest at some moment in time, and as long as the Reynolds number, or the reduced frequency of vorticity is small, the flow remains laminar (creeping motion of vorticity). As the Reynolds number/frequency of vorticity increases a wide instability range sets in, which is followed by transition and then a fully developed turbulent state. This is the visible face or large scale of the flow field, i.e. the velocity fluctuations, whereas the poorly known origin of turbulence is the invisible face or small scale of the flow field, i.e. the vorticity fluctuations.

Therefore, the origin of turbulence is a problem of boundary vorticity dynamics that needs a holistic approach of the motion process containing the linked up events: the flow surface colliding and starting vorticity creation, the non-dispersive creeping motion of vorticity (laminar flow) and the dispersive vibrating motion of vorticity (turbulent flow).

Each perturbation, small or large, is in fact a kind of impact between the flow and its boundaries exchanging the momentum between bodies within a short time of contact. As a result, the initial velocity distribution is rapidly skewed/squeezed, the vorticity is created and organizes itself into more and more concentrated structures, thus at the boundary there is a set of point-vortices. The exact solutions of the equations of inviscid motion found by Stuart [1] (e^{τ} - solutions, $\tau \in [0,2]$) can describe strong vorticity concentrations developing in skewed shear layers. The concentration level of vorticity is estimated on a natural logarithmic scale e^{τ} from e^0 - sparse/weak vorticity, up to e^2 – concentrated vorticity, where the index τ is a measure of the concentration of vorticity. In contrast to the sparse vorticity transported by the ideal fluid flow according to the laws given by Helmholtz [4], the concentrated vorticities are transported by waves, their shape depending mostly on the concentration of vorticity, Fig. 1.

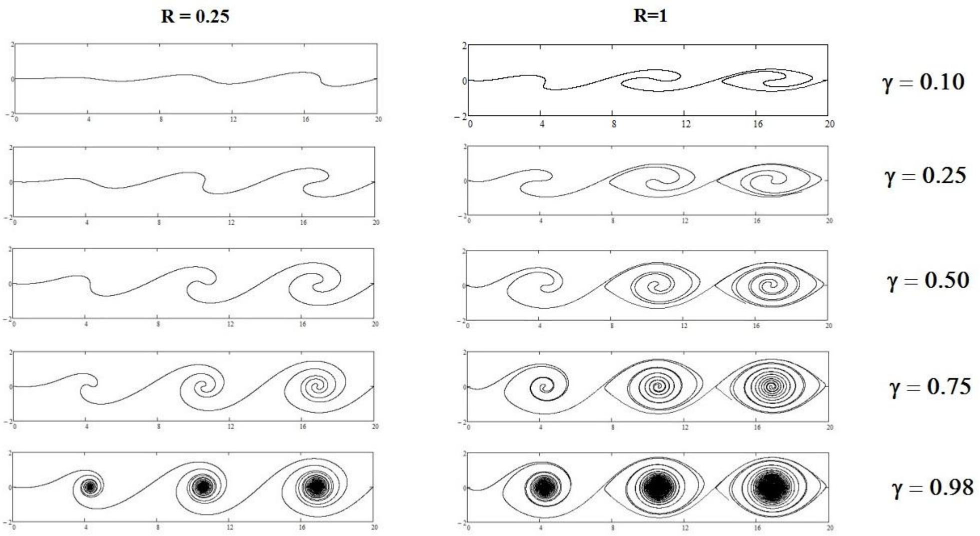

Figure 1: Effect of γ (concentration), R (circulation) parameters on streakline patterns [5].

Physically, the concentration of vorticity at boundaries is a local compression of flow inducing by the torsion of vorticity wires. The concept of torsion of the concentrated vorticity allows a better understanding of the boundary vorticity creation and its dynamics, which becomes an active one just after impact. Essentially, the boundary vorticity dynamics contains small amplitude vibrational motions generating vorticity weak waves that create the covered/hidden field of flow. The self-sustained vorticity wave onset is the mechanical origin of turbulence.

