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TWO-STEPS SHAPE OPTIMIZATION ALGORITHM IMPROVING HYDRODYNAMICS STABILITY

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Abstract. *This proceeding suggests the two-steps shape optimization algorithm improving Hydrodynamics stability, by constructing the existing shape optimization problems Problem 1 and Problem 2. In Problem 1, a dissipation energy is defined as a cost function and the stationary Navier–Stokes problem is used as a main problem. In Problem 2, the maximum value of a real part of the leading eigenvalues is defined as a cost function and the stationary Navier–Stokes problem and its eigenvalue problem are used as main problems. The initial domain is a two dimensional cavity flow* Ω_0*, where side walls and bottom walls are used as the design boundaries as the cost functions decrease. First, Problem 1 is solved to obtain an optimal domain* Ω_1*. Second, Problem 2 is demonstrated by using* Ω_1 *as the initial domain, and an optimal domain* Ω_2 *is obtained. Finally, eigenvalue problems are solved in* Ω_0 *and* Ω_1*,* Ω_2 *to depict a linear neutral curve and spectrum. As a result, it is confirmed numerically that the critical Reynolds numbers are increased throughout Problem 1 and Problem 2.*

1 INTRODUCTION

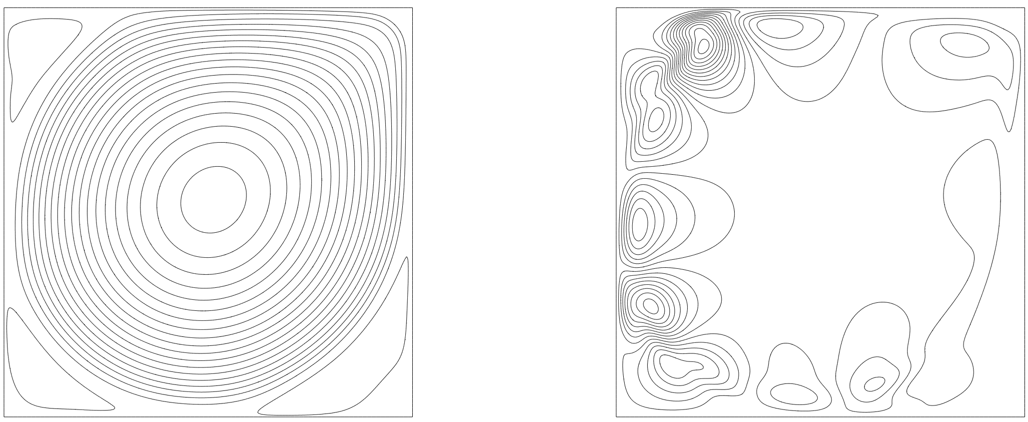
This proceeding suggests a shape optimization method improving Hydrodynamics stability with a regularization technique on the initial domain.

A shape optimization problem is generalized as a problem of optimizing the boundary design of a domain in which a boundary value problem of partial differential equations is defined, where the domain topology is fixed as that of the initial domain. In this problem, cost functions are defined as functionals of the domain and the solution of the boundary value problem. Along with computer development, numerical examinations of shape optimization problems have been conducted. Based on results of these trials, it was noted that the direct application of the gradient method often results in oscillating shapes. However, the oscillation was able to be suppressed by reduction of the degrees of freedom on the boundary.

Subsequently, it was clear that such the oscillation is caused by a lack of smoothness of the first variation of functional. To avoid oscillation without reducing the degrees of freedom, a method using the Laplace operator as smoother was proposed by H. Azegami et al. [1]. This method was called the H^1 gradient method. In the H^1 gradient method, domain variation that minimizes the objective functional is obtained as a solution to a boundary value problem of a linear elastic continuum defined in the design domain.

