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DIRECT TIME-DOMAIN INTEGRATION APPROACH FOR VISCOELASTIC SYSTEMS INVOLVING VARIOUS DAMPING MODELS

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Abstract. *Mechanical engineering systems with two or more parts of significantly different energy dissipation levels are encountered frequently in dynamical designs nowadays. Although dynamic analyses of only one exponential-like damping model in independent structures are studied by authors for decades, their counterpart for nonviscous systems involving various damping models seems not well-developed. In this paper, a time-domain analysis of various damping models subjected to arbitrary initial conditions is presented. The proposed approach is built on an extended state-space representation of the equations of motion. The method can be applied in both conditions when the damping matrices are of full rank or rank deficient in a uniform expression. The dynamic responses of the various damping models are calculated only by the original system matrices without solving the eigensolutions of the systems. This advantage particularly makes this method numerically efficient. Finally, a numerical example is offered to illustrate the accuracy of the derived method.*

1 INTRODUCTION

Damping is longtime regarded as a significant factor in the dynamic behavior of vibration systems and is a characteristic parameter describing the energy dissipation. The viscous damping model which was first proposed by Lord Rayleigh [1] has been extensively used in the past centuries for its theoretical simplicity. However, with the development of many industrial applications (such as composite structural materials, active control in spacecraft, ships, rockets and railway tracks, sandwich structures), the classical viscous damping model may not properly reveal the dissipative forces. As a result, there has been an increasing demand in recent years on nonviscous damping models with the purpose of describing dissipative forces in a more general and accurate manner compared to the limited scope offered by the viscous damping model. Majority of the nonviscous damping models, like Biot model [2, 3], Golla-Hugher-McTavish (GHM) model [4, 5], Anelastic Displacement Field (ADF) model [6, 7], are presented considering the entire velocity history via convolution integrals over some kernel functions.

Classic dynamic analysis methods for linear viscous damping cannot be directly applied in the nonviscous damping model systems. Any modifications of nonviscous model may lead to some mathematical difficulties and changes for solving the corresponding dynamic equations. In recent years, researchers have considered the dynamic responses of nonviscous damping models. Some authors developed exact state-space approaches [4-12] based on internal variables. Others proposed approximated approaches [13-20] and model reduction methods [21-26] to solve the eigenproblem for the exponential-like damping models. The previously mentioned methods are devoted to efficiently solve the nonviscous eigenproblem under different damping models. However, the solution of this eigenproblem is time-consuming. Then, a direct time-domain integration approach for exponentially damped systems was proposed by Adhikari and Wager [27]. The method used only original system matrices to compute the dynamic response which improved the efficiency significantly. Later, Cortés et al. [28] proposed a direct integration formulation for exponentially damped systems, which did not employ internal variables. Liu [29] proposed analysis method for non-viscously damped structures based on Newmark's assumption of acceleration. Pan and Wang [30] developed a frequency-domain method for exponentially damped systems using the Discrete Fourier Transform method in combination with Fast Fourier Transform which can deal with non-zero initial conditions.

The mentioned previous studies are restricted to the case of the systems with only one damping model. Nowadays, in practice, structure systems with two or more parts with remarkably different levels of energy dissipation are encountered frequently in mechanical engineering. Therefore, these damping systems often involve multiple damping models.

Although the exponential damping model, Biot model, GHM model and ADF model are physically different, they can be mathematically represented by a friction formula of rational polynomials [31]. The uniform way to express damping models can simplify the theoretical analysis of dynamic responses and be applied conveniently in multiple damping models. Recently, Li and Hu [32] have proposed a state-space approach for the analysis of linear systems with multiple damping models. The dynamic response of the system was calculated by mode-superposition method using the state-space eigensolutions which was time-consuming.

