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IDENTIFICATION OF INTERVAL FIELDS FOR SPATIAL UNCERTAINTY REPRESENTATION IN FINITE ELEMENT MODELS

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Abstract. *In the context of integrating uncertainty and variability in Finite Element (FE) models, several advanced techniques for taking both inter- (between nominally identical parts) and intra-variability (spatial variability within one part) into account have recently been introduced. In the framework of non-probabilistic variability, especially the theory of Interval Fields (IF) has been proven to show promising results. Following this approach, variability in the input parameters of the FE model is introduced as the superposition of a number of base vectors scaled by interval factors. Application of the IF concept however requires identification of these parameters. Recent work has focused on the identification of interval uncertainty for the case of inter-variability. However, to the knowledge of the authors, no such techniques for identifying interval intra-variability are present in literature.*

This work focuses on finding a solution to the inverse problem, where the variability on the output side of the model is known from measurement data, but the spatial uncertainty on the input parameters is unknown. This paper proposes a methodology to solve this inverse problem. The uncertain simulation space, created by propagating an interval field throughout an FE model, is modelled using its convex hull. The same concept is used to model the uncertainty in the measurement space. A metric to describe the discrepancy between these convex hulls, based on the difference of their volumes and overlap, is defined and minimised in order to identify the spatial variability on the input side of the model. Validation of the methodology is performed using simulated measurement data. It is shown that numerically exact identification of a simulated measured IF is possible following the proposed methodology.

1 INTRODUCTION

During the last decades, the incorporation of non-determinism in Finite Element (FE) simulations received wide acceptance among academia and the industry for taking uncertainties that are omnipresent in our everyday life, already into account in an early design phase [1]. Especially for these design purposes, the study of inter-variability (variability between different nominal identical products) as well as intra-variability (variability of model properties within a single component, also referred to as 'spatial variability') proves to be extremely valuable. In this context, non-probabilistic approaches such as Interval Finite Elements (IFE) or Fuzzy Finite Elements (FFE) were introduced, where the uncertainty is depicted as an interval or fuzzy set and thus propagated. In the IFE concept, the uncertainty is modelled as two crisp bounds on the variable between which all possible values lie, and thus propagated. The FFE concept is an extension to this principle, where the membership of a variable to an interval is considered as a continuous function, ranging from 0 (the variable is certainly no part of the interval) to 1 (the variable lies for sure in the interval). [2–5].

In the specific case of spatial uncertainty for the description of intra-variability, two extreme methodologies for the representation of the spatial character of the uncertainty are usually adopted [6, 7]. The first method consists of defining a single interval to represent a spatial varying parameter for the entire model, assuming it is homogeneous throughout the modelled geometry. The second method considers one interval per individual element in the FE model, neglecting all possible dependency throughout the model. The former method poses a serious underestimation of the complexity of the spatial nature of the uncertainty; the latter gives rise to physically infeasible realizations of the uncertainty throughout the model. Moreover, by neglecting all possible dependency, the numerical cost of evaluating the model increases drastically due to the high dimensionality of the input space. As a solution to this, the framework of Interval Fields (IF) was recently introduced [6–8]. The Interval Field concept is based on the superposition of n_b base vectors ,$\boldsymbol{\psi}_i$, which represent a set of spatial uncertainty *patterns*, scaled by independent interval factors α_i^I, which cover the uncertainty.