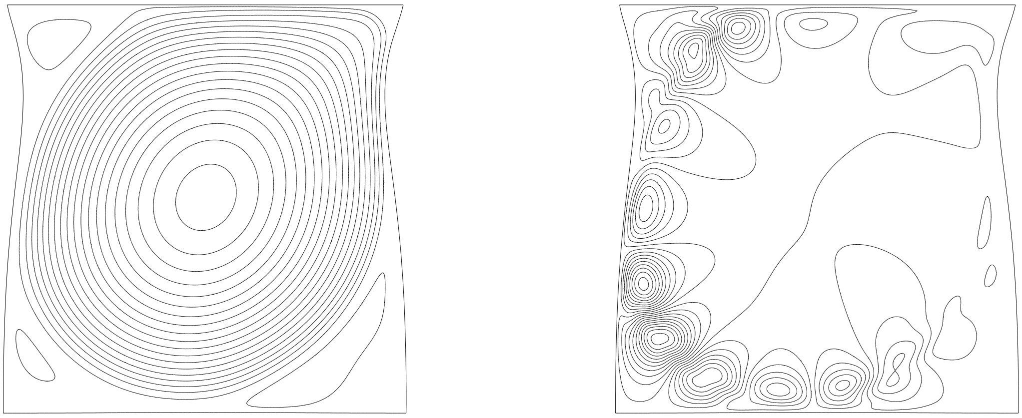
In fluid dynamics, E. Katamine et al. [2] demonstrated the applicability of the H^1 gradient method to a shape optimization problem of an isolated body in a domain defined the stationary Navier–Stokes equations as the governing equations. However, in [2], the stabilities of fluid flows in the initial domain and the optima domain were not discussed. Wherein, T. Nakazawa [3, 4] reported the possibility that the the stability could be controlled by the shape optimization problem. In particular, on the two dimensional Cavity flow Ω_0, using the stationary Navier–Stokes problem as the main problem, the maximizing and minimizing problems of the energy dissipation can make the critical Reynolds number increase and decrease, respectively.

Especially, on an optimal shape Ω_1 in the case of the minimizing problem (Problem 1), figs. 1 and 2 show stream lines of stationary flows and a real part of the leading eigenvalue on Ω_0 and Ω_1 at Re $= 11500$. Fig. 3 depicts a linear neutral curve on Ω_0 and Ω_1 and a spectrum at Re $= 11500$, and it is observed that the minimizing problem can make the critical Reynolds number larger.



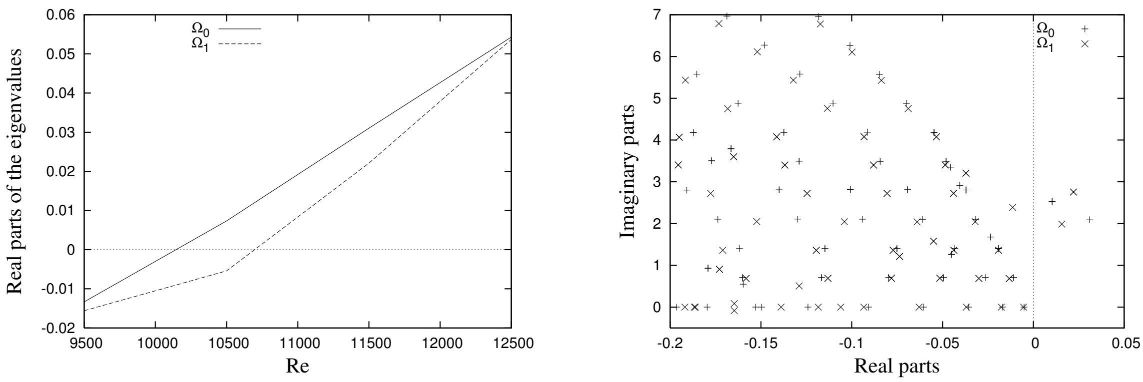
(a) (b)

Figure 1: Stream lines of (a) stationary flows and (b) a part of the leading eigenvalue on Ω_0 at Re $= 11500$.



(a) (b)

Figure 2: Stream lines of (a) stationary flows and (b) a real part of the leading eigenvalue on Ω_1 at Re = 11500.



(a) (b)

Figure 3: (a) a linear neutral curve on Ω_0 and Ω_1 and (b) a spectrum at Re = 11500.

The results of [3, 4] claim that the shape optimization problem might be able to control Hydrodynamics stability. For the aim to control Hydrodynamics stability more directly, T. Nakazawa and H. Azegami [5] suggested a new pioneering shape optimization method to make the disturbances more stable directly, in which the real parts of the eigenvalue is used as the cost function and the stationary Navier-Stokes problem and its eigenvalue problem are defined as the main problems (Problem 2). In [5], an initial domain is Poissolle flow with a sudden expansion, and the critical Reynolds number increases. In this proceeding, Problem 2 is demonstrated on Ω_0 and Ω_1 at $\mathrm{Re} = 11500$ and each linear stabilities are compared.

