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A COMPUTATIONALLY EFFECTIVE FORMULATION TO FINITE ROTATIONS - SMALL STRAINS DESCRIPTION OF BEAM ELEMENTS

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Abstract. *In the context of three-dimensional elastic frame structures analysis with small strains and in the presence of large rotations we present a computationally effective model for the Euler-Bernoulli beam element. Actually, kinematical and strain measures of the beam element are completely defined by referring to boundary nodal displacements and one finite rotation parameter solely. In particular, the director along axis of the beam is defined directly by nodal positions while directors along the principal axes of the cross-section are detected by referring to the related orthogonal plane and the used rotation parameter. The definition of local rotations, required for the evaluation of torque and flexural deformation components, is obtained by imposing rotational compatibility and equilibrium conditions across inter-element boundaries. The description of the finite three-dimensional rotations is well posed under widely applicable hypotheses. The analysis of complex spatial dome structures, where matrices with large dimension and bandwidth occur, now proves a remarkable reduction of the required arithmetical operations with respect to the classical approaches.*

1 Introduction

Considerable work has been devoted to develop models for three-dimensional elastic frame structures for small strains and in the presence of large rotations. In this context the co-rotational, with minimal set parametrizations and multiplicative representations of rotations, is the most widely exploited approach. Indeed, the large-scale calculations required by these formulations have encouraged efficient treatments of the finite rotations. Then, those treatments, typically based on the rotation vector of the Euler theorem to describe finite rotations, have an economical definition of the rotated local reference system because only three parameters are used while evaluations of the coefficients in the force vector and in the tangent stiffness matrix are inexpensive. The evolution of the co-rotational approach can be traced by referring to the works of Stuelpnagel [1], Belytschko and Hsieh [2], Goldstein [3], Argyris [4], Rankin and Nour-Omid [5], Cardona and Geradin [6], Crisfield [7], Atluri and Cazzani [8], Geradin and Rixen [9], Ibrahimbegović et al. [10] and Felippa [11].

In this context here we present an alternative and computationally effective approach. Actually, kinematical and strain measures of the beam element are completely defined by referring to boundary nodal displacements and one finite rotation parameter. In particular, the director along axis of the beam is defined directly by nodal positions while directors along the principal axes of the cross-section are detected by referring to the related orthogonal plane and the only used rotation parameter. The definition of local rotations, required for the evaluation of torque and flexural deformation components, is obtained by imposing rotational compatibility and completeness conditions across inter-element boundaries. Finally, the description of the finite three-dimensional rotations is well posed under widely applicable hypotheses.

Being the infinitesimal nodal rotations computed by vectorial operations among the adjacent elements, such a formulation requires the extra storage of an integer matrix for the node - connected elements recognition. These connections, furthermore, increase the dimension of elemental force vector and tangent stiffness matrix and lead to complex programming for the imposition of rotational boundary conditions and linked applied moments. In contrast, by retaining similar approximation properties, the discretization uses one rotational unknown for each element instead of three required by the classical approaches.

As regards beam element modeling, here we use a small strain - finite displacement formulation of a two-node finite element based on the Euler-Bernoulli beam theory. The actual configuration of the element is rigidly translated and rotated, and deformed according to linear interpolations for axial displacements and quadratic interpolations for torque and flexural modes. The nonlinear motion is recovered by referring to the nodes at the boundaries of the element with three unknown displacements per node plus one unknown rotation per element. We note that boundary conditions on rotations are imposed by assuming as known the related nodal slopes or applied moments. It follows that, as will be discussed later, treatment of rotational boundary conditions and external moments proves to be more complex with respect to typical co-rotational formulations. Furthermore, the incremental rotations are restricted to the range of validity of the described formulation. Overall the use of the presented approach requires more implementation effort but less arithmetical operations with respect to the classical one.

2 Treatment of finite rotations

In the following, we denote with Greek indices the components of vectors and matrices while Latin indices are reserved to nodes and elements identification. Let $\mathbf{g}_\alpha = \{g_{\alpha\beta}\}$ and $\hat{\mathbf{g}}_\alpha = \{\hat{g}_{\alpha\beta}\}$ be, respectively, the actual and the initial configuration of three unit mutually orthogonal vectors in the inertial reference basis $\mathbf{k}_\alpha = \{k_{\alpha\beta}\} = \{\delta_{\alpha\beta}\}$, where $\delta_{\alpha\beta}$ is the Kronecker delta. Then, matrix $\hat{\mathbf{G}} = [\hat{\mathbf{g}}_1|\hat{\mathbf{g}}_2|\hat{\mathbf{g}}_3]$ links $\hat{\mathbf{g}}_\alpha$ and $\mathbf{k}_\alpha$ vectors by $\hat{\mathbf{g}}_\alpha = \hat{\mathbf{G}}\mathbf{k}_\alpha$ while $\mathbf{G} = [\mathbf{g}_1|\mathbf{g}_2|\mathbf{g}_3]$ maps $\mathbf{k}_\alpha$ into $\mathbf{g}_\alpha$ vectors by $\mathbf{g}_\alpha = \mathbf{G}\mathbf{k}_\alpha$.