Despite the current availability of these very advanced techniques for the uncertainty propagation, the use of representative measurement data to quantify the input uncertainty of FE analyses remains nowadays the major challenge. Especially for a designer who is facing a substantial (and often beforehand unknown) variability in the realisation of his design, this knowledge is indispensable when creating reliable products. In literature, some techniques are available to improve the quality of non-deterministic analyses (see e.g., [9–11]). Techniques for updating IFE and FFE models on experimental data were introduced only very recently, based on the minimization of the difference between the extreme boundaries on simulation and measurement data. **Haag et al.** introduced in this context an inverse methodology for the identification of the parameters of FFE models [12]. **Khodaparast et al.** employed a Kriging predictor as a meta-model for the updating of interval [13] and fuzzy [14] finite element models. **Erdogan and Bakir** used a hybrid combination of a Genetic Algorithm and Particle Swarm as updating scheme in the context of the identification of uncertainty in Fuzzy Finite Element models [15]. **Fedele et al.** used an adjoint-based optimisation technique to estimate the bounds of a model parameter, based on uncertain raw measurements [16]. They handled overestimation of the interval width using the inclusion isotonicity property of interval arithmetic. Finally, **Fang et al.** employed interval response surface models for the inverse propagation of the interval uncertainty [17]. The proposed techniques however are incapable of dealing with spatially varying uncertainty, due to the assumption of homogeneity of the uncertainty throughout the model.

It is clear that in order to be representative for the actual intra-variability present in the final components, the Interval Field concept needs an accurate quantification of the driving parameters: the base vectors and interval factors for IF. In the specific case of elastostatic mechanical properties, this quantification is impossible to attain following traditional experimental approaches such as uniaxial tensile tests. This paper therefore focuses on the identification of the spatial uncertainty on the input parameters of an FE model following an inverse approach. The variability in the measurement data is represented as the convex hull in q-dimensions of the complete set of measurement replica. Identification is performed, based on the minimisation of the variability in the simulation output and this convex hull following an Interior-point minimisation scheme. The concept is demonstrated for Interval FE models, but is easily extendable to Fuzzy FE models. Simulated measurement data are used for demonstrating the developed principles.

2 The Interval Finite Element method

The goal of an Interval FE calculation is to find the bounds on the uncertainty in a general multidimensional result **y** of an FE calculation $f()$, based on an interval description of the uncertainty in a number of physical properties $\boldsymbol{\gamma}^{\mathbf{I}}$, which are defined as parameters on the input side of the model. Formally, the uncertainty space $\tilde{\mathbf{y}}$ on the output side of an interval FE calculation is defined as:

$$\tilde{\mathbf{y}} = \left\{\mathbf{y}|\mathbf{y} = f(\gamma) \wedge \gamma \in \boldsymbol{\gamma}^{\mathbf{I}}\right\} \tag{1}$$

By definition, an interval parameter x is indicated using superscript I: x^I. Vectors are expressed as lowercase boldface characters **x**, whereas matrices are depicted as uppercase boldface characters **X**. Usually, $\mathbf{y}^{\mathbf{I}}$ is calculated following an optimisation approach, where the extreme responses of each dimension $\underline{y}_i$ and $\overline{y}_i$ are determined by searching the domain, defined by $\boldsymbol{\gamma}^{\mathbf{I}}$ [3]:

$$\begin{aligned} \underline{y}_i &= \min_{\boldsymbol{\gamma}\in\boldsymbol{\gamma}^I} \quad f_i(\boldsymbol{\gamma}) \qquad i = 1, ..., d \\ \overline{y}_i &= \max_{\boldsymbol{\gamma}\in\boldsymbol{\gamma}^I} \quad f_i(\boldsymbol{\gamma}) \qquad i = 1, ..., d \end{aligned} \tag{2}$$

Herein, $y_i^I = [\underline{y}_i; \overline{y}_i]$ is the result interval scalar for the i^{th} node of the model. Solution of eq. (2) returns the smallest hypercubic approximation $\mathbf{y}^{\mathbf{I}}$ of the uncertainty in the solution set $\tilde{\mathbf{y}}$. For the remainder of the text, interval parameters are either represented using the bounds of the interval $x^I = \{\underline{x}, \overline{x}\}$ or the centerpoint $\hat{x} = \frac{\underline{x}+\overline{x}}{2}$ and interval radius $r_x = \frac{\overline{x}-\underline{x}}{2}$.

