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DIRECT EDDY CURRENT METHOD FOR VOLUMETRIC FLAWS OF CYLINDRICAL SHAPE

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Abstract. *A quasi-analytical method for the solution of direct eddy current testing problems for the case of cylindrical volumetric flaws is presented in the paper. The method is based on a simple physical assumption that the electromagnetic field induced by a coil carrying alternating current is exactly equal to zero at a sufficiently large radial distance from the coil. The axis of the coil concides with the axis of a cylindrical flaw. The method of truncated eigenfunction expansions is used to compute the change in impedance of the coil. Complex eigenvalues are computed numerically using the method which does not require initial approximation for the eigenvalue. Computations are presented for different values of the parameters of the problem. Calculated change in impedance is compared with numerical results obtained by means of Comsol Multiphysics software. Good agreement between quasi-analytical method and numerical solution is found.*

1 INTRODUCTION

Eddy currents are widely used in nondestrutive testing of electrically conducting materials. Identification of flaws in conducting media is one of the goals of nondestructive testing. This is a complicated inverse problem which can be solved if a solution of a direct problem is available. In cases where a conducting medium is unbounded in one or two spatial dimensions solutions of a direct problem can be found by the method of integral transforms [1], [2]. In practice, however, finite size of a conducting medium has to be taken into account especially in cases where the size of an excitation coil is comparable to the size of the medium. Method of truncated eigenfunction expansions (known as the TREE method in the literature) is suggested in [3], [4] in order to construct solutions of direct problems for media of finite size. The method is based on a simple physical assumption that the electromagnetic field induced by a coil carrying alternating current is exactly equal to zero at a sufficiently large radial distance from the coil. This assumption allows one to extend the class of problems which can be solved by a quasi-analytical method i.e., by the method of truncated eigenfunction expansions. Examples of the application of the TREE method to the solution of direct eddy current testing problems with cylindrical symmetry can be found in [3], [5], [6], [7]. The case of an asymmetric infinite cylindrical flaw is considered in [8]. In the present paper we present the solution of eddy current testing problem for the case of a cylindrical flaw in a two-layer plate. Such a model can be used to assess the effect of corrosion in metal coatings.

2 MATHEMATICAL FORMULATION OF THE PROBLEM

Consider an air core coil located above an electrically conducting nonmagnetic two-layer plate of finite size. A flaw in the form of a circular cylinder is located in the medium. It is assumed that the axis of the flaw coincides with the axis of the coil. An example of the flaw at the surface of a two-layer plate is shown in Fig. 1.

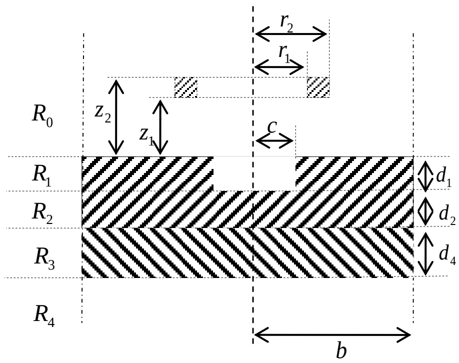

Figure 1: A coil carrying alternating current above a conducting two-layer plate with a flaw.

It is assumed that the electromagnetic field is equal to zero at a sufficiently large distance b from the axis of the coil. Using the cylindrical symmetry of the problem we assume that the vector potential has only one non-zero component in the azimuthal direction. As usual in such type of problems, we consider a single-turn coil of radius r_0 located at the distance h above the top surface of the plate. The solution for the case of a coil with finite dimensions can be found by the superposition principle. The system of the Maxwell's equations in each of the regions R_0–R_4 shown in Fig. 1 has the form:

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r}\frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} \quad = \quad -\mu_0 I \delta(r - r_0)\delta(z - h), \qquad (1)$$

$$\begin{aligned}
\frac{\partial^2 A_1}{\partial r^2} + \frac{1}{r}\frac{\partial A_1}{\partial r} - \frac{A_1}{r^2} + \frac{\partial^2 A_1}{\partial z^2} - j\omega\sigma\mu_0 A_1 &= 0, && (2)\\
\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r}\frac{\partial A_2}{\partial r} - \frac{A_2}{r^2} + \frac{\partial^2 A_2}{\partial z^2} - j\omega\sigma_1\mu_0 A_2 &= 0, && (3)\\
\frac{\partial^2 A_3}{\partial r^2} + \frac{1}{r}\frac{\partial A_3}{\partial r} - \frac{A_3}{r^2} + \frac{\partial^2 A_3}{\partial z^2} - j\omega\sigma_2\mu_0 A_3 &= 0, && (4)\\
\frac{\partial^2 A_4}{\partial r^2} + \frac{1}{r}\frac{\partial A_4}{\partial r} - \frac{A_4}{r^2} + \frac{\partial^2 A_4}{\partial z^2} &= 0, && (5)
\end{aligned}$$

where σ_1 and σ_2 are the electrical conductivities of regions R_2 and R_3, respectively, $\sigma = \sigma_1$ if $c < r < b$ and $\sigma = 0$ if $0 \le r < c$.

The boundary conditions at $r = b$ are

$$A_i|_{r=b} = 0, \quad i = 0, 1, ..., 4. \tag{6}$$

The boundary conditions at $r = c$ have the form

$$A_1^{air}|_{r=c} = A_1^{con}|_{r=c}, \quad \frac{\partial A_1^{air}}{\partial r}|_{r=c} = \frac{\partial A_1^{con}}{\partial r}|_{r=c}, \tag{7}$$

where we use the notations A_1^{air} and A_1^{con} in regions $0 \le r < c$ and $c < r < b$, respectively.

The conditions at the surfaces $z = 0$, $z = -d_1$, $z = -d_3$ and $z = -d_5$, where $d_3 = d_1 + d_2$, $d_5 = d_3 + d_4$ are

$$A_0|_{z=0} = A_1^{air}|_{z=0}, \quad \frac{\partial A_0}{\partial z}|_{z=0} = \frac{\partial A_1^{air}}{\partial z}|_{z=0} \quad 0 \le r < c, \tag{8}$$

$$A_0|_{z=0} = A_1^{con}|_{z=0}, \quad \frac{\partial A_0}{\partial z}|_{z=0} = \frac{\partial A_1^{con}}{\partial z}|_{z=0} \quad c < r < b. \tag{9}$$

$$A_1^{air}|_{z=-d_1} = A_2|_{z=-d_1}, \quad \frac{\partial A_1^{air}}{\partial z}|_{z=-d_1} = \frac{\partial A_2}{\partial z}|_{z=-d_1} \quad 0 \le r < c. \tag{10}$$

$$A_1^{con}|_{z=-d_1} = A_2|_{z=-d_1}, \quad \frac{\partial A_1^{con}}{\partial z}|_{z=-d_1} = \frac{\partial A_2}{\partial z}|_{z=-d_1} \quad c < r < b. \tag{11}$$

$$A_2|_{z=-d_3} = A_3|_{z=-d_3}, \quad \frac{\partial A_2}{\partial z}|_{z=-d_3} = \frac{\partial A_3}{\partial z}|_{z=-d_3}, \tag{12}$$

$$A_3|_{z=-d_5} = A_4|_{z=-d_5}, \quad \frac{\partial A_3}{\partial z}|_{z=-d_5} = \frac{\partial A_4}{\partial z}|_{z=-d_5}. \tag{13}$$