To obtain the updated treatment of rotations we refer to the $\mathbf{g}_{\alpha(k)} = \mathbf{G}_{(k)}\mathbf{k}_\alpha$, $\mathbf{G}_{(k)} = \mathbf{E}_{(k)}\bar{\mathbf{G}}_{(k)}$, expression for the actual configuration of the $\mathbf{g}_\alpha$ orthonormal triad at the k-th step. Vectors $\mathbf{e}_{\alpha(k)}$ defining $\mathbf{E}_{(k)} = [\mathbf{e}_1|\mathbf{e}_2|\mathbf{e}_3]$ represent the incremental rotation from the $\bar{\mathbf{g}}_{\alpha(k)}$ previously computed configuration. The subsequent $k+1$ step, afterward, refers to the $\bar{\mathbf{G}}_{(k+1)} = \mathbf{E}^*_{(k)}\bar{\mathbf{G}}_{(k)}$ updated configuration with the $\mathbf{e}^*_{\alpha(k)}$ established configuration of $\mathbf{e}_\alpha$. The process is initialized by $\bar{\mathbf{G}}_{(0)} = \hat{\mathbf{G}}$.

Classically, the treatment of rotations is based on the recursive composition $\mathbf{g}_{\alpha(k)} = \mathbf{E}_{(k)}\bar{\mathbf{G}}_{(k)}\hat{\mathbf{g}}_\alpha$, $\mathbf{E}_{(k)} = \mathbf{E}(\boldsymbol{\psi}_{(k)})$, where ψ_α components of $\boldsymbol{\psi}$ are the unknown rotation parameters. Following the description given before, $\mathbf{E}_{(k)}$ is the incremental rotation matrix which maps the updated frame $\bar{\mathbf{g}}_{\alpha(k)}$ into the actual frame $\mathbf{g}_{\alpha(k)}$ while $\bar{\mathbf{G}}_{(k)} = \bar{\mathbf{G}}_{(k)}(\bar{\boldsymbol{\psi}}_{(k)})$ maps the initial frame $\hat{\mathbf{g}}_\alpha$ into the updated frame $\bar{\mathbf{g}}_{\alpha(k)}$. Based on the rotation vector $\boldsymbol{\psi} = \varphi\boldsymbol{\phi}$, $\boldsymbol{\phi}^T\boldsymbol{\phi} = 1$, of the Euler theorem to describe finite rotations, a representation of rotation operators is:

$$\mathbf{G}(\boldsymbol{\psi}) = \mathbf{I} + \frac{\sin\varphi}{\varphi}\boldsymbol{\psi}_\times + \frac{1-\cos\varphi}{\varphi^2}\boldsymbol{\psi}^2_\times. \tag{1}$$

where $\mathbf{I}$ is the identity matrix. In (1) $\boldsymbol{\psi}_\times$ denotes the skew symmetric tensor obtained by the components of vector $\boldsymbol{\psi}$:

$$\boldsymbol{\psi}_\times = \mathbf{Skew}(\boldsymbol{\psi}) = \begin{bmatrix} 0 & -\psi_3 & \psi_2 \\ \psi_3 & 0 & -\psi_1 \\ -\psi_2 & \psi_1 & 0 \end{bmatrix}. \tag{2}$$

The $\boldsymbol{\psi} = \mathbf{axial}(\boldsymbol{\psi}_\times)$ is the converse operation of (2) that extracts the $\boldsymbol{\psi}$ vector from the skew symmetric tensor $\boldsymbol{\psi}_\times$.

In the use of the $\mathbf{G} = \mathbf{E}\bar{\mathbf{G}}$ composition of rotation operators, however, we stress that $\boldsymbol{\psi}_G \neq \boldsymbol{\psi} + \bar{\boldsymbol{\psi}}$ successive rotations cannot be obtained by simply adding their corresponding rotation vectors but the solution of the inverse $\boldsymbol{\psi}_G = G^{-1}(\mathbf{G})$ problem is required. Such a problem is defined as the operation of obtaining the $\boldsymbol{\psi}_G$ rotation vector based on the knowledge of the $\mathbf{G}$ rotation matrix and can be solved by the no ill-conditioning Spurrier algorithm [12]. By referring to the established configuration $\boldsymbol{\psi}^*_{(k)}$, afterwards, the updated configuration is achieved in the subsequent $k+1$ step as

$$\mathbf{G} = \mathbf{E}^*_{(k)}\bar{\mathbf{G}}_{(k)}, \quad \boldsymbol{\psi}_G = G^{-1}(\mathbf{G}), \quad \bar{\mathbf{G}}_{(k+1)} = \mathbf{G}(\boldsymbol{\psi}_G). \tag{3}$$

The process is initialized by $\bar{\mathbf{G}}_{(0)} = \mathbf{I}$.