Based on the problems mentioned above, a time-domain approach for various damping models is proposed. The presented time-domain approach can be applied in both conditions when the damping matrices are of full rank or rank deficient. The advantage of the proposed method over the traditional mode superposition method is that the final recurrence formulas to be solved contain nothing but a linear combination of the system matrices. It will be demonstrated by a numerical example that the proposed method shares the same accuracy with exact values but is of greater efficiency.

The paper is organized as follows: in Section 2, we will quickly review some preliminary concepts and definitions of various damping models and its corresponding state-space formalism. A direct time-domain approach for various damping models is proposed to calculate the dynamic response of the systems in Section 3. We briefly summarize the method in Section 4 to make it convenient to be coded up. The results of a numerical example are presented in Section 5 in order to assess the performance of the proposed method. The paper ends with some conclusions in section 6.

2 REVIEW OF THE STATE-SPACE FORMALISM

The equations of motion of an N-degree-of-freedom linear system with various damping models can be expressed by

$$\mathbf{M\ddot{u}}(t)+\mathbf{C_0\dot{u}}(t)+\sum_{k=1}^{n}\mathbf{C}_k\int_{-\infty}^{t}g_k(t-\tau)\frac{\partial\mathbf{u}(\tau)}{\partial\tau}d\tau+\mathbf{Ku}(t)=\mathbf{f}(t) \tag{1}$$

together with the initial conditions

$$\mathbf{u}(t=0)=\mathbf{u_0}\in\mathbb{R}^N \quad \text{and} \quad \mathbf{\dot{u}}(t=0)=\mathbf{\dot{u}_0}\in\mathbb{R}^N \tag{2}$$

Here $\mathbf{u}(t)\in\mathbb{R}^{N\times N}$ is the displacement vector, $\mathbf{M}\in\mathbb{R}^{N\times N}$ is the mass matrix, $\mathbf{K}\in\mathbb{R}^{N\times N}$ is the stiffness matrix, $\mathbf{C_0}\in\mathbb{R}^{N\times N}$ is the frequency-independent damping matrix, $\mathbf{C}_k\in\mathbb{R}^{N\times N}$ are the coefficient matrices of frequency-dependent damping, $g_k(t)$ is the kth damping kernel function, $\mathbf{f}(t)$ is the forcing vector.

In fact, any causal model which can make the energy dissipation function nonnegative, may be considered as a candidate for a damping model. So far, several physically realistic mathematical forms of damping models are available in the literature. The Biot model [2, 3] can be given by

$$g_k(t)=a_0\delta(t)+\sum_{k=1}^{m}a_k\exp(-b_kt) \tag{3}$$

The GHM model [4, 5] and ADF model [6, 7]can be respectively expressed as

$$g_k(t)=\sum_{k=1}^{m}\alpha_k\frac{\hat{b}_{2k}e^{-\hat{b}_{1k}t}-\hat{b}_{1k}e^{-\hat{b}_{2k}t}}{\hat{b}_{2k}-\hat{b}_{1k}} \text{ and } g_k(t)=\sum_{k=1}^{m}\Delta_k\exp(-\Omega_kt) \tag{4}$$

where

$$\hat{b}_{1k} = \hat{\omega}_k \hat{\zeta}_k - \hat{\omega}_k \sqrt{\hat{\zeta}_k^2 - 1}, \hat{b}_{2k} = \hat{\omega}_k \hat{\zeta}_k + \hat{\omega}_k \sqrt{\hat{\zeta}_k^2 - 1}$$

The uniform way of expressing damping models including the exponential damping model, Biot model, GHM model and ADF model can be represented by a fraction formula of rational polynomials [32]

$$G_k(s) = \frac{c_{p_k} s^{p_k} + c_{p_{k-1}} s^{p_{k-1}} + \cdots + c_0}{d_{q_k} s^{q_k} + d_{q_{k-1}} s^{q_{k-1}} + \cdots + d_0} \tag{5}$$