Concomitantly with the vorticity creation the impact process induces microstructure changes of the flow properties resulting in a time dependent shear stress, $\nu = \nu(t)$, known as the thixotropic behavior of the flowing fluid. It is experimentally shown that the transient viscosities follow the line of the complex viscosity versus angular frequency [5]. This behavior can be described by a Klein-Gordon like wave equation [6]

$$\frac{d^2\nu(x)}{dx^2} = \frac{1}{U_e^2}\left(\omega_0^2 - \omega^2\right)\nu(x), \tag{5}$$

where ν_0 is steady shear viscosity and $\omega_0 = \nu_0^{-1}$ is the natural angular frequency of fluid, defined as the first zero of the Fourier coefficient B(ω) of a square pulse/impact. Thus, the solutions of Eq. 5, describe well enough the dual behavior of the thixotropic fluid, as a reactive medium, $\omega < \omega_0$, at impact inducing exponential waves (without energy dissipation), and as a dispersive medium which can support sinusoidal waves for ω above the natural frequency ν_0^{-1}. Equation (5) also shows that the microstructure takes time to respond to the flow/stress and its elastic response at low frequency is faster as the flow velocity U_e increases.

The disturbed post-impact flow is a boundary-layer type flow which is relaxed through a complicated wave system, which transports concentrated vorticity from boundaries to the flow field and rebuilds the flow microstructure. There is a non-dispersive transport of vorticity performed by exponential waves in the form of the laminar flows dominated by the frictional shear stress and a dispersive one which involves lightly damped sinusoidal waves by dry friction in turbulent flows. Hence, it is evident that the analysis of the impact-relaxation process requests another constitutive relation to describe the intricate behavior of viscous fluid. For the thixotropic fluid, such a relationship is a shear compliance defined as

$$\frac{1}{\rho} p_{torsion,w} \equiv \omega_w \nu = U_e^2 \text{ on } \partial \mathrm{B}, \tag{6}$$

where $p_{torsion,w}$ is the torsion pressure at wall, $\omega_w = e^\tau$ is the vorticity at a two-dimensional wall ($\partial\mathrm{B}$), τ is a torsion/concentration index $\tau \in [0,2]$, and $\nu(t)$ denotes the change of viscosity during the post-impact flow which is able to adjust itself continuously.

A non-steady fluid system involves an oscillating behavior of its opposite, intrinsic properties (vorticity and viscosity) and suddenly excited it decays as a big damped harmonic oscillator. The evolution is slow and to visualize its full way a plotting on suited scales is necessary. Using the exponential scales and measure units e and ν_0^{-1}, the shear compliance, Eq. 6, can be written as

$$\begin{aligned} &e^{\tau}\left(\nu_0^{-1}\right)^{1/\tau} = \mathrm{Re}_x \text{ for } \mathrm{Re}_x < \mathrm{Re}_c \text{ (laminar flow)} \\ &e^{1/\tau}\left(\nu_0^{-1}\right)^{\tau} = \mathrm{Re}_x \text{ for } \mathrm{Re}_x \geq \mathrm{Re}_c \text{ (turbulent flow)} \end{aligned}, \tag{7}$$

where the critic Reynolds number $\mathrm{Re}_c = e^2\nu_0^{-1}$ is the non-rolling condition for concentrated vorticity, which separates the non-periodic creeping motion of vorticity inducing laminar flow, from the torsional vibration motion of vorticity generating turbulent flow, and ν_0^{-1} is the natural frequency of the thixotropic fluid. Equations (7) express the conservation of boundary vorticity in laminar flow, its dispersion in turbulent flow, respectively.

For the above $\mathrm{Re}_x = \nu_0^{-1}$ the transient flow in the neighborhood of the wall vibrates as a continuous and homogeneous string carrying transverse vorticity waves which permanently disperse vorticity. Figure 2 illustrates the wave system induced by the flow-boundary impact and the dispersion mechanism of concentrated vorticity that displays a wide frequency/Reynolds number spectrum from the low indifference Reynolds number $\mathrm{Re}_{lind} = e^2\nu_0^{-1/2}$ - the onset of the weakest waves (TSW-Tolmien-Schlichting waves) up to the high indifference

Reynolds number $\mathrm{Re}_{hind} = e^{1/2}\nu_0^{-2}$. For the last Re the wave system becomes a slightly damped one with the resonance close to the natural frequency ν_0^{-1}, emitting acoustic waves. For above $\mathrm{Re}_x = e^2\nu_0^{-1}$ the transitional flow displays a strong beat phenomenon with varicose aspect of vorticity where its frequency $\mathrm{Re}_x^{1/2}$, is far from the resonance frequency. For above $\mathrm{Re}_x = e^{2/3}\nu_0^{-3/2}$, the vorticity is broken down in contra-rotating fragments/flocks and its frequency approaches the resonance frequency where the flow is full turbulent and it can be statistically described. The Reynolds number controls the wave system playing a role of tuning button that switches the frequency band from concentrated vorticity and small Re/low frequency and long wavelength to dispersed vorticity and high Re/high frequency and short wavelength.