In the presented finite rotations formulation let respectively $\overset{i}{\mathbf{u}}$ and $\overset{j}{\mathbf{u}}$ displacement vectors at the i and j boundary nodes of the element with lengh h, while discrete operators s and d rest defined by $s\mathbf{u} = (\overset{i}{\mathbf{u}} + \overset{j}{\mathbf{u}})/2$ and $d\mathbf{u} = (\overset{j}{\mathbf{u}} - \overset{i}{\mathbf{u}})/h$. Then, by evaluating the vector connecting the i and j nodes, we refer to

$$\mathbf{p}_1 = d\mathbf{u} + \hat{\mathbf{g}}_1, \quad \check{\mathbf{g}}_1 = \mathbf{p}_1/||\mathbf{p}_1|| \tag{4}$$

for the first actual vector of the orthonormal basis. By referring to the given unit vector $\bar{\mathbf{g}}_2$ we define the vectors

$$\mathbf{p}_2 = \bar{\mathbf{g}}_2 - \check{\mathbf{g}}_1 \cdot \bar{\mathbf{g}}_2 \;\; \check{\mathbf{g}}_1, \qquad \check{\mathbf{g}}_2 = \mathbf{p}_2/||\mathbf{p}_2|| \tag{5}$$
$$\check{\mathbf{g}}_3 = \check{\mathbf{g}}_1 \times \check{\mathbf{g}}_2$$

such that the basis $\breve{\mathbf{G}} = [\breve{\mathbf{g}}_1|\breve{\mathbf{g}}_2|\breve{\mathbf{g}}_3]$ is orthonormal and coincides with the principal axes of the beam element. The additional rotation in the reference space characterized by the angle r about the $\breve{\mathbf{g}}_1$ axis

$$\mathbf{G}_r = \mathbf{I} + \sin r \breve{\mathbf{g}}_{1\times} + (1-\cos r)\breve{\mathbf{g}}_{1\times}^2 \tag{6}$$

completes now the definition of the rotation operator $\mathbf{G} = \mathbf{G}_r\breve{\mathbf{G}}$ that links $\mathbf{g}_\alpha$ and $\mathbf{k}_\alpha$ vectors.

We note that, in the definition of the $\breve{\mathbf{g}}_2$ vector in (5), the vectors $\breve{\mathbf{g}}_1$ and $\bar{\mathbf{g}}_2$ have to form a linearly independent set. However this is not an actual limitation for the incremental rotation $\mathbf{E}_{(k)}$. Furthermore, rotation $\mathbf{G}_r$ defined in (6) preserves the linearity of the vector space operations because it has as axis of rotation the component $\breve{\mathbf{g}}_1 = \mathbf{g}_1$ of $\mathbf{G}$.

3 Kinematics and energetic quantities of the beam element

Let ξ be the referential coordinate along the beam element centerline $-h/2 \leq \xi \leq +h/2$. Global displacement vector $\mathbf{u}(\xi)$ of the e element is composed of rigid and deformation components by

$$\overset{e}{\mathbf{u}}(\xi) = \overset{e}{\mathbf{u}} + (\overset{e}{\mathbf{g}}_1 - \overset{e}{\hat{\mathbf{g}}}_1)\xi + \overset{e}{\mathbf{G}}\overset{e}{\tilde{\mathbf{u}}}(\xi), \tag{7}$$

where

$$\begin{aligned} \overset{e}{\tilde{u}}_1(\xi) &= \overset{e}{\varepsilon}\xi \\ \overset{e}{\tilde{u}}_2(\xi) &= -\frac{h^2}{8}\overset{e}{\chi}_3 - \frac{h^2}{24}\overset{e}{\gamma}_2\xi + \frac{1}{2}\overset{e}{\chi}_3\xi^2 + \frac{1}{6}\overset{e}{\gamma}_2\xi^3 \\ \overset{e}{\tilde{u}}_3(\xi) &= \frac{h^2}{8}\overset{e}{\chi}_2 + \frac{h^2}{24}\overset{e}{\gamma}_3\xi - \frac{1}{2}\overset{e}{\chi}_2\xi^2 - \frac{1}{6}\overset{e}{\gamma}_3\xi^3. \end{aligned} \tag{8}$$

Then, local transverse rotations in the $\overset{e}{\mathbf{G}}$ basis are computed by

$$\begin{aligned} \overset{e}{\varphi}_2(\xi) &= -\overset{e}{\tilde{u}}_{3,\xi}(\xi) = -\frac{h^2}{24}\overset{e}{\gamma}_3 + \overset{e}{\chi}_2\xi + \frac{1}{2}\overset{e}{\gamma}_3\xi^2 \\ \overset{e}{\varphi}_3(\xi) &= \overset{e}{\tilde{u}}_{2,\xi}(\xi) = -\frac{h^2}{24}\overset{e}{\gamma}_2 + \overset{e}{\chi}_3\xi + \frac{1}{2}\overset{e}{\gamma}_2\xi^2, \end{aligned} \tag{9}$$

while local axial rotation is now defined by the expression

$$\overset{e}{\varphi}_1(\xi) = \overset{e}{\chi}_1\xi + \frac{1}{3}\overset{e}{\gamma}_1\xi^2. \tag{10}$$

By evaluating above relations for nodal coordinates we deduce that

$$\overset{e}{\varepsilon} = \overset{e}{\mathbf{g}}_1 \cdot \overset{e}{\mathbf{p}}_1 - 1 \tag{11}$$

is the expression of the axial deformation and

$$\overset{e}{\boldsymbol{\chi}} = d\overset{e}{\boldsymbol{\varphi}}, \qquad \overset{e}{\boldsymbol{\gamma}} = \frac{6}{h^2}s\overset{e}{\boldsymbol{\varphi}}, \tag{12}$$

are the expressions of torque and shear deformations as a function of nodal displacement and local rotations.