The previous equation can be rewritten as the following form based on the theory of minimal realization

$$G_k(s) = \frac{c_{p_k} s^{p_k} + c_{p_{k-1}} s^{p_{k-1}} + \cdots + c_0}{d_{q_k} s^{q_k} + d_{q_{k-1}} s^{q_{k-1}} + \cdots + d_0} = \mathbf{P}_k^T (\mathbf{E}_k - s\mathbf{W}_k)^{-1} \mathbf{Q}_k \tag{6}$$

where $\mathbf{E}_k, \mathbf{W}_k \in \mathbb{R}^{q_k \times q_k}$ and vectors $\mathbf{P}_k, \mathbf{Q}_k \in \mathbb{R}^{q_k \times q_k}$.

It should be noted that the coefficient of the highest order denominator can be normalized to 1, which has

$$d_{q_k} = 1 \tag{7}$$

The other matrices and vectors can be expressed as

$$\mathbf{P}_k = \begin{Bmatrix} c_{q_k-1} - d_{q_k-1} c_{q_k} \\ c_{q_k-2} - d_{q_k-2} c_{q_k} \\ \vdots \\ c_0 - d_0 c_{q_k} \end{Bmatrix} \left(\forall j > p_k, c_j = 0\right), \mathbf{E}_k = \begin{bmatrix} -d_{q_k-1} & \cdots & -d_1 & -d_0 \\ 1 & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \ddots & \mathbf{0} & \vdots \\ 0 & \mathbf{0} & 1 & 0 \end{bmatrix}, \mathbf{W}_k = \mathbf{I}_{q_k}, \mathbf{Q}_k = \begin{Bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$$

In the actual mechanical structure, the damping materials are only a small part of the whole system. As a result, the coefficient matrix of frequency-dependent damping is rank deficient. We assume that in general

$$r_k = rank(\mathbf{C}_k) \le N, \forall k = 1, \cdots n \tag{8}$$

Thus, the coefficient matrix $\mathbf{C}_k$ has the rank-revealing decomposition

$$\mathbf{C}_k = \mathbf{L}_k \mathbf{R}_k^T \tag{9}$$

where $\mathbf{L}_{k,} \mathbf{R}_k \in \mathbb{R}^{N \times r_k}$ are of full column rank r_k .

Eq.(1) can be represented in the first-order form as

$$\mathbf{B}\dot{\mathbf{z}}(t)=\mathbf{A}\mathbf{z}(t)+\mathbf{r}(t) \tag{10}$$

where

$$\mathbf{B}=\begin{bmatrix}\mathbf{C}_0+\mathbf{L}\mathbf{E}^{-1}\mathbf{R}^T & \mathbf{M} & \mathbf{L}\mathbf{E}^{-1}\mathbf{W}\\ \mathbf{M} & 0 & 0\\ \mathbf{E}^{-1}\mathbf{R}^T & 0 & \mathbf{E}^{-1}\mathbf{W}\end{bmatrix},\mathbf{A}=\begin{bmatrix}-\mathbf{K} & 0 & 0\\ 0 & \mathbf{M} & 0\\ 0 & 0 & \mathbf{I}_y\end{bmatrix},\mathbf{z}(t)=\begin{bmatrix}\mathbf{u}(t)\\ \mathbf{v}(t)\\ \mathbf{y}(t)\end{bmatrix},\mathbf{r}(t)=\begin{bmatrix}\mathbf{f}(t)\\ 0\\ 0\end{bmatrix}$$

$$\mathbf{v}(t)=\dot{\mathbf{u}}(t),\ \mathbf{y}(t)=\ell^{-1}\left[s(\mathbf{E}-s\mathbf{W})^{-1}\mathbf{R}^T\right]\mathbf{u}(t)$$