In the case of spatial interval uncertainty, the input of the IFE model is given as an Interval Field (IF) $\boldsymbol{\gamma}_{\mathbf{F}}^{\mathbf{I}}(\boldsymbol{r})$. The description of an IF is based on the superposition of n_b base vectors $\boldsymbol{\psi}_i$, which represent a set of spatial uncertainty patterns, scaled by independent interval scalars α_i^I, which cover the uncertainty. In this way the dependency between locally defined intervals is given by the base vectors, while the local intervals themselves remain independent. Moreover, the dimensionality of the input space is reduced drastically as compared to defining a local interval for each node of the model. This favours the numerical cost of an evaluation of the IF FE model. Formally, an interval field for the description of spatial uncertainty in the input parameters $\boldsymbol{\gamma}_{\mathbf{F}}^{\mathbf{I}}(\mathbf{r})$ is expressed in the explicit form as:

$$\boldsymbol{\gamma}_{\mathbf{F}}^{\mathbf{I}}(\mathbf{r}) = \sum_{i=1}^{n_b} \alpha_i^I \boldsymbol{\psi}_i(\mathbf{r}) \tag{3}$$

3 Identification of interval fields

The application of the interval field approach for the representation of spatial uncertainty requires the identification of its constituting parameters. The methodology starts from the expert knowledge of the analyst on the amount and shape of the base functions necessary to represent the interval field at the input side of the model and incorporates measurement data to identify the amount of uncertainty, which is captured by the interval vector $\boldsymbol{\alpha}^I$.

The propagation of the uncertainty in the physical parameters $\boldsymbol{\gamma}_{\mathbf{F}}^{\mathbf{I}}(\mathbf{r})$ of an FE model typically yields a d-dimensional uncertainty space $\tilde{\mathbf{y}}$, of which $\mathbf{Y}_s \in \mathbb{R}^{d\times q}$ is the matrix containing all q propagated realisations of $\boldsymbol{\gamma}_{\mathbf{F}}^{\mathbf{I}}(\mathbf{r})$. In the context of identifying the input interval field, a quantification of $\mathbf{Y}_s$ is necessary to steer the updating procedure. Therefore its convex hull $\mathbf{C}_s$ and corresponding volume V_s are calculated. In general, the convex hull of a finite set of vectors in Euclidean space is defined as the set of all convex combinations of these vectors. It may also be shown that the convex hull is the smallest possible convex set, encompassing the vectors in $\mathbf{Y}_s$. The uncertainty space, spanned by the computed realizations in $\mathbf{Y}_s$, may contain any combination of output quantities, ranging from nodal output quantities such as displacements, stresses or temperatures to global quantities such as eigenfrequencies and -modes.

The measurement data are obtained trough an experimental procedure where the model is physically tested in analogy to the FE model under consideration. The scattered measurement points, having the same dimensionality d as $\tilde{\mathbf{y}}$, are grouped in $\mathbf{Y}_m \in \mathbb{R}^{d\times t}$, with t the number of test replica. In general, the dimensionality of the experimental and simulation data is not identical. However, through the usage of data interpolation for the measurement data or model reduction techniques for the simulation model, this condition can be achieved. Analogous to the simulation data, the convex hull $\mathbf{C}_m$ and its corresponding volume V_m are calculated. The QHULL library is used to compute the convex hulls. This library makes use of the "Quickhull" algorithm, as developed by **Barber et. al** [18].

Identification of $\boldsymbol{\alpha}^I$ is made through the minimisation of a cost function $\delta(\boldsymbol{\alpha}^I)$, expressing the discrepancy between the output space of the IFE simulation $\mathbf{Y}_s$ and repeated measurement data $\mathbf{Y}_m$. $\delta(\boldsymbol{\alpha}^I)$ is calculated, based on the geometrical properties of the d-dimensional convex hulls $\mathbf{C}_m$ and $\mathbf{C}_s$. More specifically, the difference between the volumes V_m and V_s of the respective hulls and the volume V_o of the intersection $\mathbf{C}_o$ of $\mathbf{C}_m$ and $\mathbf{C}_s$ is taken into account. Due to the convexity of $\mathbf{C}_m$ and $\mathbf{C}_s$, $\mathbf{C}_o$ is also convex. When there is no overlap (i.e. $\mathbf{C}_o = \emptyset$) present between $\mathbf{C}_m$ and $\mathbf{C}_s$, a descent direction for the optimisation problem is ensured by incorporating the squared L_2 norm of the difference between the center of gravity of respectively $\mathbf{C}_m$ ($\mathbf{c}_m$) and $\mathbf{C}_s$ ($\mathbf{c}_s$). Formally, $\delta(\boldsymbol{\alpha}^I)$ is defined as:

