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RAYLEIGH METHOD APPLIED TO A 46-M-HIGH CONCRETE MAST

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Abstract. *In this study, an analytical approach based on the Rayleigh method is adopted to calculate the first resonant frequency of a 46-m-high concrete mobile phone mast system, by considering the geometric stiffness, functions of the concentrated forces, and self-weight of the structure. It is important to bear in mind that actual structures are more complex than simple systems such as beams and columns because the properties of actual structures vary with their length. The mast is done in concrete its nonlinear behavior is linearized reducing the flexural stiffness and the ground is taken into account as distributed springs. Under geometric nonlinearity, the vibration frequency of the fundamental mode is calculated analytically and, for comparison, a finite element method (FEM)-based computer simulation is performed. Finally, the structural stiffness is evaluated. The results of the analytical approach are found to differ slightly from those of the FEM-based model.*

1 INTRODUCTION

Lee [1] relates that because beams are basic structural components, studies on free vibration of linear elastic beams under various conditions have been extensively reported for several decades. Accurate prediction of the natural frequencies of structures or structural components is important, especially in the design of structures subjected to dynamic loads. In various structural engineering fields, including civil, architectural, mechanical, and aeronautical engineering, tapered members, for example, are frequently used in structural designs for economical, aesthetic and geometrical reasons, goes on he.

The dynamic bending of such systems, which mingles with the flexural vibration of beams/columns, was initially investigated by Daniel Bernoulli in the late 18th century. The vibrations of Euler–Bernoulli beams have long been studied, since around 1750, Strutt [2]. As a beam, a column represents a continuous structural member, and its vibrations are governed by nonlinear partial differential equations, relates Norouzi [3], for which exact analytical solutions cannot be found, Awrejcewicz [4].

To investigate of the behavior of a continuous system with infinite degrees of freedom requires discretization techniques in which the structure is transformed into subsystems defined by points called joints. The discretization of differential equations was initialized by Euler through the first order discrete approximation. The Euler method is a one-step discrete method. Eventhough, this method provides a very low computational accuracy, one still used such a method in numerical computations because it is very simple to use, affirm Guo [5]. In fact, the discretization methods lead to the resolution of an algebraic problem for an initial continuous eigenvalue problem. The possibility to solve automatically the algebraic problem using a computer makes the discretization approach advantageous as compared to the initial continuous one, comments Challamel et al [6].

However, continuum systems can be associated with a system having a single degree of freedom, thereby restricting the mode in which the system will deform, and express the properties of such a system as functions of the generalized coordinates. This technique was adopted by Rayleigh [7] in his classic study of elastic system vibration, and his equations were found to be valid in the entire domain of the problem. However, actual structures are more complex than simple beams and columns, because their properties vary with their length and apply that method require a special attention. In order to evaluate the application of the Rayleigh method to an actual structure was selected one piece of concrete with high slenderness, for which the frequency of the first vibration mode was calculated analytically and by a computational model using the finite element method (FEM)**.**

2 CHARACTERISTICS OF THE 46-M-HIGH CONCRETE MAST

The structure is a concrete pole with high slenderness (slenderness of 334) with 46 m high. It has a hollow circular section of external diameter ϕ_{ext} and thickness of the wall of each segment *t*. The geometric details can be seen in shown in Figure 1 and Figure 2. In the structure there is a set of antennas and one platform on top, which totalize a mass of 1097.76 kg. There is even ladder, cables and guardrail, from 6 up to 46 m of height, which gives the structure an additional mass distributed of 40 kg/m. The lateral soil was represented by distributed spring stiffness (K_m) equal to 2668.93 kN/m^3. The density adopted for reinforced concrete foundation was 2500 kg/m^3, while the superstructure is considered to 2600 kg/m^3 concrete to be centrifuged. The foundation is of the kind shaft, relatively deep, with the following characteristics: base diameter 140 cm, shaft diameter 80 cm, length of shaft 580 cm and 20 cm base height. The concrete resistance is 45 MPa for the post, and 20 MPa for the foundation. Because the material has a nonlinear behavior or physical nonlinearity of concrete for is necessary take it into account. What will be done following the recommendations from Brazilian Association for Standardization [8], which preconize um reducer factor of 0.5.