2 Formulation of problem

2.1 Initial Domain

A Cartesian coordinate system is used and a position vector is generally denoted by $\boldsymbol{x} = (x,y) \in \mathbb{R}^2$. An initial domain $\Omega_0 = \left\{(x,y) \in \mathbb{R}^2 \mid ([0,1]\times[0,1])\right\}$ is considered, and the top boundary and the wall boundary are set as $\Gamma_{\mathrm{top}} = \{(x,y) \mid 0 \le x \le 1,\ y = 1\}$ and $\Gamma_{\mathrm{wall}} = \partial\Omega \backslash \Gamma_{\mathrm{top}}$.

2.2 Domain Variation

Using the initial domain, the domain variation is defined in the following way. Let $\boldsymbol{i}+\boldsymbol{\phi} : \mathbb{R}^2 \to \mathbb{R}^2$ be a bi-Lipschitz transform and D be the set of $\boldsymbol{\phi}$, where $\boldsymbol{i}$ denotes identity mapping. For a $\boldsymbol{\phi} \in D$, let a varied domain $\Omega(\boldsymbol{\phi})$ and boundary $\Gamma_{\mathrm{wall}}(\boldsymbol{\phi})$ be defined respectively as $(\boldsymbol{i}+\boldsymbol{\phi})(\Omega_0)$ and $(\boldsymbol{i}+\boldsymbol{\phi})(\Gamma_{\mathrm{wall}})$. The function space of the Fréchet derivative with respect to arbitrary domain variation $\boldsymbol{\phi}$ is defined as

$$X = \left\{\boldsymbol{\phi} \in H^1(\mathbb{R}^2;\mathbb{R}^2) \mid \boldsymbol{\phi} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{top}}\right\}.$$

The shape derivatives of functionals are obtained as follows. Let ζ be a real-valued function of $\varphi \in C^1(D;H^2(\mathbb{R}^2;\mathbb{R}))$ and $\nabla\varphi(\boldsymbol{\phi})$, and

$$L(\boldsymbol{\phi}) = \int_{\Omega(\boldsymbol{\phi})} \zeta(\varphi(\boldsymbol{\phi}), \nabla\varphi(\boldsymbol{\phi}))dx.$$

Using the shape derivative ζ' of ζ, the shape derivative JL of L is given as

$$JL(\boldsymbol{\phi}) = \int_{\Omega(\boldsymbol{\phi})} \zeta' dx + \int_{\Gamma_{\mathrm{wall}}(\boldsymbol{\phi})} \zeta \boldsymbol{\nu}\cdot\boldsymbol{\psi} d\gamma,$$

where the deformed boundary is denote by $\Gamma_{\mathrm{wall}}(\boldsymbol{\phi}) = \partial\Omega(\boldsymbol{\phi})\backslash\Gamma_{\mathrm{top}}$ and where $\boldsymbol{\nu}$ denotes an outward unit normal vector on the boundary ([1] Eq. (18), p.274).

2.3 Main problems

The stationary Navier–Stokes equation and the equation of continuity in non-dimensional form are considered, where $(\hat{\boldsymbol{u}}, \hat{p}) \in \hat{U}\times\hat{Q}$ denote non dimensional stationary velocity and pressure and

$$\hat{U} = \left\{\hat{\boldsymbol{u}} = (\hat{u},\hat{v}) \in H^1(\Omega(\boldsymbol{\phi});\mathbb{R}^2) \mid \hat{\boldsymbol{u}} = \boldsymbol{u}_{\mathrm{D}} \text{ on } \partial\Omega(\boldsymbol{\phi})\right\},$$

$$\hat{Q} = \left\{\hat{q} \in L^2(\Omega(\boldsymbol{\phi});\mathbb{R}) \mid \int_{\Omega(\boldsymbol{\phi})} \hat{q} dx = 0\right\}.$$