$$\delta(\boldsymbol{\alpha}^I) = \left(\left(1 - \frac{V_s}{V_m}\right)^2 + \left(1 - \frac{V_o}{V_m}\right)^2 + \|\mathbf{c}_m - \mathbf{c}_s\|_2^2\right)^{1/2} \tag{4}$$

The goal interval vector $\boldsymbol{\alpha}^{I,*}$ is finally identified according to:

$$\boldsymbol{\alpha}^{I,*} = argmin\ \delta(\boldsymbol{\alpha}^I) \tag{5}$$

4 VALIDATION

Validation the developed methodology is performed using a dynamic model of a 1-dimensional cantilever beam, shown in figure 1. A finite element model containing 10 quadratic shell elements with 4 nodes is constructed and evaluated for the first 10 eigenfrequencies f_e. Without the loss of generality, it is assumed that the stiffness of the beam is subjected to spatial uncertainty, which is modelled using the Interval Field concept, with $\mathbf{r}_1 = 2$ and $\mathbf{r}_2 = 7$, with locally defined intervals $\boldsymbol{\alpha}_1^I$ and $\boldsymbol{\alpha}_2^I$ capturing the uncertainty. Hyperbolic base functions $\boldsymbol{\psi}_i(\mathbf{r})$ are assumed, which are known *a priori*. This leads to a simulation space $\mathbf{Y}_s \in \mathbb{R}^{q \times d}$, with $d = 10$. In this specific case, the Interval Field FE model is solved using the vertex method, leading to $q = 2^2 = 4$ realisations of the simulation space.

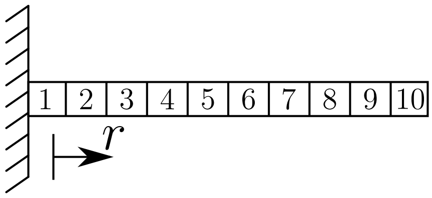

Figure 1: Illustration of the Finite-Element model of the cantilever beam on which the methodology is validated. **r** indicates the positive direction along the beam, the numbers in the finite elements indicate the element numbers.

For benchmarking purposes, measurement data is simulated using an *a priori* defined interval field, based on known local intervals $\boldsymbol{\alpha}^I \in \mathbb{IR}^2$, located at $\mathbf{r}_1$ and $\mathbf{r}_2$. Concretely, the goal local intervals are defined as $\alpha_1^I = [1.35e6; 1.55e6]$ and $\alpha_2^I = [1.65e6; 1.85e6]$.

The measurement space $\mathbf{Y}_m \in \mathbb{R}^{t \times d}$ is generated by performing $t = 500$ Monte Carlo simulations, while assuming a uniform distribution between the extreme boundaries of the local intervals $\boldsymbol{\alpha}^I$. Thus, 500 deterministic fields on the stiffness are solved for the first 10 f_e, in order to construct the 10-dimensional measurement space.

The results of the identification procedure are shown in table 1. It can be seen that the proposed methodology is capable of determining the goal interval field, with a local maximum error of 0.042%. This error could be improved by tightening the convergence criteria on the optimization process or propagating more Monte Carlo samples for the representation of the measurement data. However, because it is already negligible, it can be defined as *numerically exact*. The error listed in table 1 is defined as:

Table 1: Results of the identification procedure on an Interval Field capturing the spatial variability in the cantilever beam example.