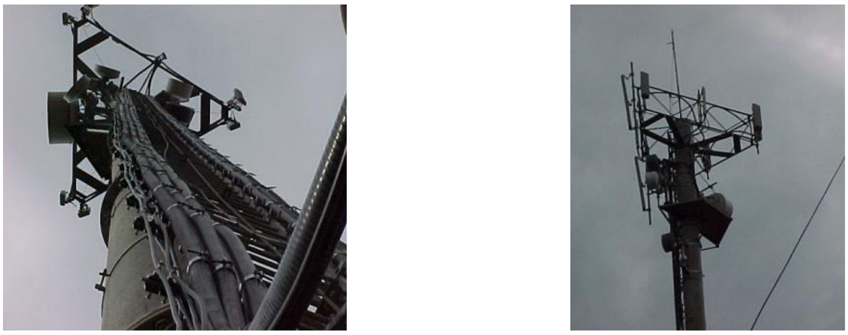

Figure 1: Mobile phone mast system (46 m high).

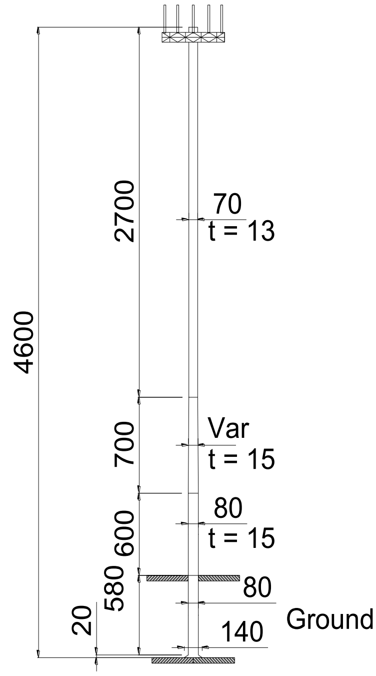

Figure 2: Geometric of the 46-m-high concrete mast (measurements in centimeters).

3 SYNTHETIC MATHEMATICAL DESCRIPTION

Below is described synthetically the mathematical bases of the Rayleigh Method. The basic concept underlying that method is the principle of conservation of energy in mechanical systems; therefore, it is applicable to linear and nonlinear structures according to Clough [9]. Temple [10] suggested that the fundamental principles developed by Rayleigh are applicable to both linear and nonlinear continuous systems with finite degrees of freedom. The purpose is to determine the fundamental period of vibration and to analyze the stability of elastic systems with the precision required for engineering problems. Toward this end, the principle of virtual work must be described by adequately chosen generalized coordinates and by a functional form that describes the first mode of vibration. At the end of the calculation, the equation of motion is written in terms of the generalized coordinates, from which one can extract the generalized elastic and geometric properties of the system. However, it is important to have in mind that for actual structure the integrals obtained by the method should be resolved within the limits established for each interval, i.e., the generalized properties calculated for each segment of the structure, with special attention to the geometric stiffness because each part must consider the normal force distributed in its range as well as forces that operate in the upper segments.

Consider the well-known trigonometric function

$$\phi(x) = 1 - \cos\left(\frac{\pi x}{2L}\right). \quad (1)$$

Applying the principle of virtual work and its adequate derivations, the dynamic properties of interest of the system can be obtained as follows. The conventional stiffness is given by

$$K_0 = \int_0^L EI\left(\frac{d^2\phi(x)}{dx^2}\right)^2 dx, \quad (2)$$

where E is the elastic modulus and I is the inertia of the section. The geometrical stiffness appears as a function of the axial load, which includes the self-weight, and it is expressed as

$$K_g = \int_0^L N(x)\left(\frac{d\phi(x)}{dx}\right)^2 dx, \quad (3)$$

where $N(x)$ is a function of the normal force given by $N(x) = \left[m_0 + \bar{m}\left(L - x\right)\right]g$.