The weak form is written as

$$\int_{\Omega(\boldsymbol{\phi})} \left\{ ((\hat{\boldsymbol{u}} \cdot \boldsymbol{\nabla}) \hat{\boldsymbol{u}}) \cdot \hat{\boldsymbol{w}} - \hat{p} \boldsymbol{\nabla} \cdot \hat{\boldsymbol{w}} + \frac{1}{\mathrm{Re}} (\boldsymbol{\nabla} \hat{\boldsymbol{u}}^{\mathrm{T}}) \cdot (\boldsymbol{\nabla} \hat{\boldsymbol{w}}^{\mathrm{T}}) - \hat{q} \boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}} \right\} dx = 0,$$

for all $(\hat{\boldsymbol{w}}, \hat{q}) \in \hat{W} \times \hat{Q}$, where $(\hat{\boldsymbol{w}}, \hat{q})$ represents trial functions for velocity $\hat{\boldsymbol{u}}$ and pressure $\hat{p}$, and

$$\hat{W} = \left\{ \hat{\boldsymbol{w}} \in H^1(\Omega(\boldsymbol{\phi}); \mathbb{R}^2) \mid \hat{\boldsymbol{w}} = \boldsymbol{0} \text{ on } \partial\Omega(\boldsymbol{\phi}) \right\}.$$

A set of velocity fields $\bar{\boldsymbol{u}} \in \bar{U}$ and pressure $\bar{p} \in \bar{Q}$ for the disturbance

$$\bar{U} = \left\{ \bar{\boldsymbol{u}} \in H^1(\Omega(\boldsymbol{\phi}); \mathbb{C}^2) \mid \bar{\boldsymbol{u}} = \boldsymbol{0} \text{ on } \partial\Omega(\phi) \right\},$$
$$\bar{Q} = \left\{ \bar{q} \in L^2(\Omega(\boldsymbol{\phi}); \mathbb{C}) \mid \int_{\Omega(\boldsymbol{\phi})} \bar{q} dx = 0 \right\},$$

are introduced. Based on the linear stability theory, for an eigenvalue $\lambda \in \mathbb{C}$, an eigenvalue equation in weak form is written as

$$\int_{\Omega(\boldsymbol{\phi})} \Big\{ \lambda \bar{\boldsymbol{u}} \cdot \bar{\boldsymbol{w}}^c + ((\hat{\boldsymbol{u}} \cdot \boldsymbol{\nabla}) \bar{\boldsymbol{u}}) \cdot \bar{\boldsymbol{w}} + ((\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla}) \hat{\boldsymbol{u}}) \cdot \bar{\boldsymbol{w}}^c - \bar{p} \boldsymbol{\nabla} \cdot \bar{\boldsymbol{w}} + \frac{1}{\mathrm{Re}} (\boldsymbol{\nabla} \bar{\boldsymbol{u}}^{\mathrm{T}}) \cdot (\boldsymbol{\nabla} \bar{\boldsymbol{w}}^{c\mathrm{T}})$$
$$- \bar{q}^c \boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}} \} dx = 0,$$

for all $(\bar{\boldsymbol{w}}, \bar{q}) \in \bar{W} \times \bar{Q}$, where $(\bar{\boldsymbol{w}}, \bar{q})$ represents trial functions for $\bar{\boldsymbol{u}}$ and $\bar{p}$, and

$$\bar{W} = \left\{ \bar{\boldsymbol{w}} \in H^1(\Omega(\boldsymbol{\phi}); \mathbb{C}^2) \mid \bar{\boldsymbol{w}} = \boldsymbol{0} \text{ on } \partial\Omega(\boldsymbol{\phi}) \right\}.$$

3 Shape optimization problem

The minimization problem of the maximum value of the real parts of the leading eigenvalues (Problem 2) is formulated as