In the following, we refer to the m and p elements respectively adjacent of the current e element at the nodes i and j. Let, furthermore, $\overset{em}{\hat{\mathbf{P}}} = \overset{e}{\hat{\mathbf{G}}} \cdot \overset{m}{\hat{\mathbf{G}}}$ and $\overset{ep}{\hat{\mathbf{P}}} = \overset{e}{\hat{\mathbf{G}}} \cdot \overset{p}{\hat{\mathbf{G}}}$ rotation matrices that carry out the representations of vector respectively in the $\overset{m}{\hat{\mathbf{g}}}_\alpha$ and $\overset{p}{\hat{\mathbf{g}}}_\alpha$ basis with respect to $\overset{e}{\hat{\mathbf{g}}}_\alpha$. By referring to the $\overset{ei}{\boldsymbol{\varphi}}$ and $\overset{ej}{\boldsymbol{\varphi}}$ nodal local rotations, nodal director components are now calculated by the vectorial operations

$$\overset{ei}{\mathbf{G}} = \overset{e}{\mathbf{G}}(\mathbf{I} + \overset{ei}{\boldsymbol{\varphi}}_\times), \qquad \overset{ej}{\mathbf{G}} = \overset{e}{\mathbf{G}}(\mathbf{I} + \overset{ej}{\boldsymbol{\varphi}}_\times). \tag{13}$$

We note that the first order accuracy of the (13) representations leads to local evaluations consistent with the small strains hypotheses. Rotational compatibility conditions at the boundary nodes, then, are obtained by imposing

$$\overset{m}{\mathbf{G}}(\mathbf{I} + \overset{mj}{\boldsymbol{\varphi}}_\times) = \overset{e}{\mathbf{G}}(\mathbf{I} + \overset{ei}{\boldsymbol{\varphi}}_\times)\overset{em}{\hat{\mathbf{P}}}, \qquad \overset{p}{\mathbf{G}}(\mathbf{I} + \overset{pi}{\boldsymbol{\varphi}}_\times) = \overset{e}{\mathbf{G}}(\mathbf{I} + \overset{ej}{\boldsymbol{\varphi}}_\times)\overset{ep}{\hat{\mathbf{P}}}. \tag{14}$$

Now we also define respectively the local rotations $\overset{em}{\boldsymbol{\theta}}$ and $\overset{ep}{\boldsymbol{\theta}}$ connecting directly the $\overset{m}{\mathbf{g}}_\alpha$ and $\overset{p}{\mathbf{g}}_\alpha$ bases with $\overset{e}{\mathbf{g}}_\alpha$ by the relations:

$$\overset{e}{\mathbf{G}} = \overset{m}{\mathbf{G}}\overset{em}{\hat{\mathbf{P}}} \cdot (\mathbf{I} + \overset{em}{\boldsymbol{\theta}}_\times), \qquad \overset{e}{\mathbf{G}} = \overset{p}{\mathbf{G}}\overset{ep}{\hat{\mathbf{P}}} \cdot (\mathbf{I} + \overset{ep}{\boldsymbol{\theta}}_\times). \tag{15}$$

Then by (14) and (15) we can evaluate at the first order:

$$\overset{mj}{\boldsymbol{\varphi}} = \overset{em}{\hat{\mathbf{P}}} \cdot (\overset{ei}{\boldsymbol{\varphi}} + \overset{em}{\boldsymbol{\theta}}), \qquad \overset{pi}{\boldsymbol{\varphi}} = \overset{ep}{\hat{\mathbf{P}}} \cdot (\overset{ej}{\boldsymbol{\varphi}} + \overset{ep}{\boldsymbol{\theta}}), \tag{16}$$

with

$$\overset{em}{\boldsymbol{\theta}}_\times = \overset{em}{\hat{\mathbf{P}}}\overset{m}{\mathbf{G}} \cdot \overset{e}{\mathbf{G}} - \mathbf{I}, \qquad \overset{ep}{\boldsymbol{\theta}}_\times = \overset{em}{\hat{\mathbf{P}}}\overset{p}{\mathbf{G}} \cdot \overset{e}{\mathbf{G}} - \mathbf{I}. \tag{17}$$

We note that rotations $\overset{em}{\boldsymbol{\theta}}$ and $\overset{ep}{\boldsymbol{\theta}}$ are in function of the assumed nodal displacements and elemental rotations unknowns. To recover expressions for the rotational deformations in (12), here we impose equilibrium equations or consistence conditions at the nodes.