In addition, the solution is bounded as $z \to \pm\infty$:

$$A_0|_{z\to+\infty} = 0, \quad A_4|_{z\to-\infty} = 0. \tag{14}$$

3 PROBLEM SOLUTION

Problem (1)-(14) is solved by the method of separation of variables. It is convenient to represent the solution in region R_0 in the form

$$\begin{aligned}
A_{00}(r,z) &= \sum_{i=1}^{\infty} (D_{1i}\, e^{-\lambda_i z} J_1(\lambda_i r), && (15)\\
A_{01}(r,z) &= \sum_{i=1}^{\infty} (D_{2i}\, e^{-\lambda_i z} + D_{3i}\, e^{\lambda_i z}) J_1(\lambda_i r), && (16)
\end{aligned}$$

where $\lambda_i = \alpha_i / b$ and α_i are the roots of the equation

$$J_1(\alpha_i) = 0. \tag{17}$$

Here A_{00} and A_{01} represent the solutions in subregions $R_{00} = \{0 < z < h\}$ and $R_{01} = \{z > h\}$, respectively. The vector potential is continuous at $z = h$:

$$A_{00}|_{z=h} = A_{01}|_{z=h}. \tag{18}$$

Another condition at $z = h$ can be obtained from equation (1). Integrating (1) with respect to z from $h - \varepsilon$ to $h + \varepsilon$ and considering the limit as $\varepsilon \to +0$ we obtain

$$\frac{\partial A_{01}}{\partial z}|_{z=h} - \frac{\partial A_{00}}{\partial z}|_{z=h} = -\mu_0 I \delta(r - r_0). \tag{19}$$

Using (15), (16), (18) and (19) we obtain

$$A_{00}(r,z) = \sum_{i=1}^{\infty} (D_{2i}\, e^{-\lambda_i z} J_1(\lambda_i r) + \frac{\mu_0 I r_0}{b^2} \sum_{i=1}^{\infty} \frac{J_1(\lambda_i r_0) J_1(\lambda_i r)\, e^{\lambda_i (h-z)}}{\lambda_i J_0^2(\lambda_i b)}, \tag{20}$$

$$A_{01}(r,z) = \sum_{i=1}^{\infty} (D_{2i}\, e^{-\lambda_i z} J_1(\lambda_i r) + \frac{\mu_0 I r_0}{b^2} \sum_{i=1}^{\infty} \frac{J_1(\lambda_i r_0) J_1(\lambda_i r)\, e^{\lambda_i (z-h)}}{\lambda_i J_0^2(\lambda_i b)}. \tag{21}$$

The solution in region R_1 can be written in the form

$$A_1^{air}(r,z) = \sum_{i=1}^{\infty} (D_{4i} J_1(p_i r)\, e^{p_i z} + D_{5i} J_1(p_i r)\, e^{-p_i z}, \tag{22}$$

$$\begin{aligned} A_1^{con}(r,z) = & \sum_{i=1}^{\infty} [(D_{6i} J_1(q_i r) + D_{7i} Y_1(q_i r))\, e^{p_i z} \\ & + (D_{8i} J_1(q_i r) + D_{9i} Y_1(q_i r))\, e^{-p_i z}], \end{aligned} \tag{23}$$

where $p_i = \sqrt{q_i^2 + j\omega\sigma_1\mu_0}$. Using (22), (23) and the first condition in (7) we obtain

$$A_1^{air}(r,z) = \sum_{i=1}^{\infty} J_1(p_i r) T_1(q_i c)[\hat{D}_{6i}\, e^{p_i z} + \hat{D}_{8i}\, e^{-p_i z}], \tag{24}$$

$$A_1^{con}(r,z) = \sum_{i=1}^{\infty} J_1(p_i c) T_1(q_i r)[\hat{D}_{6i}\, e^{p_i z} + \hat{D}_{8i}\, e^{-p_i z}], \tag{25}$$

where

$$T_1(q_i r) = J_1(q_i r) Y_1(q_i b) - J_1(q_i b) Y_1(q_i r)$$

and $\hat{D}_{6i}$, $\hat{D}_{8i}$ are arbitrary constants. Using the second condition in (7) the following equation is obtained

$$q_i T_1'(q_i r) J_1(p_i c) = p_i J_1'(p_i c) T_1(q_i c). \tag{26}$$

Equation (26) is used to calculate complex eigenvalues p_i and the corresponding values q_i.

The solution in region R_2 is

$$A_2(r,z) = \sum_{i=1}^{\infty} (D_{10i}\, e^{p_{1i} z} + D_{11i}\, e^{-p_{1i} z}) J_1(\lambda_i r), \tag{27}$$

where $p_{1i} = \sqrt{\lambda_i^2 + j\omega\sigma_1\mu_0}$.