$$\mathbf{E}=diag(\mathbf{I}_{r_1}\otimes\mathbf{E}_1,\mathbf{I}_{r_2}\otimes\mathbf{E}_2,\cdots,\mathbf{I}_{r_n}\otimes\mathbf{E}_n),\mathbf{W}=diag(\mathbf{I}_{r_1}\otimes\mathbf{W}_1,\mathbf{I}_{r_2}\otimes\mathbf{W}_2,\cdots,\mathbf{I}_{r_n}\otimes\mathbf{W}_n)$$

$$\mathbf{L}=\left[\mathbf{L}_1(\mathbf{I}_{r_1}\otimes\mathbf{Q}_1)^T,\mathbf{L}_2(\mathbf{I}_{r_2}\otimes\mathbf{Q}_2)^T,\cdots,\mathbf{L}_n(\mathbf{I}_{r_n}\otimes\mathbf{Q}_n)^T\right]$$

$$\mathbf{R}=\left[\mathbf{R}_1(\mathbf{I}_{r_1}\otimes\mathbf{P}_1)^T,\mathbf{R}_2(\mathbf{I}_{r_2}\otimes\mathbf{P}_2)^T,\cdots,\mathbf{R}_n(\mathbf{I}_{r_n}\otimes\mathbf{P}_n)^T\right]$$

The symbol $\otimes$ in the previous expression denotes the Kronecker product.

3 DIRECT TIME-DOMAIN APPROACH

First, assume the displacement and the velocities vectors at random time *t* can be expressed by the linear approximations

$$\mathbf{u}(t)=\mathbf{u}_j(1-\frac{t}{h})+\mathbf{u}_{j+1}\frac{t}{h} \tag{11}$$

and

$$\mathbf{v}(t)=\mathbf{v}_j(1-\frac{t}{h})+\mathbf{v}_{j+1}\frac{t}{h} \tag{12}$$

The internal variables $\mathbf{y}(t)$ can also be written in a linear manner

$$\mathbf{y}(t)=\mathbf{y}_j(1-\frac{t}{h})+\mathbf{y}_{j+1}\frac{t}{h} \tag{13}$$

The step size $h=T/\mathrm{N}_d$, where T equals the time which the response calculation is required and N_d is the number of divisions in the time axis. In order to simplify the calculation, in this paper, we only consider the constant time-steps. To get the totally discretized one-step formulation which allows $\mathbf{z}_{j+1}$ to be calculated by $\mathbf{z}_j$, the last and the next step considers the integration of the Eq.(10) with respect to the time interval with $t/h\in\left[j,j+1\right]$. The trapezoidal rule of numerical integration is used.

$$\left[\mathbf{B}-\frac{h}{2}\mathbf{A}\right]\mathbf{z}_{j+1}=\left[\mathbf{B}+\frac{h}{2}\mathbf{A}\right]\mathbf{z}_{j}+\mathbf{i}_{r},\mathbf{i}_{r}=\int_{jh}^{(j+1)h}\mathbf{r}(t)\,\mathrm{d}t \tag{14}$$

where

$$\mathbf{z}_0^T=\left[\mathbf{u}_0^T,\mathbf{v}_0^T,\mathbf{0}^T,\cdots,\mathbf{0}^T\right] \tag{15}$$

Eq.(14) can be used to compute $\mathbf{z}(t)$ at discrete time.

Combining Eq.(10) and the second row of Eq.(14), we can get the following expression

$$\mathbf{v}_{j+1}=\frac{2}{h}[\mathbf{u}_{j+1}-\mathbf{u}_{j}]-\mathbf{v}_{j} \tag{16}$$

The third row in Eq.(14) is used to express the internal variables inters of the displacements as

$$\mathbf{y}_{j+1}=(\mathbf{E}^{-1}\mathbf{W}-\frac{h}{2}\mathbf{I}_{y})^{-1}[-\mathbf{E}^{-1}\mathbf{R}^{T}(\mathbf{u}_{j+1}-\mathbf{u}_{j})+(\mathbf{E}^{-1}\mathbf{W}+\frac{h}{2}\mathbf{I}_{y})\mathbf{y}_{j}] \tag{17}$$