Twisting and bending moments $\mathbf{m}_\alpha$ at the i node due to the e element can be expressed by the formula

$$\overset{ei}{\mathbf{m}}_\alpha = \overset{e}{H}_{\alpha\alpha}\overset{ei}{\varphi}_{\alpha,\xi}\overset{ei}{\mathbf{g}}_\alpha. \tag{18}$$

In (18) we denote with $\mathbf{H} = \mathbf{diag}[GJ_1, EJ_2, EJ_3]$ the elastic constitutive matrix of torsional and bending stiffnesses about the principal axes. Let $\overset{i}{\mathbf{m}}$ the applied vector moment at the i node, then related equilibrium equation is:

$$\overset{ei}{\mathbf{m}} = \overset{i}{\mathbf{m}} + \overset{mj}{\mathbf{m}}, \qquad \overset{ei}{\mathbf{m}} = \sum_\alpha \overset{e}{H}_{\alpha\alpha}\overset{ei}{\varphi}_{\alpha,\xi}\overset{ei}{\mathbf{g}}_\alpha, \quad \overset{mj}{\mathbf{m}} = \sum_\alpha \overset{m}{H}_{\alpha\alpha}\overset{mj}{\varphi}_{\alpha,\xi}\overset{mj}{\mathbf{g}}_\alpha. \tag{19}$$

For clarity of presentation, in the following we refer to the case $\overset{i}{\mathbf{m}} = \mathbf{0}$. Then, for the β component of equilibrium equation (19) we have

$$\overset{m}{H}_{\beta\beta}\overset{mj}{\varphi}_{\beta,\xi} = \sum_\alpha \overset{e}{H}_{\alpha\alpha}\overset{ei}{\varphi}_{\alpha,\xi}\overset{mj}{\mathbf{g}}_\beta \cdot \overset{ei}{\mathbf{g}}_\alpha. \tag{20}$$

By proceeding in a similar way for the equilibrium equation at the j node and by writing relations (20) in the matrix form we obtain

$$\overset{m}{\mathbf{H}}\overset{mj}{\boldsymbol{\varphi}}_{,\xi} = \overset{mj}{\mathbf{G}} \cdot \overset{ei}{\mathbf{G}}\overset{e}{\mathbf{H}}\overset{ei}{\boldsymbol{\varphi}}_{,\xi}, \qquad \overset{p}{\mathbf{H}}\overset{pi}{\boldsymbol{\varphi}}_{,\xi} = \overset{pi}{\mathbf{G}} \cdot \overset{ej}{\mathbf{G}}\overset{e}{\mathbf{H}}\overset{ej}{\boldsymbol{\varphi}}_{,\xi}. \tag{21}$$

By inserting compatibility conditions $\overset{ei}{\mathbf{G}}\overset{em}{\hat{\mathbf{P}}} = \overset{mj}{\mathbf{G}}$ and $\overset{ej}{\mathbf{G}}\overset{ep}{\hat{\mathbf{P}}} = \overset{pi}{\mathbf{G}}$ in the (21) relations we have:

$$\overset{m}{\mathbf{H}}\overset{mj}{\boldsymbol{\varphi}}_{,\xi} = \overset{em}{\hat{\mathbf{P}}} \cdot \overset{e}{\mathbf{H}}\overset{ei}{\boldsymbol{\varphi}}_{,\xi}, \qquad \overset{p}{\mathbf{H}}\overset{pi}{\boldsymbol{\varphi}}_{,\xi} = \overset{ep}{\hat{\mathbf{P}}} \cdot \overset{e}{\mathbf{H}}\overset{ej}{\boldsymbol{\varphi}}_{,\xi}. \tag{22}$$

In the cases of constant bending state, $\overset{e}{\boldsymbol{\gamma}} = \mathbf{0}$ and $\overset{m}{\boldsymbol{\gamma}} = \mathbf{0}$ in (12) expressions imply that $\overset{ej}{\boldsymbol{\varphi}} = -\overset{ei}{\boldsymbol{\varphi}}$ and $\overset{mi}{\boldsymbol{\varphi}} = -\overset{mj}{\boldsymbol{\varphi}}$, respectively, and

$$\overset{ei}{\boldsymbol{\varphi}}_{,\xi} = \overset{e}{\boldsymbol{\chi}} = -\frac{2}{\overset{e}{h}}\overset{ei}{\boldsymbol{\varphi}}, \qquad \overset{mj}{\boldsymbol{\varphi}}_{,\xi} = \overset{m}{\boldsymbol{\chi}} = \frac{2}{\overset{m}{h}}\overset{mj}{\boldsymbol{\varphi}}. \tag{23}$$

Finally, by (16), (22) and (23), for the i node we obtain

$$\overset{em}{\mathbf{A}}\overset{ei}{\boldsymbol{\varphi}} = -\frac{1}{\overset{m}{h}}\overset{em}{\hat{\mathbf{P}}}\overset{m}{\mathbf{H}}\overset{em}{\hat{\mathbf{P}}} \cdot \overset{em}{\boldsymbol{\theta}}, \qquad \overset{em}{\mathbf{A}} = \frac{1}{\overset{e}{h}}\overset{e}{\mathbf{H}} + \frac{1}{\overset{m}{h}}\overset{em}{\hat{\mathbf{P}}}\overset{m}{\mathbf{H}}\overset{em}{\hat{\mathbf{P}}} \cdot . \tag{24}$$