Similarly, the solution in region R_3 can be written in the form

$$A_3(r,z) = \sum_{i=1}^{\infty} (D_{12i}\, e^{p_{2i}z} + D_{13i}\, e^{-p_{2i}z}) J_1(\lambda_i r), \tag{28}$$

where $p_{2i} = \sqrt{\lambda_i^2 + j\omega\sigma_2\mu_0}$.

Finally, the bounded solution in region R_4 is

$$A_4(r,z) = \sum_{i=1}^{\infty} D_{14i}\, e^{\lambda_i z} J_1(\lambda_i r). \tag{29}$$

Unknown constants in (20)–(25), (27)–(29) can be found from the boundary conditions (8)–(13). It can be shown that the corresponding system of equations can be reduced to system of two linear algebraic equations with respect to the unknowns $\hat{D}_{6i}$ and $\hat{D}_{8i}$. It is neccesary to truncate the infinite series to a finite number of terms, m. Computational details for the solution of similar problems can be found elsewhere (see, for example, [3] and [5]).

Assuming that the coefficients $\hat{D}_{6i}$ and $\hat{D}_{8i}$ are calculated (this can be done using any linear solver) we obtain the induced vector potential of the single-turn coil in the form

$$A_0^{ind}(r,z,r_0,h) = \sum_{k=1}^{m} D_{2k}\, e^{-\lambda_k z} J_1(\lambda_k r). \tag{30}$$

Induced vector potential for the coil of finite dimensions can be computed as follows

$$A_{coil}^{ind}(r,z) = \int_{r_1}^{r_2}\int_{z_1}^{z_2} A_0^{ind}(r,z,r_0,h)\,dr_o dh. \tag{31}$$

The induced change in impedance of the coil is given by the formula (see [3]):

$$Z_{ind} = \frac{2\pi j\omega N}{I(r_2-r_1)(z_2-z_1)} \int_{r_1}^{r_2}\int_{z_1}^{z_2} r A_{coil}^{ind}(r,z)\,dr dz, \tag{32}$$

where N is the number of turns in the coil.

Using (30)–(32) we obtain

$$\begin{aligned} Z_{ind} \;&=\; \frac{2\pi j\omega N^2\mu_0}{(r_2-r_1)^2(z_2-z_1)^2)} \sum_{i=1}^{n} \frac{e^{-\lambda_i z_1} - e^{-\lambda_i z_2}}{\lambda_i^3} \int_{\lambda_i r_1}^{\lambda_i r_2} yJ_1(y)\,dy \\ &\times \sum_{k=1}^{n} Y_{ik} \frac{e^{-\lambda_k z_1} - e^{-\lambda_k z_2}}{\lambda_k^3} \int_{\lambda_k r_1}^{\lambda_k r_2} \xi J_1(\xi)\,d\xi. \end{aligned} \tag{33}$$

The coefficients Y_{ik} in (33) are bulky and are not shown here for brevity.

4 NUMERICAL RESULTS

Calculations of the change in impedance of the coil are performed with Mathematica. Integrals in (33) are evalauted in closed form using Bessel and Struve functions (see [9]). Complex eigenvlaues (the roots of equation (26)) are computed using the method described in [10] and [11]. Numerical results are presented for the following values of the parameters of the problem: $z_1 = 0.3$ mm, $z_2 = 2.6$ mm, $\sigma_1 = 3$ Ms/m, $\sigma_2 = 7$ Ms/m, $d_1 = 0.4$ mm, $d_2 = 0.5$ mm,

$d_3 = 0.8$ mm, $c = 2.2$ mm. The real amd imaginary parts of the change in impedance for the seven frequencies of the excitation current (from 1 kHz to 7 kHz with a step size of 1 kHz) are shown in Figs. 2 and 3. The inner and outer radii of the coil in Fig. 2 are $r_1 = 1.5$ mm and $r_2 = 2.5$ mm, respectively. The same problem is also solved numerically using Comsol Multiphysics software. The details of the finite element modeling of similar problems with Comsol Multiphysics can be found in [12].