The essential difference between laminar flow and turbulent flow is given by the difference between the behaviors of fluid as linear viscoelastic-reactive medium and nonlinear thixotropic-dispersive medium. That is, while both are time effects, the former is in the linear region, where the microstructure responds but remains unchanged and the latter takes place in the nonlinear region where the microstructure is broken down by deformation as well as responding to it.

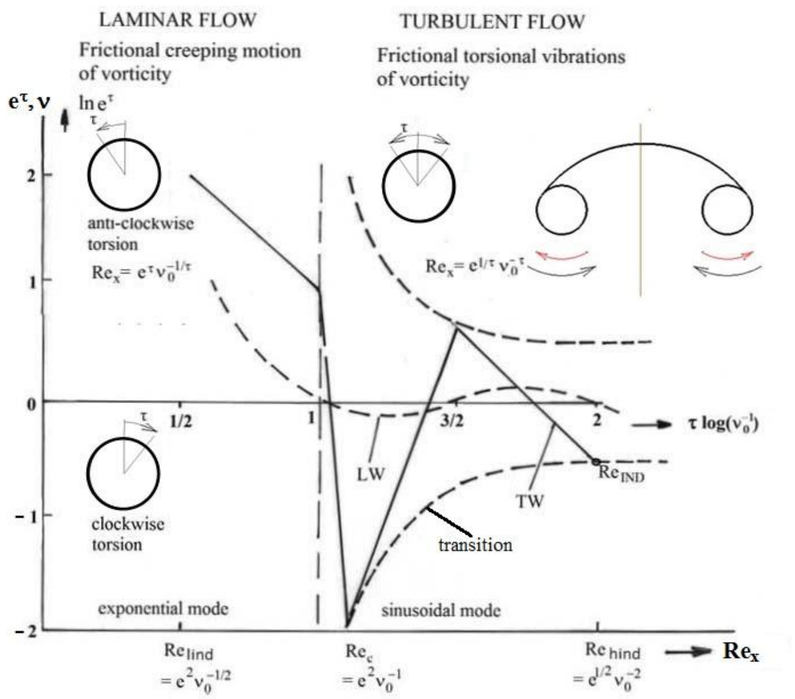

Figure 2: Transverse waves (TW) modulated in amplitude on a longitudinal carrying wave (LW) and dispersion mechanism of concentrated vorticity by shear waves [8].

4 RESULTS FOR ZERO-PRESSURE GRADIENT FLOW

The zero-pressure gradient flows are simple shear flows (Prandtl boundary-layer flow and Couette flow) where the shear stress has a prevailing role everywhere in the flow field. The reason to treat these flows is because of fundamental importance for turbulent flows close to walls in general, far beyond these particular cases. It will be seen that the flow regions of the chosen turbulent flows close to wall have universal importance, so that, up to some yet to be specified conditions, the results can be carried over to the regions close to the wall of general turbulent flows. That is, the Prandtl and Couette flows are the extreme examples concerning the visual perception of turbulence: one sidewall-bounded flows (boundary-layers flows) with visible velocity fluctuations, and two sidewall-bounded flows (shear flows) involving invisible velocity fluctuations. As a matter of fact, the Couette flow is also called no-fluctuation shear flow.

In this context, the motion equations contain an irrotational potential $\nabla\varphi = U_\infty x$ (outside a shear layer) and a vector potential $\mathbf{A} = \psi(x, y)\mathbf{k}$ (inside and normal to shear layer) where ψ is the stream function defined by

$$u = \frac{\partial\psi}{\partial y}, v = -\frac{\partial\psi}{\partial x}, \tag{8}$$

Using the approximations and notations of the boundary-layer flow [7]

$$\psi = \sqrt{2\nu x U_\infty} f(\eta), \delta(x) \approx \left(\frac{x\nu}{U_\infty}\right)^{1/2}, \eta = \frac{y}{\delta(x)}, \tag{9}$$

where $f(\eta)$ is the dimensionless stream function, $\delta(x)$ is a scaled measure of the boundary-layer thickness (up to the approximation $u = 0.99U_\infty$) and $\frac{u}{U_\infty} = f'(\eta)$ is the similarity law of the velocity profile, the boundary layer equations and their boundary equations become one ODE for the stream equation

$$\text{Prandtl flow}\begin{cases} k_w f''' + ff'' = 0 \\ \eta = 0 : f = 0, f' = 0 \\ \eta \to \infty : f' = 1 \end{cases} \qquad \text{Couette flow}\begin{cases} k_w f''' + ff'' = 0 \\ \eta \to +\infty : f' = 1 \\ \eta = 0 : f = 0 \\ \eta \to -\infty : f' = 0 \end{cases}. \tag{10}$$