Similarly, for the j node:

$$\overset{ep}{\mathbf{A}}\overset{ej}{\boldsymbol{\varphi}} = -\frac{1}{\overset{p}{h}}\overset{ep}{\hat{\mathbf{P}}}\overset{p}{\mathbf{H}}\overset{ep}{\hat{\mathbf{P}}} \cdot \overset{ep}{\boldsymbol{\theta}}, \qquad \overset{ep}{\mathbf{A}} = \frac{1}{\overset{e}{h}}\overset{e}{\mathbf{H}} + \frac{1}{\overset{p}{h}}\overset{ep}{\hat{\mathbf{P}}}\overset{p}{\mathbf{H}}\overset{ep}{\hat{\mathbf{P}}} \cdot . \tag{25}$$

By the (24) and (25) definitions of the local rotations we can obtain the computation of the (12) element beam deformations in function of the assumed unknowns. We note that the $\mathbf{A}$ matrices are defined by fixed quantities of the problem. Then such quantities can be computed, for each node, at the preprocessing procedure.

If we want to achieve the complete consistence at the generic i node, together with (16) expressions we must impose the $\sum_\alpha \overset{ei}{\varphi}_{\alpha,\xi}\overset{ei}{\mathbf{g}}_\alpha = \sum_\alpha \overset{mj}{\varphi}_{\alpha,\xi}\overset{mj}{\mathbf{g}}_\alpha$ and $\sum_\alpha \overset{ei}{\varphi}_{\alpha,\xi\xi}\overset{ei}{\mathbf{g}}_\alpha = \sum_\alpha \overset{mj}{\varphi}_{\alpha,\xi\xi}\overset{mj}{\mathbf{g}}_\alpha$ continuity conditions that, at the first order of approximation, lead to the relations:

$$\overset{mj}{\boldsymbol{\varphi}}_{,\xi} = \overset{em}{\hat{\mathbf{P}}} \cdot \overset{ei}{\boldsymbol{\varphi}}_{,\xi} \qquad \text{and} \qquad \overset{mj}{\boldsymbol{\varphi}}_{,\xi\xi} = \overset{em}{\hat{\mathbf{P}}} \cdot \overset{ei}{\boldsymbol{\varphi}}_{,\xi\xi} \tag{26}$$

respectively, while for the j node

$$\overset{pi}{\boldsymbol{\varphi}}_{,\xi} = \overset{ep}{\hat{\mathbf{P}}} \cdot \overset{ej}{\boldsymbol{\varphi}}_{,\xi} \qquad \text{and} \qquad \overset{pi}{\boldsymbol{\varphi}}_{,\xi\xi} = \overset{ep}{\hat{\mathbf{P}}} \cdot \overset{ej}{\boldsymbol{\varphi}}_{,\xi\xi}. \tag{27}$$

After algebraic manipulations by the above expressions the following relations are obtained:

$$-\frac{2\overset{m}{h}{}^2 + 3\overset{m}{h}\overset{e}{h} + \overset{e}{h}{}^2}{\overset{e}{h}{}^2} s\overset{e}{\boldsymbol{\varphi}} + \frac{\overset{m}{h} + \overset{e}{h}}{2} d\overset{e}{\boldsymbol{\varphi}} = \overset{em}{\boldsymbol{\theta}}, \qquad -\frac{2\overset{p}{h}{}^2 + 3\overset{p}{h}\overset{e}{h} + \overset{e}{h}{}^2}{\overset{e}{h}{}^2} s\overset{e}{\boldsymbol{\varphi}} - \frac{\overset{p}{h} + \overset{e}{h}}{2} d\overset{e}{\boldsymbol{\varphi}} = \overset{ep}{\boldsymbol{\theta}}. \tag{28}$$

Also here, by the (28) definitions of the local rotations, we can achieve the computation of the element beam deformations in function of the assumed unknowns.

4 Boundary conditions, multiple connections and solution scheme

Boundary conditions imposition implies a case depending implementation of the solution process. In effect, conditions involving rotations and moments given at boundary nodes are imposed by specializing related elements by following the formulation described in Section

3 because we have worked with global rotation and curvature definitions. Furthermore, we remark that only the external work of forces can be defined in the described formulation so that moments can be modelled as forces following the motion of points of the beam element. In particular, let vector $\overset{n}{\mathbf{m}}$ be the spatially fixed moment applied in the n node of the e element and $\overset{en}{\mathbf{g}}_\alpha$ the related nodal basis. As described in [13], we refer to three force vectors $\mathbf{p}_{(\alpha)}$ applied to n and compute the external work as

$$W_{\mathbf{m}} == \sum_{\alpha} \mathbf{p}_{(\alpha)} \cdot (\overset{en}{\mathbf{g}}_\alpha - \overset{en}{\hat{\mathbf{g}}}_\alpha). \tag{29}$$