If is necessary to consider the soil participation on the vibration of the system, one way is to associate it as distributed springs along of the height, for that should take the Eq.(4), where $k(x)$ is the spring parameter.

$$K_m = \int_0^L k(x)\phi(x)^2 dx. \quad (4)$$

The total generalized mass is given by $M = m_0 + m,$ where m_0 is the tip lumped mass at the element joint and

$$m = \int_0^L \bar{m}\left[\phi(x)\right]^2 dx, \text{ where } \bar{m} \text{ is the mass per unit length} \quad (5)$$

Finally, the natural cyclical frequency should be calculated as

$$\omega = \sqrt{\frac{K}{M}}, \text{since } K = K_0 - K_g + K_m, \quad (6)$$

For further details on the development of this specific analytical procedure, readers may refer to our previous work, Wahrhaftig [11].

4 RESULTS AND DISCUSSION

4.1 Geometric definitions and parameters of the Rayleigh method

The following ordered referring to the heights in the structure are defined: $\mathrm{L_1 = 0.2\ m}$, $\mathrm{L_2 = 6.0\ m}$, $\mathrm{L_3 = 12.0\ m}$, $\mathrm{L_4 = 19.0\ m}$, $\mathrm{L_5 = 46.0\ m}$. On the base foundation, one has: $\mathrm{D_1 = 140\ cm}$, $\mathrm{A_1 = \frac{\pi}{4} D_1^{\,2}}$, $\mathrm{I_1 = \frac{\pi}{64} D_1^{\,4}}$. On the shaft $\mathrm{D_2 = 80\ cm}$, $\mathrm{A_2 = \frac{\pi}{4} D_2^{\,2}}$, $\mathrm{I_2 = \frac{\pi}{64} D_2^{\,4}}$. The variable diameter on the segment between the base and the shaft is obtained by linear interpolation on the variable segment, doing: $\mathrm{D_1(x) = \frac{D_2 - D_1}{L_1} x + D_1}$. So, the area and inertia of the section are given by $\mathrm{A_1(x) = \frac{\pi}{4} D_1(x)^2}$ and $\mathrm{I_1(x) = \frac{\pi}{64} D_1(x)^4}$. Being D_3 and e_3 the external diameter and thickness of the initial section, one has $D_3 = 80$ cm and $t_3 = 13$ cm. Then, the internal diameter, area and inertia of the section are: $d_3 = D_3 - 2t_3$, $A_3 = \frac{\pi}{4}\left(D_3^{\ 2} - d_3^{\ 2}\right)$ and $I_3 = \frac{\pi}{64}\left(D_3^{\ 4} - d_3^{\ 4}\right)$. Similarly, D_5, t_5, d_5, A_5 and I_5 are the external diameter, the thickness, area and inertia to the last segment. Between these two segments there is a segment with variable section, of thickness equal to the previous ($t_4 = t_3$), such diameter can be obtained by linear interpolation similarly as done previously, using the expression $D_4(x) = \frac{D_4 - D_3}{L_4 - L_3}\left(x - L_3\right) + D_3$. Therefore, $d_4(x) = D_4(x) - 2t_4$, $A_4(x) = \frac{\pi}{64}\left(D_4(x)^4 - d_4(x)^4\right)$ and $I_4(x) = \frac{\pi}{64}\left(D_4(x)^4 - d_4(x)^4\right)$ are, respectively, the internal diameter , the area and the inertia of the variable section on that correspondent segment.

The parameters of the analytical procedure are listed in Table 1.

Table 1. Parameters of analytical procedure.