This nonlinear third order equation and the three boundary conditions completely determine its solution, if the mute constant k_w can be known a priori. Blasius found a solution of Eq. 10 for $k_w = 2$, known as Blasius equation, in the form of a series expansion for $\eta \to 0$ and an asymptotic expansion for $\eta \to \infty$, the two forms being matched at a suitable value of η. But, Eq. 10 describes a more general phenomenon, that of the transverse standing vorticity/shear waves, called solitons, which retain their identity upon a collision.

The vorticity soliton identified here as Blasius soliton, depends on the function k_w directly related to Re_x via the shear compliance, Eq. 7 , which is responsible for coupling between the autonomous fast motion of vorticity and the velocity field of the non-autonomous slower flow,

$$\begin{aligned} &k_w = e^\tau \text{ for } \mathrm{Re}_x < \mathrm{Re}_c \text{ (weak coupling)} \\ &k_w = (\log \mathrm{Re}_x)^{-1} \text{ for } \mathrm{Re}_x \geq \mathrm{Re}_c \text{ (strong coupling)} \end{aligned}. \tag{11}$$

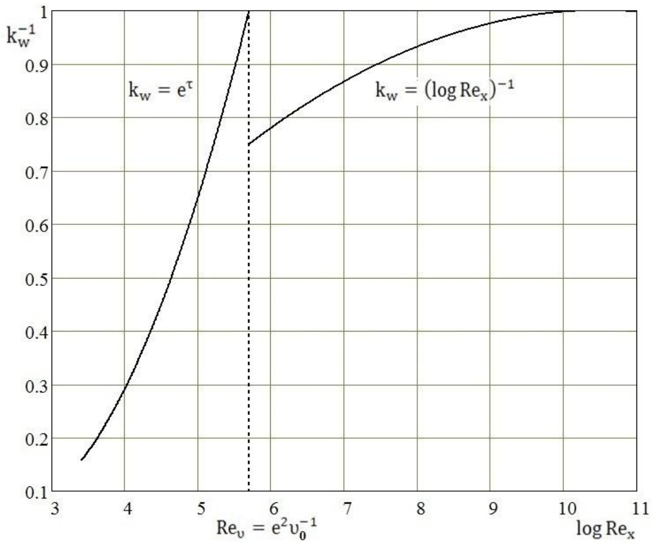

Figure 3: Coupling function $k_w(\mathrm{Re}_x)$ [8].

Figure 3 shows the variation of the coupling function $k_w(\mathrm{Re}_x)$, that as a matter of fact is a vorticity creation boundary condition, with a physical support, for any wall-bounded flow. This natural boundary condition can be interested in the numerical methods using vorticity-stream function/velocity formulations [8]. Now, justly Re_x known the solutions of Blasius soliton can be easy computed by a standard shooting technique.

Figure 4 shows the solution for the vorticity waves close to the ends of the Reynolds spectrum, i.e. flow field at small scale for Prandtl flow. In contrast to the strong shock/pressure waves, the weaker vorticity waves propagate under the form of the three wave packet: shear stress wave f'', elastic wave f''' and dispersion wave f^{iv}. The vorticity wave packet is a superimposing of three waves, each wave having different roles depending on the Reynolds number (the flow type), that is while the laminar flows are dominated by the shear stress waves induced by the creeping motion of vorticity (without energy dissipation), in turbulent flows the dispersive sinusoidal waves, induced by torsional vibrations of vorticity, are the key mechanism of the turbulence phenomenon.