The variation of the functional $W_{\mathbf{m}}$ is carried out on the $\overset{en}{\mathbf{g}}_\alpha$ vectors by considering $\mathbf{p}_{(\alpha)}$ as constants. Then, after variation of (29), we define the $\mathbf{p}_{(\alpha)}$ force vectors by

$$\mathbf{p}_{(\alpha)} = -\frac{1}{2} \sum_{\beta\kappa} e_{\alpha\beta\kappa} p_\beta \overset{en}{\mathbf{g}}_\kappa, \qquad \sum_{\alpha} \overset{en}{\mathbf{g}}_\alpha \times \mathbf{p}_{(\alpha)} = \overset{n}{\mathbf{m}}. \tag{30}$$

As can be observed in (30), applied forces are such that the resulting moment in the n node is the given $\overset{n}{\mathbf{m}}$ vector. Simple algebraic manipulations, finally, lead to the $p_\alpha = \overset{n}{\mathbf{m}}^T \overset{en}{\mathbf{g}}_\alpha$ components. We note that, the definitions given in (29) and (30) imply that the external force vectors are a function of the assumed unknowns.

In the case of more elements connected to the i node, the relations in (19) of the (*EF*) equilibrium based formulation become

$$\overset{ei}{\mathbf{m}} = \overset{i}{\mathbf{m}} + \sum_{m} \overset{mj}{\mathbf{m}}, \quad \text{or} \quad \sum_{\alpha} \overset{e}{H}_{\alpha\alpha} \overset{ei}{\varphi}_{\alpha,\xi} \overset{ei}{\mathbf{g}}_\alpha = \overset{i}{\mathbf{m}} + \sum_{m} \sum_{\alpha} \overset{m}{H}_{\alpha\alpha} \overset{mj}{\varphi}_{\alpha,\xi} \overset{mj}{\mathbf{g}}_\alpha. \tag{31}$$

Being $\overset{ei}{\mathbf{G}} \overset{em}{\hat{\mathbf{P}}} = \overset{mj}{\mathbf{G}}$, $\forall m$ and by assuming $\overset{mi}{\boldsymbol{\varphi}} = -\overset{mj}{\boldsymbol{\varphi}}$, $\forall m$ we obtain:

$$\overset{em}{\mathbf{A}} \overset{ei}{\boldsymbol{\varphi}} = -\sum_{m} \frac{1}{\overset{m}{h}} \overset{em}{\hat{\mathbf{P}}} \overset{m}{\mathbf{H}} \overset{em}{\hat{\mathbf{P}}} \cdot \overset{em}{\boldsymbol{\theta}}, \qquad \overset{em}{\mathbf{A}} = \frac{1}{\overset{e}{h}} \overset{e}{\mathbf{H}} + \sum_{m} \frac{1}{\overset{m}{h}} \overset{em}{\hat{\mathbf{P}}} \overset{m}{\mathbf{H}} \overset{em}{\hat{\mathbf{P}}} \cdot, \tag{32}$$

and the analogous for the j node. From the implementation point of the view, at the current e element, multiple connections are carried out by repeating m times the assembling of internal force vectors and stiffness matrices for each connected element. For the (*CF*) consistence based formulation, instead, a master element must be assigned at the multiply connected node as reference of related continuity conditions.

The definition of equilibrium equations is based on the classical stationary problem for the energy functional. Because a multiplicative approach is here exploited, admissible variational formulation and linearization must be carried out at the solution point in the respect of the $\delta\mathbf{G} = \delta\boldsymbol{\varphi}_\times \mathbf{G}$ consistent condition, where $\delta\boldsymbol{\varphi}$ is the spatial component of the angular variation. A predictor-corrector scheme as described in [14] for the equilibrium path individualization is used in the analysis. It is characterized by a predictor step obtained by the linear extrapolation of the previously computed two solution points when $k > 0$, while the first order asymptotic extrapolation is used when $k = 0$. Furthermore, the corrector is accomplished by a Newton method based corrector scheme with minimization of the distance between the approximate and equilibrium points as a constraint equation.

5 Numerical examples

Equilibrium curve of the framed dome shown in Figure 1 and analyzed in Battini [15], Kouhia and Tuomala [16], was calculated with both *EF* and *CF* approaches. By assuming ε=0 or ε=0.0001 two different behaviours of the structure have been considered. The dome has been modelled by using N_{mb}=4,8 elements for each member, i.e. totally 18 N_{mb} elements.