If we denote $\mathbf{a}_1=(\mathbf{E}^{-1}\mathbf{W}-\frac{h}{2}\mathbf{I}_y)^{-1},\mathbf{a}_2=(\mathbf{E}^{-1}\mathbf{W}+\frac{h}{2}\mathbf{I}_y)$, Eq.(17) can be expressed as

$$\mathbf{y}_{j+1}=\mathbf{a}_1[-\mathbf{E}^{-1}\mathbf{R}^T(\mathbf{u}_{j+1}-\mathbf{u}_j)+\mathbf{a}_2\mathbf{y}_j] \tag{18}$$

Substituting Eq.(16) into the first row of Eq.(14), we can receive a totally discretized one-step formulation of the displacements $\mathbf{u}_{j+1}$

$$\mathbf{a}_3\mathbf{u}_{j+1}=(\mathbf{a}_3-h\mathbf{K})\mathbf{u}_j+2\mathbf{M}\mathbf{v}_j+\mathbf{a}_4\mathbf{y}_j-\mathbf{a}_4\mathbf{y}_{j+1} \tag{19}$$

where

$$\mathbf{a}_3=(\mathbf{C}_0+\mathbf{L}\mathbf{E}^{-1}\mathbf{R}^T+\frac{2}{h}\mathbf{M}+\frac{h}{2}\mathbf{K}) \tag{20}$$

$$\mathbf{a}_4=\mathbf{L}\mathbf{E}^{-1}\mathbf{W} \tag{21}$$

Substituting Eq.(18) into Eq.(19), we have

$$\begin{aligned}\left[\mathbf{a}_3-\mathbf{a}_4\mathbf{a}_1\mathbf{E}^{-1}\mathbf{R}^T\right]\mathbf{u}_{j+1}&=\left[\mathbf{a}_3-\mathbf{a}_4\mathbf{a}_1\mathbf{E}^{-1}\mathbf{R}^T-h\mathbf{K}\right]\mathbf{u}_j+2\mathbf{M}\mathbf{v}_j\\&+\left[\mathbf{a}_4-\mathbf{a}_4\mathbf{a}_1\mathbf{a}_2\right]\mathbf{y}_j+\int_{jh}^{(j+1)h}\mathbf{f}(t)\,\mathrm{d}t\end{aligned} \tag{22}$$

Eq.(22) together with Eqs.(16), (18) allow $\mathbf{u}_{j+1}$ to be calculated by means of $\mathbf{u}_j$. The above process can be applied in both cases when $\mathbf{C}_k$ matrices are rank deficient or of full rank

4 SUMMARY OF THE METHOD

Based on the procedure mentioned above, the time-domain response of a various damping models can be obtained followed by the next steps:

(1) Transform each various damping model into the formalism of Eq.(5) and obtain the corresponding computing matrices $\mathbf{P}_k, \mathbf{Q}_k, \mathbf{E}_k, \mathbf{W}_k, \mathbf{L}_k, \mathbf{R}_k$

(2) Compute matrices $\mathbf{E}, \mathbf{W}, \mathbf{L}, \mathbf{R}$

(3) Select a sufficiently small time step size h and construct the computing matrices

$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ according to Eqs.(18)-(21)

(4) Calculate the matrices as

$$\mathbf{S}_1 = \left[\mathbf{a}_3 - \mathbf{a}_4\mathbf{a}_1\mathbf{E}^{-1}\mathbf{R}^T\right] \tag{23}$$

$$\mathbf{S}_2 = \left[\mathbf{a}_3 - \mathbf{a}_4\mathbf{a}_1\mathbf{E}^{-1}\mathbf{R}^T - h\mathbf{K}\right] = \mathbf{S}_1 - h\mathbf{K} \tag{24}$$

$$\mathbf{S}_3 = \left[\mathbf{a}_4 - \mathbf{a}_4\mathbf{a}_1\mathbf{a}_2\right] \tag{25}$$