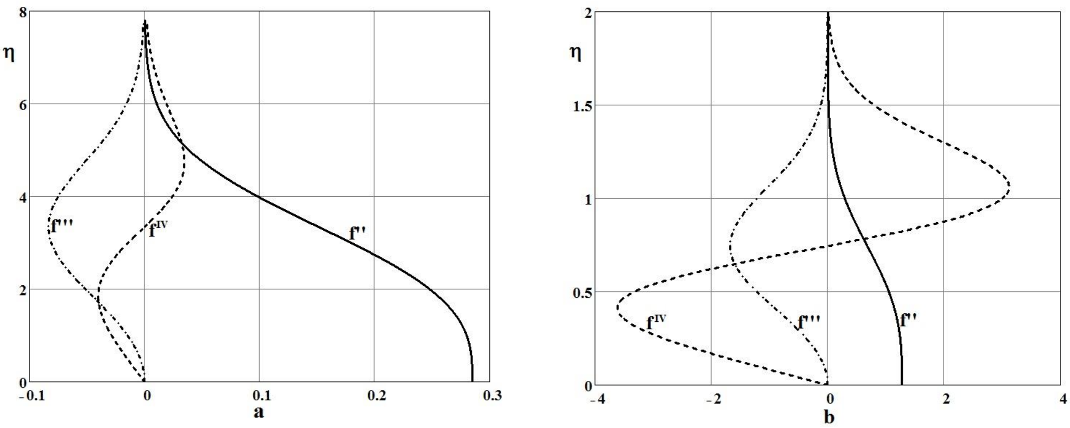

Figure 4: Blasius soliton for T – waves, (Prandtl flow): f'' - shear wave, f''' - elastic wave, f^{iv} –dispersion wave for a) $k_w = e$ and b) $k_w = e^{-2}$.

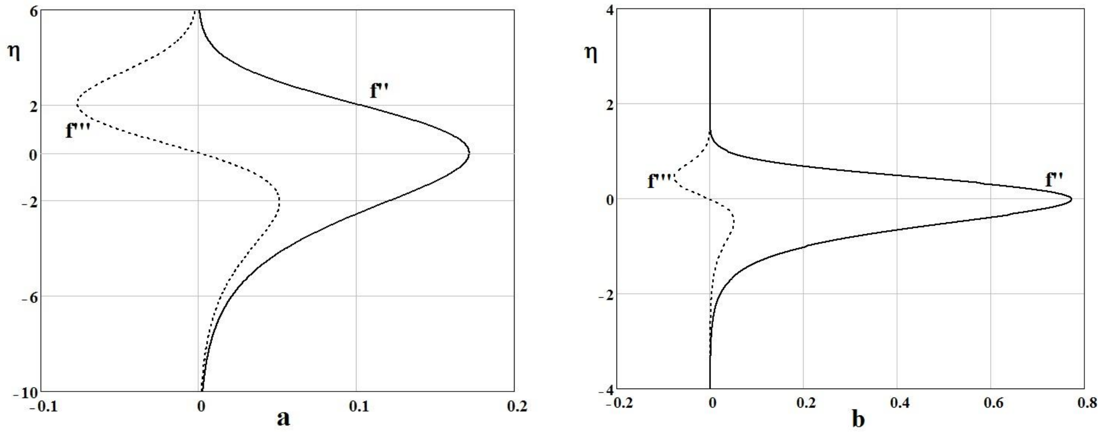

Figure 5: Blasius soliton for T – waves, (Couette flow): *f''* - shear wave, *f'''* - elastic wave, for a) $k_w = e$ and b) $k_w = e^{-2}$.

Figure 5 shows the vorticity field for Couette flow, where the wave packet contains only shear stress wave (f") and elastic wave (f''') because of the stronger restriction on the flow bounded by two sidewalls. The suppression of the dispersion wave leads to a pressure increase in the flow field as the Reynolds number increases. At $\mathrm{Re} = 2\mathrm{Re}_c^{1/2} \simeq 1400$ the concentrated vorticity (e^2) is broken down in two contra-rotating fragments, the wall torsion pressure is recovered in the flow field and the velocity field is regularized, Fig. 6b2. The onset of turbulence is the vorticity jump itself where the vorticity (its concentration) is halved and the energy storage at impact as torsion pressure (suction) is recovered as dynamic pressure of flow, Fig. 6.