	$\underline{\boldsymbol{\alpha}}_1^I$ $[Pa]$	$\overline{\boldsymbol{\alpha}}_1^I$ $[Pa]$	$\underline{\boldsymbol{\alpha}}_2^I$ $[Pa]$	$\overline{\boldsymbol{\alpha}}_2^I$ $[Pa]$
Goal	$1.35 \cdot 10^6$	$1.65 \cdot 10^6$	$1.55 \cdot 10^6$	$1.85 \cdot 10^6$
Initial	$1.00 \cdot 10^6$	$1.1 \cdot 10^6$	$1.81 \cdot 10^6$	$2.30 \cdot 10^6$
Initial error (%)	15.5%	35.5%	9.7%	24.3%
Result	$1.356 \cdot 10^6$	$1.652 \cdot 10^6$	$1.554 \cdot 10^6$	$1.849 \cdot 10^6$
Error (%)	0.042%	0.025%	0.001%	0.007%

Figure 2 shows the convergence plots. Absolute convergence is obtained after 37 iterations of the optimization process, whereas the locally defined input intervals do not change significantly after 22 iterations. The discrepancy metric δ, defined in equation 4 decreases over the optimization from 8.99 to 0.017.

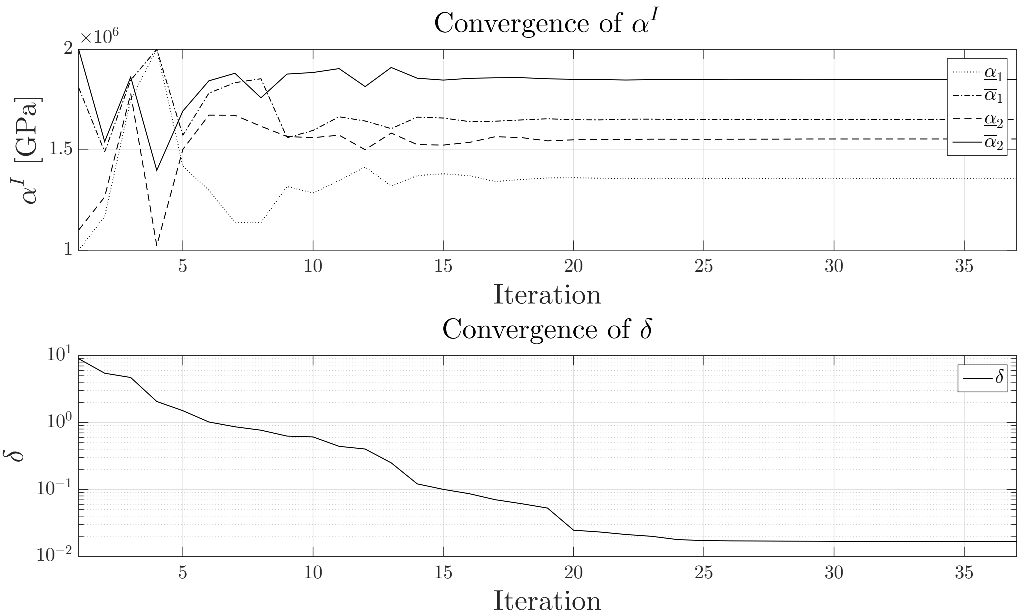

Figure 2: Convergence of the updating procedure. As can be noted, convergence is approximately obtained after 20 iterations. Absolute convergence is only obtained after 37 iterations.

Finally, figure 3 shows a selection of the convex hull over 4 sets of eigenfrequencies before and after the updating procedure. The accurate identification of the interval field encompassing the measurement data is also evident from these plots.