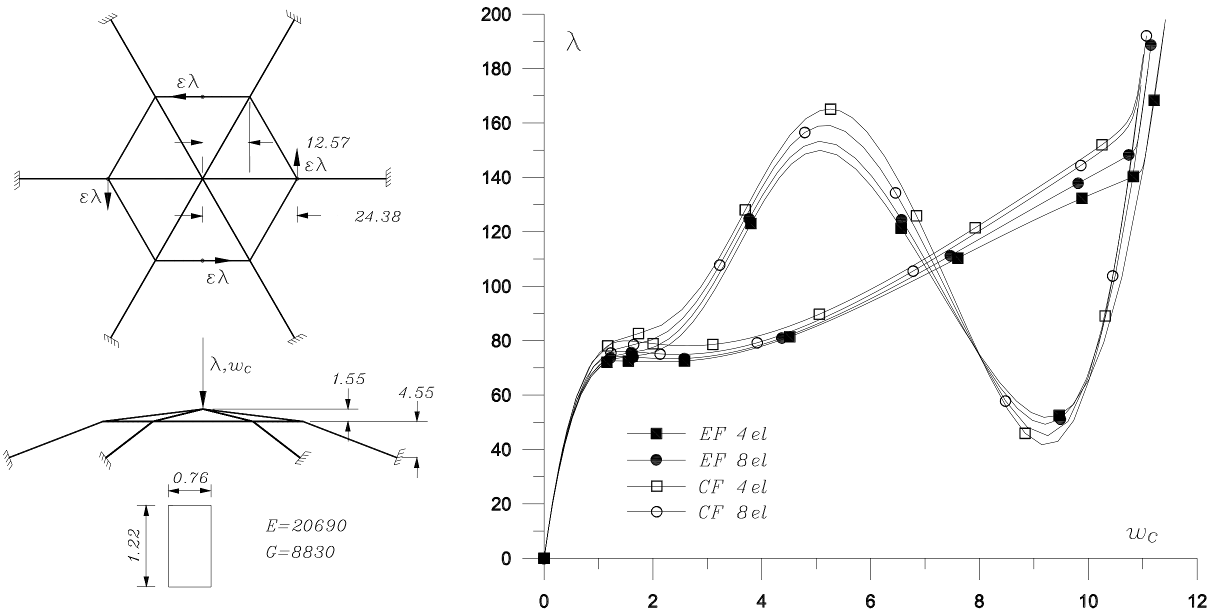

Figure 1: Framed dome: problem definition and equilibrium paths.

In Figure 1 the fundamental path and the secondary path branching out at the lowest bifurcation point are also displayed. Significant deformed configurations are shown in Figure 2 for both loadings. In particular, the perfect case (a) is characterized by a symmetric behaviour while the post-buckling mode (b) is a rotation around the central vertical axis. The detected computational performances indicate that the presented formulation requires more implementation effort but less arithmetical operations with respect to the classical rotation vector based approach.

6 Conclusions

In the hypothesis of large displacements and rotations and small strains, a technique to analyze the behaviour of three-dimensional finite element beam frames has been presented. The approach is based on an updated Lagrangian description of rotations and the presented formulations do not use angle measures. By adopting the Euler-Bernoulli beam model, a computationally effective beam element is obtained because kinematical and strain measures are completely defined by referring to boundary nodal displacements and one finite rotation parameter solely. The treatment of rotational boundary conditions and external moments proves to be more complex with respect to the co-rotational formulations. Nevertheless, the description of the finite three-dimensional rotations is well posed under widely applicable hypotheses, while the analysis of complex spatial dome structures, where matrices with large dimension and bandwidth occur, now proves a remarkable reduction of the required arithmetical operations with respect to the classical approaches.

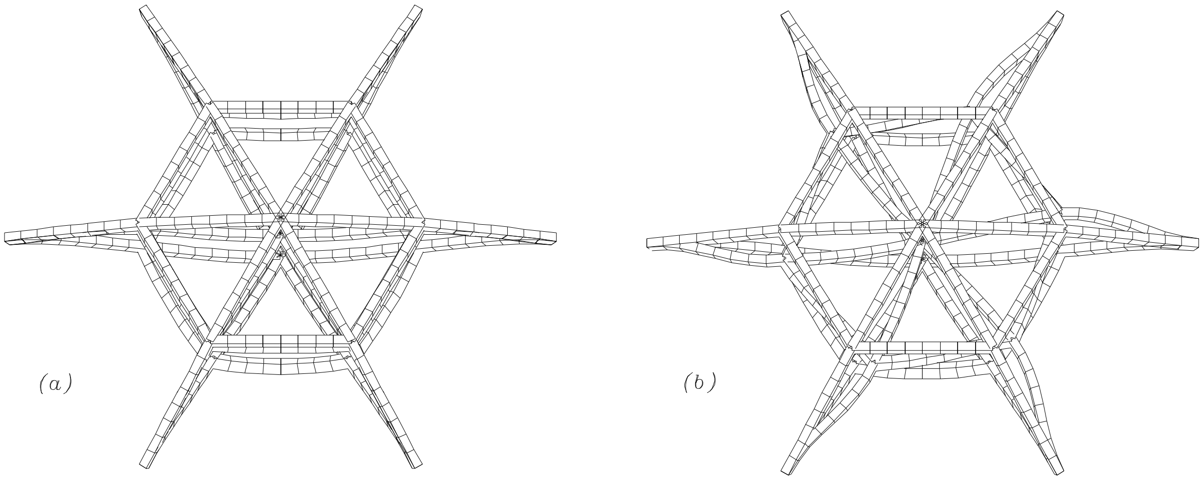

Figure 2: Framed dome: deformed configurations for N_{mb}=8.

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