(5) Solve for displacements

$$\mathbf{S}_1\mathbf{u}_{j+1} = \mathbf{S}_2\mathbf{u}_j + 2\mathbf{M}\mathbf{v}_j + \mathbf{S}_3\mathbf{y}_j + \int_{jh}^{(j+1)h} f(t)\,\mathrm{d}t \tag{26}$$

Velocities

$$\mathbf{v}_{j+1} = \frac{2}{h}\left[\mathbf{u}_{j+1} - \mathbf{u}_j\right] - \mathbf{v}_j \tag{27}$$

Auxiliary variable

$$\mathbf{y}_{j+1} = \mathbf{a}_1\left[-\mathbf{E}^{-1}\mathbf{R}^T(\mathbf{u}_{j+1} - \mathbf{u}_j) + \mathbf{a}_2\mathbf{y}_j\right] \tag{28}$$

5 NUMERICAL EXAMPLE

A four DOF system with two different damping models is considered as shown in Fig.1.

The equations of motion of this model system can be expressed by Eq.(1). The system matrices are given by

$$\mathbf{M}=\begin{bmatrix} m & & & \\ & m & & \\ & & m & \\ & & & m \end{bmatrix}, \mathbf{K}=\begin{bmatrix} 2k_e & -k_e & & \\ -k_e & 2k_e & -k_e & \\ & -k_e & 2k_e & -k_e \\ & & -k_e & 2k_e \end{bmatrix}, G_1(s)=a_0+\sum_{k=1}^{2}\frac{a_k}{s+b_k},$$

$$\mathbf{C}_1=\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, G_2(s)=\sum_{j=1}^{2}\frac{\mu_j}{s+\mu_j}, \mathbf{C}_2=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Here $m=1\text{kg}, k\ =100\,\text{N/m}, a_0=1.497\times10^{-2}, a_1=1.0132\times10^{4}, a_2=1.2022\times10^{4}$, $b_1=5.5893, b_2=102.59, \mu_1=5$ and $\mu_2=100$. It should be noted that $G_1(s)$ is a Biot damping model and $G_2(s)$ is an exponential damping model both with two series. Obviously, both the damping coefficient matrices are rank deficiency with $r_1=rank(\mathbf{C}_1)=1\le 4$ and $r_2=rank(\mathbf{C}_2)=1\le 4$. The order of the system matrices expressed by Eq.(10) in the state-space can be obtained as $m=2\times4+(2\times1+2\times1)=12$.