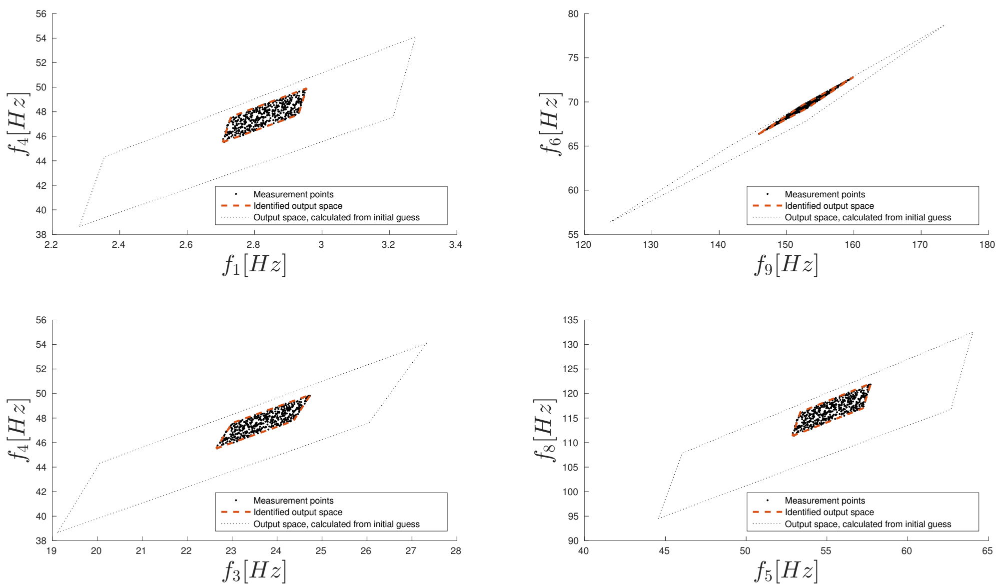

Figure 3: Illustration of the extreme realizations of the input field $\gamma_F^I(\mathbf{r})$ before and after identification, as well as the *goal* input field as a function of the element number.

5 CONCLUSIONS AND FUTURE WORK

In order to represent spatial uncertainty in the physical model parameters constituting the input of a Finite Element (FE) model, the Interval Field theorem proves to deliver accurate results with a limited set of data. The use of this concept however, requires the identification of its driving parameters, the interval factor $\boldsymbol{\alpha}^I$ and a number n_b of base vectors $\boldsymbol{\psi}_i(\mathbf{r})$. A general methodology to identify $\boldsymbol{\alpha}^I$, based on the incorporation of measurement data was introduced. The concept of the identification is based on the convex properties of the uncertainty space, stemming from propagating the input interval field through the interval FE solver on the one hand, and the calculation of the convex hull of the measurement data on the other hand. A measure of discrepancy is defined, and identification is performed by minimising this measure.

The methodology was validated on an industrially sized case study of an axisymmetric model of a pressure vessel using simulated measurement data. It was found that $\boldsymbol{\alpha}^I$ could be identified up to *numerically exact*, using the proposed methodology, within a limited number of iterations.

Future work will be aimed at extending the developed methodology towards the identification of base vectors $\boldsymbol{\psi}_i(\mathbf{r})$ and dimensionality of input space n_b, based on the defined discrepancy measure. Also techniques for the reduction of the output dimensionality will be considered for speeding up the updating process. Finally, also the incorporation of real life measurement data, e.g. coming from full-field strain measurement techniques for elastostatic applications or eigenfrequencies for structural dynamic applications is foreseen in the near future.

REFERENCES

[1] D. Moens and D. Vandepitte, "Recent advances in non-probabilistic approaches for non-deterministic dynamic finite element analysis," *Archives of Computational Methods in Engineering*, vol. 13, no. 3, pp. 389–464, 2006.

[2] L. Farkas, D. Moens, D. Vandepitte, and W. Desmet, "Fuzzy finite element analysis based on reanalysis technique," *Structural Safety*, vol. 32, pp. 442–448, nov 2010.

[3] D. Moens and M. Hanss, "Non-probabilistic finite element analysis for parametric uncertainty treatment in applied mechanics: Recent advances," *Finite Elements in Analysis and Design*, vol. 47, no. 1, pp. 4–16, 2011.