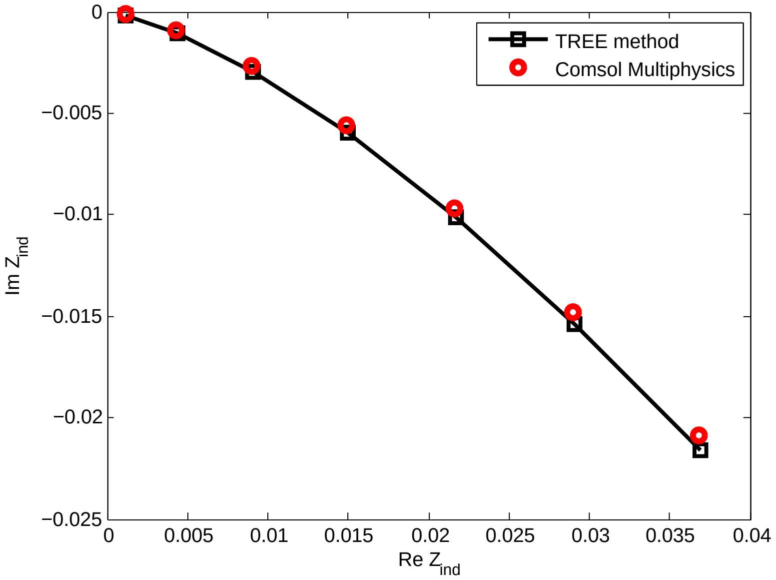

Figure 2: The change in impedance of the coil with $r_1 = 1.5$ mm, $r_2 = 2.5$ mm.

The change in impedance of the coil is also shown in Fig. 3 for the case $r_1 = 2.5$ mm, $r_2 = 4.5$ mm (the other parameters are the same as in Fig. 2).

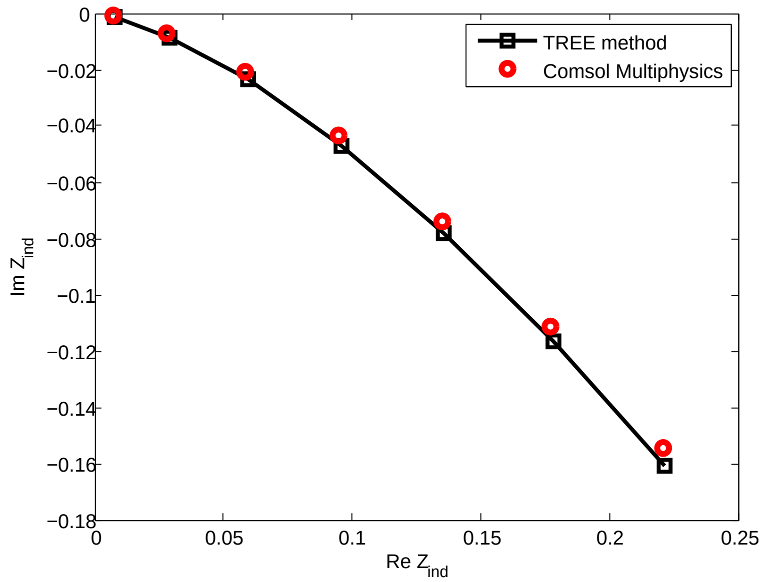

Figure 3: The change in impedance of the coil with $r_1 = 2.5$ mm, $r_2 = 4.5$ mm.

As can be seen from the graphs good agreement is found between the results obtained by the TREE method and finite element modeling with Comsol Multiphysics.

5 CONCLUSION

The model described in the paper represents the continuation of the authors' work related to the solution of direct eddy current testing problems for the case of cylindrical flaws. Good agreement is found between quasi-analytical solution and finite element modeling for all cases considered. The results obtained in the paper can be generalized for the case of magnetic materials (assuming that the magnetic permeability of a medium is constant).

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