Parameter	Value
Elastic modulus of the mast concrete reduced	E = 18782971011 MPa
Elastic modulus of the foundation concrete reduced	E_1 = 12521.98 MPa
Density of structure and foundation concrete	ρ = 2600 kg/m^3, ρ_1 = 2500 kg/m^3
Lumped mass at the top	m_0 = 1097.76 kg
Additional distributed mass per unit height	$\bar{m}_e$ = 40 kg/m
Spring parameter	k_s = 2668.93 kN/m^3
Gravitational acceleration	g = 9.806650 m/s^2

4.2 Calculation of generalized mass of the Rayleigh method

Sub-indices in Roman numerals are introduced from this point on to prevent ambiguity in notation. The generalized mass is calculated using the following integrals.

$$m_1 = \int_0^{L_1} m_I(x)\phi(x)^2\,dx\text{, where } m_I(x) = \left(\frac{A_2 - A_1}{L_1} + A_1\right)\rho_1;$$

$$m_2 = \int_{L_1}^{L_2} m_{II}\phi(x)^2\,dx\text{, where } m_{II} = A_2\rho_1;\ m_3 = \int_{L_2}^{L_3} m_{III}\phi(x)^2\,dx\text{, where } m_{III} = A_3\rho + \bar{m}_e;$$

$$m_4 = \int_{L3}^{L_4} m_{IV}(x)\phi(x)^2\,dx\text{, where } m_{IV} = A_4(x)\rho + \bar{m}_e \text{ and } A_4(x) = \frac{\pi}{4}\left(D_4(x)^2 - d_4(x)^2\right); \tag{7}$$

$$\text{and } m_5 = \int_{L_4}^{L} m_V\phi(x)^2\,dx\text{, where } m_V = A_5\rho + \bar{m}_e.$$

The generalized distributed mass is given by $m = \sum_{i=1}^{5} m_i$ and the total generalized mass is given by $M = m_0 + m$. The total generalized mass of the structure is 7848.06 kg.

4.3 Calculation of generalized geometric stiffness by Rayleigh method

To compute the generalized geometric stiffness, it is necessary to determine the normal forces for the parts defined in the geometry. From the top to the bottom of the structure, the axial forces are

$$F_0 = m_0 g,\ F_5 = \int_{L_4}^{L} m_V g dx,\ F_4 = \int_{L_3}^{L_4} m_{IV}(x) g dx,$$
$$F_3 = \int_{L_2}^{L_3} m_{III} g dx,\ F_2 = \int_{L1}^{L_2} m_{II} g dx,\ \text{and}\ F_1 = \int_{0}^{L_1} m_I(x) g dx, \tag{8}$$

where g is the gravitational acceleration. The generalized axial force F is given by $F = \sum_{i=0}^{5} F_i$, and the geometric stiffness is calculated as follows:

$$K_{g5} = \int_{L_4}^{L} \left[F_0 + m_V (L - x) g \left(\frac{d}{dx}\phi(x) \right)^2 \right],$$
$$K_{g4} = \int_{L_3}^{L_4} \left[F_0 + F_5 + m_{IV}(x)(L_4 - x) g \left(\frac{d}{dx}\phi(x) \right)^2 \right],$$
$$K_{g3} = \int_{L_2}^{L_3} \left[F_0 + F_5 + F_4 + m_{III}(L_3 - x) g \left(\frac{d}{dx}\phi(x) \right)^2 \right], \tag{9}$$
$$K_{g2} = \int_{L_2}^{L_3} \left[F_0 + F_5 + F_4 + F_3 + m_{II}(L_2 - x) g \left(\frac{d}{dx}\phi(x) \right)^2 \right],\ \text{and}$$
$$K_{g1} = \int_{L_2}^{L_3} \left[F_0 + F_5 + F_4 + F_3 + F_3 + m_I(x)(L_1 - x) g \left(\frac{d}{dx}\phi(x) \right)^2 \right].$$

Thus, the generalized geometric stiffness K_g of the structure is given by $K_g = \sum_{i=1}^{5} K_{gi}$. The generalized geometric stiffness of the structure is 2.630744 kN/m.