Find Ω_2
that minimizes $f = 2\mathrm{Real}[\lambda]$
subject to $(\hat{\boldsymbol{u}}, \hat{p}, \bar{\boldsymbol{u}}, \bar{p}) \in \hat{U} \times \hat{Q} \times \bar{U} \times \bar{Q}$ such that Eqs. (1) and (1) ,

where $\lambda, \bar{\boldsymbol{u}}$ depict an eigenvalue and an eigenfunction for the disturbance with the maximum value of the real parts of the leading eigenvalues. The shape derivative of the Lagrange function L is evaluated by application of the Lagrange multiplier method, where L is written as

$$L(\hat{\boldsymbol{u}}, \hat{p}, \hat{\boldsymbol{w}}, \hat{q}, \lambda, \bar{\boldsymbol{u}}, \bar{p}, \bar{\boldsymbol{w}}, \bar{q}) =$$
$$2\mathrm{Real}[\lambda]$$
$$- \int_{\Omega(\boldsymbol{\phi})} \left\{ ((\hat{\boldsymbol{u}} \cdot \boldsymbol{\nabla}) \hat{\boldsymbol{u}}) \cdot \hat{\boldsymbol{w}} - \hat{p} \boldsymbol{\nabla} \cdot \hat{\boldsymbol{w}} + \frac{1}{\mathrm{Re}} (\boldsymbol{\nabla} \hat{\boldsymbol{u}}^{\mathrm{T}}) \cdot (\boldsymbol{\nabla} \hat{\boldsymbol{w}}^{\mathrm{T}}) - \hat{q} \boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}} \right\} dx$$
$$- \int_{\Omega(\boldsymbol{\phi})} \{ h(\hat{\boldsymbol{u}}, \bar{\boldsymbol{u}}, \bar{p}, \bar{\boldsymbol{w}}, \bar{q}) + h(\hat{\boldsymbol{u}}, \bar{\boldsymbol{u}}, \bar{p}, \bar{\boldsymbol{w}}, \bar{q})^c \} dx,$$

where

$$h(\hat{\boldsymbol{u}}, \bar{\boldsymbol{u}}, \bar{p}, \bar{\boldsymbol{w}}, \bar{q}) = \lambda \bar{\boldsymbol{u}} \cdot \bar{\boldsymbol{w}} + ((\hat{\boldsymbol{u}} \cdot \boldsymbol{\nabla}) \bar{\boldsymbol{u}}) \cdot \bar{\boldsymbol{w}} + ((\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla}) \hat{\boldsymbol{u}}) \cdot \bar{\boldsymbol{w}}$$
$$- \bar{p} \boldsymbol{\nabla} \cdot \bar{\boldsymbol{w}} + \frac{}{\mathrm{e}} (\boldsymbol{\nabla} \bar{\boldsymbol{u}}^{\mathrm{T}}) \cdot (\boldsymbol{\nabla} \bar{\boldsymbol{w}}^{c\mathrm{T}}) - \bar{q}^c \boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}}.$$

The shape derivative JL for L is taken to obtain a sensitivity and main problem (Eqs. (1) and (1)) and its adjoint problems considering KKT condition, and adjoint equations for eq. (1) is obtained as

$$\int_{\Omega(\boldsymbol{\phi})}\Big\{-\lambda\bar{\boldsymbol{u}}'\cdot\bar{\boldsymbol{w}}^c-((\bar{\boldsymbol{w}}^c\cdot\boldsymbol{\nabla})\hat{\boldsymbol{u}})\cdot\bar{\boldsymbol{u}}'+\left(\left(\nabla\hat{\boldsymbol{u}}^{\mathrm{T}}\right)\bar{\boldsymbol{w}}^c\right)\cdot\bar{\boldsymbol{u}}'-\bar{p}'\boldsymbol{\nabla}\cdot\bar{\boldsymbol{w}}^c+\frac{1}{\mathrm{Re}}(\boldsymbol{\nabla}\bar{\boldsymbol{u}}'^{\mathrm{T}})\cdot(\boldsymbol{\nabla}\bar{\boldsymbol{w}}^{c\mathrm{T}})$$
$$-\bar{q}^c\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}'\Big\}\,dx=0,$$

for all $(\bar{\boldsymbol{u}}',\bar{p}')\in\bar{U}\times\bar{Q}$, where $(\cdot)'$ represents shape derivative. And adjoint equations for eq. (1) is written as