[4] W. Desmet, D. Vandepitte, M. De Munck, W. Verhaeghe, and D. Moens, "Recent Advances in Interval and Fuzzy Finite Element," in *Proceedings of the 1st International Symposium on Uncertainty Quantification and Stochastic Modeling*, no. 2007, (Sao Sebastiao, Brazil), 2012.

[5] M. Hanss, "The transformation method for the simulation and analysis of systems with uncertain parameters," *Fuzzy Sets and Systems*, vol. 130, pp. 277–289, sep 2002.

[6] W. Verhaeghe, W. Desmet, D. Vandepitte, and D. Moens, "Random field expansion with interval correlation length using interval fields," in *Eurodyn 2011*, (Leuven), pp. 2662 – 2667, apr 2011.

[7] W. Verhaeghe, W. Desmet, D. Vandepitte, and D. Moens, "Interval fields to represent uncertainty on the output side of a static FE analysis," *Computer Methods in Applied Mechanics and Engineering*, vol. 260, no. 0, pp. 50–62, 2013.

[8] W. Verhaeghe, W. Desmet, D. Vandepitte, and D. Moens, "Interval fields: Capturing spatial uncertainty within a non-probabilistic Framework," in *Safety, Reliability, Risk and Life-Cycle Performance of Structures and Infrastructures - Proceedings of the 11th International Conference on Structural Safety and Reliability, ICOSSAR 2013*, no. 2, pp. 2963–2969, 2013.

[9] Q. Rui, H. Ouyang, and H. Y. Wang, "An efficient statistically equivalent reduced method on stochastic model updating," *Applied Mathematical Modelling*, vol. 37, pp. 6079–6096, apr 2013.

[10] D. Straub and I. Papaioannou, "Bayesian Updating with Structural Reliability Methods," *Journal of Engineering Mechanics, Trans. ASCE*, vol. 141, no. 3, pp. 1–34, 2015.

[11] I. Boulkaibet, L. Mthembu, T. Marwala, M. I. Friswell, and S. Adhikari, "Finite element model updating using the shadow hybrid Monte Carlo technique," *Mechanical Systems and Signal Processing*, vol. 52-53, no. 1, pp. 115–132, 2015.

[12] T. Haag, M. Hanss (sup.), D. Moens (cosup.), and L. Gaul (cosup.), *Forward and Inverse Fuzzy Arithmetic for Uncertainty Analysis with Applications to Structural Mechanics*. Phd thesis, Universität Stuttgart, 2012.

[13] H. H. Khodaparast, J. E. Mottershead, and K. J. Badcock, "Interval model updating with irreducible uncertainty using the Kriging predictor," *Mechanical Systems and Signal Processing*, vol. 25, pp. 1204–1206, may 2011.

[14] H. Haddad Khodaparast, Y. Govers, S. Adhikari, M. Link, M. I. Friswell, J. E. Mottershead, J. Sienz, and H. H. Khodaparast, "Fuzzy model updating and its application to the DLR AIRMOD test structure," in *Proceedings of the International Conference on Uncertainty in Structural Dynamics, USD 2014*, pp. 4509–4522, 2014.

[15] Y. Serhat Erdogan and P. Gundes Bakir, "Inverse propagation of uncertainties in finite element model updating through use of fuzzy arithmetic," *Engineering Applications of Artificial Intelligence*, vol. 26, no. 1, pp. 357–367, 2013.

[16] F. Fedele, R. L. Muhanna, N. Xiao, and R. L. Mullen, "Interval-Based Approach for Uncertainty Propagation in Inverse Problems," *Journal of Engineering Mechanics*, vol. 4, pp. 1–7, may 2014.

[17] S.-E. Fang, Q.-H. Zhang, and W.-X. Ren, "An interval model updating strategy using interval response surface models," *Mechanical Systems and Signal Processing*, vol. 60-61, pp. 909–927, 2015.

[18] C. B. Barber, D. P. Dobkin, and H. Huhdanpaa, "The quickhull algorithm for convex hulls," *ACM Transactions on Mathematical Software*, vol. 22, no. 4, pp. 469–483, 1996.