4.4 Calculation of generalized conventional stiffness by Rayleigh method

The elastic geometric components are

$$K_{01} = \int_{0}^{L_1} E_1 I_1(x) \left(\frac{d^2}{dx^2}\phi(x) \right)^2 dx,\ K_{02} = \int_{L_1}^{L_2} E_1 I_2 \left(\frac{d^2}{dx^2}\phi(x) \right) dx,$$
$$K_{03} = \int_{L_2}^{L_3} EI_3 \left(\frac{d^2}{dx^2}\phi(x) \right)^2 dx,\ K_{04} = \int_{L_3}^{L_4} EI_4(x) \left(\frac{d^2}{dx^2}\phi(x) \right)^2 dx, \tag{10}$$
$$\text{and}\ K_{05} = \int_{L_4}^{L} EI_5 \left(\frac{d^2}{dx^2}\phi(x) \right)^2 dx.$$

The generalized elastic stiffness is given by the sum of the above components, as $K_0 = \sum_{i=1}^{5} K_{0i}$ .The generalized conventional stiffness is 8.068271 kN/m.

4.5 Calculation of generalized spring stiffness by Rayleigh method

Once that k_s is the spring factor, such value, for this specific case, is k_s = 2669 kN/m^3, the distributed springs, in the first and second segments are given by $k_1(x) = k_s D_1(x)$ and $k_2(x) = k_s D_2$. The generalized spring stiffness, designed by K_m, is calculated by means of follow equations:

$$K_m = \int_0^{L_1} k_1(x)\phi(x)^2\,dx + \int_{L_1}^{L_2} k_2(x)\phi(x)^2\,dx. \tag{11}$$

So, the generalized stiffness of the structure can be finally determined by the algebraic sum of the precedents parcels, so that is the total stiffness obtained by: $K = K_0 - K_g + K_m$. The generalized total stiffness is 6.560661 kN/m.

4.6 Calculation of frequency by Rayleigh method

The frequency of the first mode calculated numerically using Eq. (8) is 0.132477 Hz. If the geometric stiffness is not considered, linear analysis, the frequency is 0.172238 Hz.

4.7 Solution from a computational model based on the fem

The formulation corresponding to the finite element method (FEM) in relation to the mathematical procedure is the *geometric nonlinear formulation*. Then, the problem representing the modal analysis is resolved as

$$\det\left[\left[K\right] - \omega^2\left[M\right]\right] = 0 \tag{12}$$

with [K] being the stiffness matrix, composed by the conventional and geometric parcels and [M] is the mass matrix. In the classical modal analysis, this information is related to the natural frequencies of the system (eigenvalues), clears Shiki [12]. For comparison, a computational model based on the FEM was developed according to SAP2000 (Integrated software for structural analysis and design, Analysis Reference Manual, Computer and Structures, Inc., Berkeley, California, USA), a commercial software packed. The forces and parameters listed in Table 1 were applied to the model with the corresponding masses. The material was assumed to be isotropic, homogeneous, elastically linear, and with the parameters used in the analytical investigation in addition to a Poisson's ratio of 0.2. The spring value also was assigned to the foundation. The structures analyzed were modeled using 40 bar elements with constant and variable cross sections, as appropriate.

The modal analysis was performed over the stiffness came from a nonlinear static preprocessing. The first natural frequency calculated by FEM without geometric nonlinearity is 0.169961 Hz, and with geometric and material nonlinearity is 0.141285 Hz.

5 CONCLUSIONS

- This study employed an analytical procedure based on the Rayleigh method to calculate the initial undamped frequency of a 46-m-high concrete mast under geometric nonlinearity. Computational modeling based on finite element method (FEM) was performed for comparison.

- It was observed that the geometric stiffness represents 32.61% of the conventional stiffness. This implies that the reduction in the stiffness reduces the frequency in 15.51%. The same analysis using the FEM represents a reduction of 30.28%.

- When calculated analytically by the Rayleigh method, under geometric nonlinearity, the vibration frequency of the first natural mode was only 3% greater than that obtained by the FEM using a compatible formulation (0.145517 Hz for the former and 0.141285 Hz for latter). When computed without the geometric stiffness component, this frequency was 0.172238 Hz and 0.184065 Hz for both methods. So, one notes that the analytically solution based on the Rayleigh method has a good approximation with the nonlinear formulation based on the geometric matrix from the FEM. The results showed that there were no absolute differences between both solutions.

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