$$\int_{\Omega(\boldsymbol{\phi})}\Big\{-((\hat{\boldsymbol{u}}\cdot\boldsymbol{\nabla})\hat{\boldsymbol{w}})\cdot\hat{\boldsymbol{u}}'+\left(\left(\nabla\hat{\boldsymbol{u}}^{\mathrm{T}}\right)\hat{\boldsymbol{w}}\right)\cdot\hat{\boldsymbol{u}}'-\hat{p}'\boldsymbol{\nabla}\cdot\hat{w}+\frac{1}{\mathrm{Re}}(\boldsymbol{\nabla}\hat{\boldsymbol{u}}'^{\mathrm{T}})\cdot(\boldsymbol{\nabla}\hat{\boldsymbol{w}}^{\mathrm{T}})-\hat{q}\boldsymbol{\nabla}\cdot\hat{\boldsymbol{u}}'$$
$$+2\mathrm{Real}\left[-((\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla})\bar{\boldsymbol{w}})\cdot\hat{\boldsymbol{u}}'+\left(\left(\nabla\bar{\boldsymbol{u}}^{\mathrm{T}}\right)\bar{\boldsymbol{w}}\right)\cdot\hat{\boldsymbol{u}}'\right]\Big\}\,dx=0,$$

for all $(\hat{\boldsymbol{u}}',\hat{p}')\in\hat{U}\times\hat{Q}$. Solving these problems, substitute main variables $(\hat{\boldsymbol{u}},\hat{p},\bar{\boldsymbol{u}},\bar{p})$ and adjoint variables $(\hat{\boldsymbol{w}},\hat{q},\bar{\boldsymbol{w}},\bar{q})$ into JL, JL is reduced to

$$JL=\int_{\Gamma_{wall}(\boldsymbol{\phi})}G\boldsymbol{\psi}\cdot\boldsymbol{v}\,d\gamma,$$

where G represents the sensitivity for Problem 2

$$G=-\frac{1}{\mathrm{Re}}(\boldsymbol{\nabla}\hat{\boldsymbol{u}}'^{\mathrm{T}})\cdot(\boldsymbol{\nabla}\hat{\boldsymbol{w}}^{\mathrm{T}})+2\mathrm{Real}\left[-\frac{1}{\mathrm{Re}}(\boldsymbol{\nabla}\bar{\boldsymbol{u}}'^{\mathrm{T}})\cdot(\boldsymbol{\nabla}\bar{\boldsymbol{w}}^{\mathrm{T}})\right].$$

See more detail in Section 3 of T. Nakazawa and H. Azegami [5]. For reshaping a domain, H^1 gradient method is used .

4 RESULTS and CONCLUSIONS

Problem 2 is addressed on Ω_0 and Ω_1. Fig. 4 depicts cost function f with reshaping steps, and Ω_{real} and Ω_2 represent f for using Ω_0 and Ω_1 as initial domains. From fig. 4, in the case of using Ω_0 as an initial domain, f is not decreasing, and on the other hand f using Ω_1 as an initial domain is minimizing. As a result, Problem 1 might regularize a geometrical shape of Ω_0 for Problem 2.

Fig. 5 shows stream lines of stationary flows and a real part of the leading eigenvalue on Ω_2 at $\mathrm{Re}=11500$, and fig. 6 a spectrum at $\mathrm{Re}=11500$. Finally, a linear neutral curve on Ω_0 and Ω_1, Ω_2 is shown in fig. 7, from which it is confirmed numerically that the critical Reynolds numbers are increased throughout Problem 1 and Problem 2.

By the way, the sensitivity derived in Problem 2 is evaluated by using the only disturbance with a maximum value of real parts of the leading eigenvalues, and after the second bifurcation, Problem 2 ignores all the disturbances except the most unstable one, which is not reasonable in the sense of Hydrodynamics stability. Therefore, it is needed to construct a shape optimization problem considering all the unstable disturbances as one of future works.