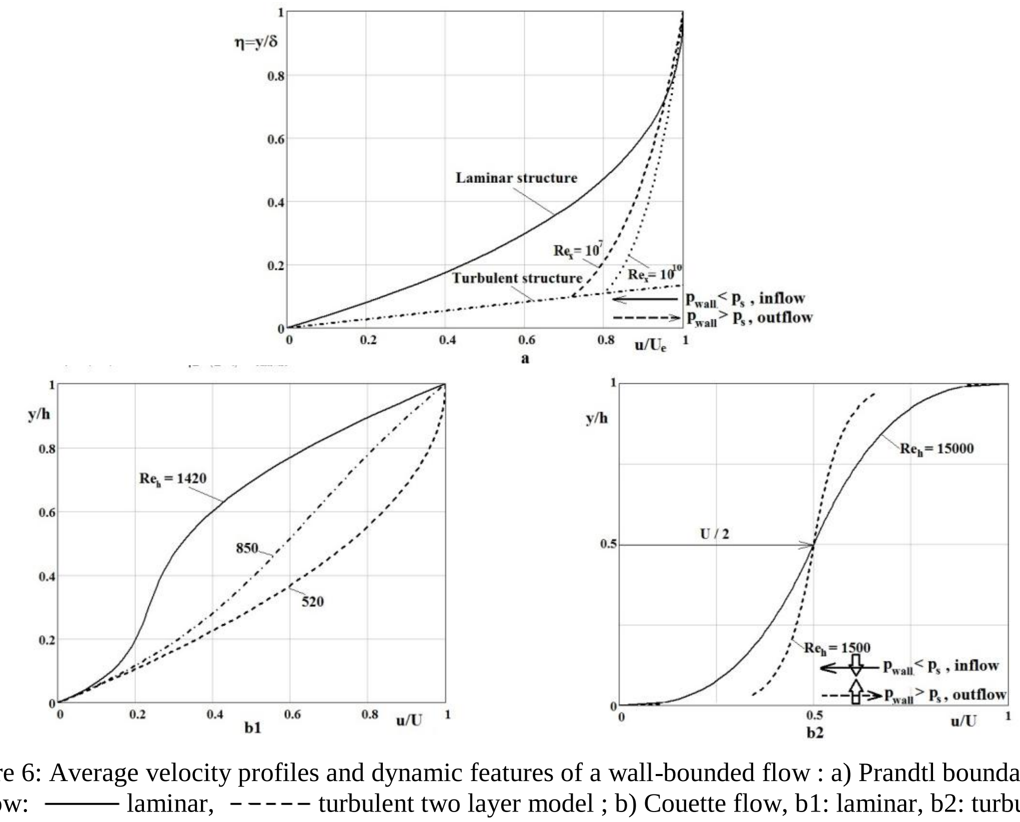

Figure 6: Average velocity profiles and dynamic features of a wall-bounded flow : a) Prandtl boundary layer flow: ——— laminar, – – – – – turbulent two layer model ; b) Couette flow, b1: laminar, b2: turbulent.

Figures 6(a, b) show the flow field at large scale marking also the vibration tendencies of flow near wall when the Reynolds number exceeds its critic value. The analysis of the wall-bounded flow both at small scale (vorticity field) and large scale (velocity field) points out the self-sustained wave mechanism of turbulence, that is similar to synthetic jets generated by pulsed jets with zero net-mass flux and a small flow-momentum consumption. The thixotropic fluid in the turbulent wall-bounded flows operates as a diaphragm that alternatively sucks inflow and ejects outflow in a periodic manner, creating discrete/concentrated vortical structures followed by their dispersion by wall friction and transport by flow.

5 CONCLUSIONS

Using the concept of torsion of the concentrated vorticity associated with the hypothesis of thixotropic/nonlinear viscous fluid a vorticity wave mechanism is devised for dispersion/transport of concentrated boundary vorticity. The main results obtained from the investigation of the vorticity wave system, triggered off by the flow-boundary collision, can be summarized as follows:

- The boundary vorticity creation is result of the flowing fluid - boundary collision that is a weak viscous shock of its evolution is governed by solitary shear waves called solitons, called in the paper Blasius solitons, analogous to KdV (Korteweg-de Vries) solitons formed in shallow water.
- There is a wide frequency/Reynolds spectrum from $\mathrm{Re}_x = e^2 \nu_0^{-1/2}$ (TSW) up to $\mathrm{Re}_x = e^{1/2} \nu_0^{-2}$ (acoustic waves) where the wave system operates like a big slightly damped oscillator close to the natural frequency of the fluid.
- The interested region in research lies in the audible frequency range, i.e. below $\mathrm{Re}_x = e^{2/3} \nu_0^{-3/2}$ where the breaking down process of concentrated vorticity by torsion vibrations remains a black box of turbulence.
- The origin of turbulence is one of mechanical nature where vibrations of concentrated boundary vorticity (transverse vorticity waves) are induced by the slightly damped oscillations (longitudinal compressing/expanding waves) of the fluid excited at the beginning of motion. The self-sustaining mechanism of fluctuations in turbulent shear flows consisting of creation, dispersion and transport of vorticity is similar to a typical device generating synthetic jets.
- The dual concept of the torsional concentrated vorticity – thixotropic fluid is vital for any paradigm in research of turbulence.

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