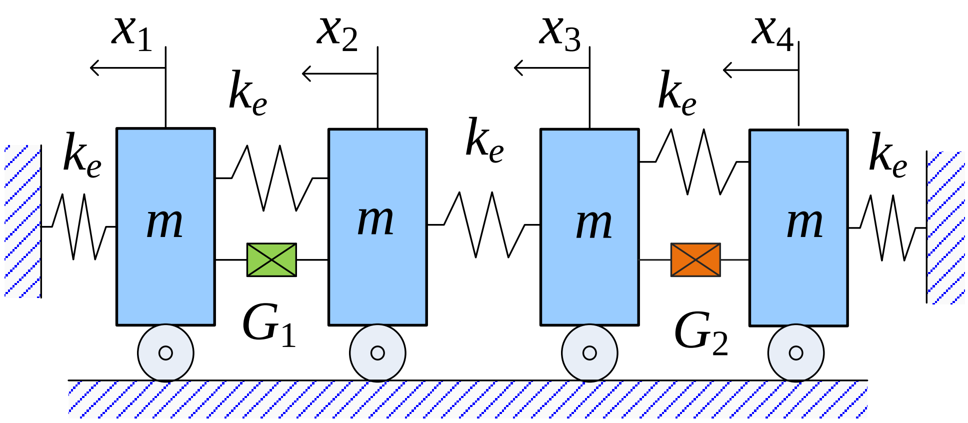

Figure 1： A four DOF system with two different damping models

We consider the dynamic response of the system subjected to a unit impulse excitation at the first DOF. The unit impulse excitation can be equally transformed into an unit initial displacement according to the theorem of impulse, that is, $\mathbf{f}(i\omega)=\mathbf{0}, \mathbf{u}_0=\mathbf{0}, \mathbf{v}_0=\{1,0,0,0\}^{\mathrm{T}}$. The response of the four masses are obtained using the above proposed direct time-domain approach with time step $h=0.01s$.

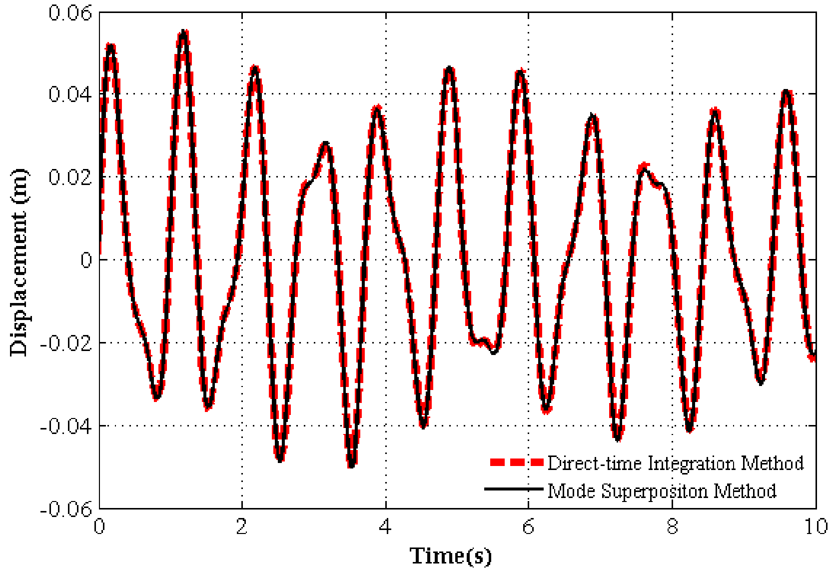

Figure 2： Displacement at the first DOF.

Fig.2 shows the response of the first DOF as a function of time. The results of the system on the same initial condition are obtained using the exact state-space mode superposition method and shown in the same figure. The eigenvalues and eigenvectors of the four DOF system are list in paper [32]. However, for large-scaled problems, it is often unnecessary and impossible to obtain all the eigenpairs of the system, the high-order mode truncation problem is frequently encountered and leads to errors [33-36]. But for the proposed direct-time domain method for various damping models, we need not compute the eigensolutions which will be a significant advantage over the traditional superposition method.

It is shown that the results obtained using both approaches match with excellent accuracy. The time-domain method proposed in this paper has a higher efficiency because the computation of state-space eigensolutions is avoided.

6 CONCLUSIONS

In this paper, a direct time-domain formulation of linear viscoelastic systems with various damping models has been presented. The various damping models described by a fraction formula of rational polynomials are attempting to express different damping models in a uniform way. The proposed method is on the base of an extended state-space formulation of various viscoelastic damping models. The method can be utilized in both conditions when the damping matrices $\mathbf{C}_k$ are of full rank or rank deficient. The advantage of the proposed method over the traditional mode superposition method is that the final recurrence formulas to be solved contain noting but a linear combination of the system matrices $\mathbf{M},\mathbf{C}_k,\mathbf{K}$ and time-stepping scheme *h*. We do not need to compute the eigenvalues and handle the non-viscous modes of the viscoelastic models. Thus, although the problem is originally organized in the extended state-space, the final solution is in effect implemented in the original reduced physical space. Finally, a four DOF system with two different damping models is used to show the validity of the proposed method.

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