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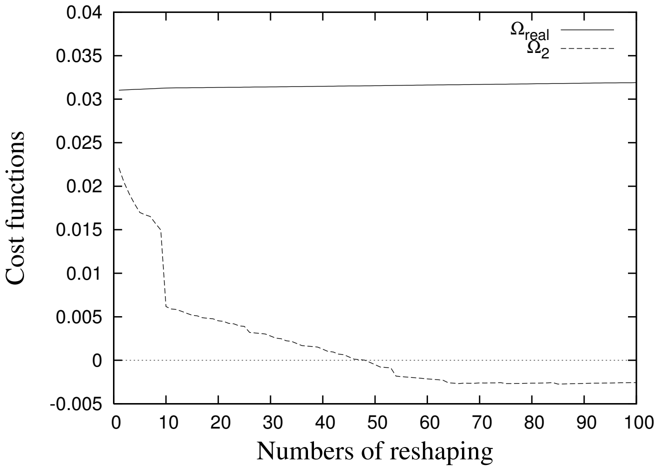
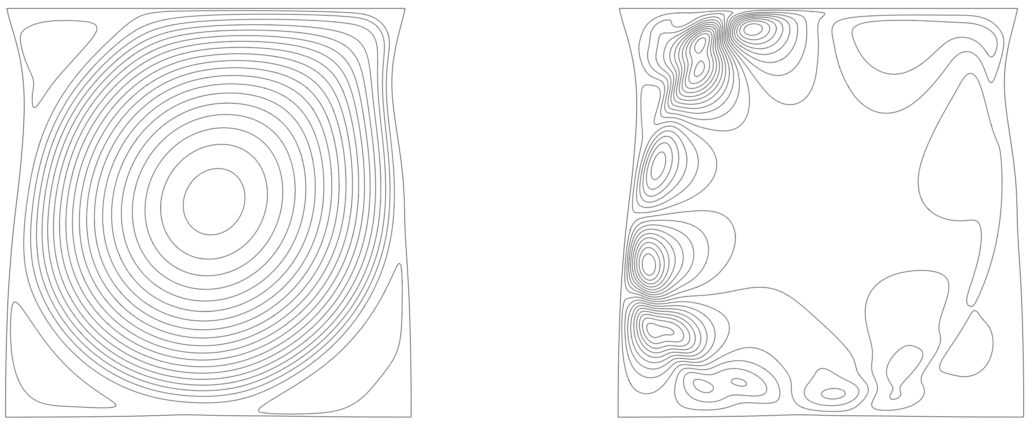


Figure 4: Cost functions with reshaping steps on Ω_0 and Ω_1 at $\mathrm{Re} = 11500$.



(a) (b)

Figure 5: (a) Stream lines of (a) stationary flows and (b) a real part of the leading eigenvalue on Ω_2 at $\mathrm{Re} = 11500$.

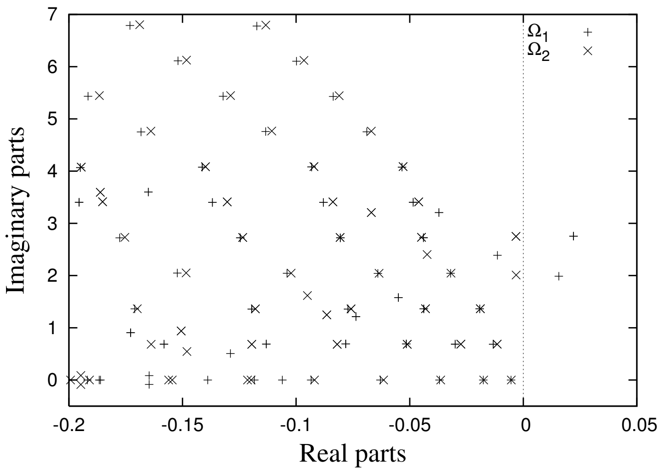


Figure 6: Spectrum on and Ω_2 at $\mathrm{Re} = 11500$.

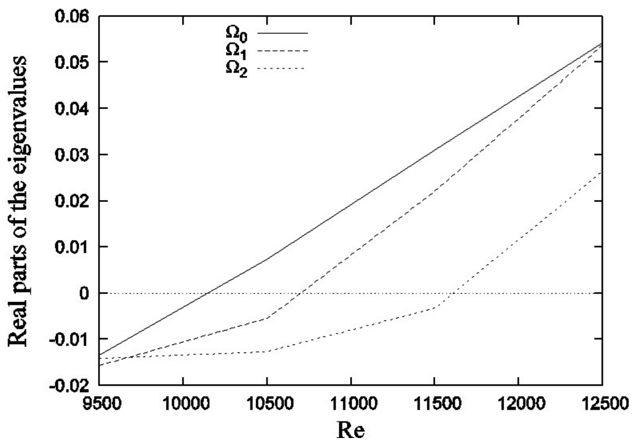


Figure 7: Cost functions with reshaping steps on Ω_0, Ω_1 and Ω_2 at Re $= 11500$.

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