ECCOMAS Congress 2016
VII European Congress on Computational Methods in Applied Sciences and Engineering
M. Papadrakakis, V. Papadopoulos, G. Stefanou, V. Plevris (eds.)
Crete Island, Greece, 5–10 June 2016

THE STRUCTURAL SYMMETRY WITHIN THE CONTEXT OF NONLOCAL ELASTICITY

A.A. Pisano[1], **P. Fuschi**[1]

[1] Dept. PAU, University Mediterranea of Reggio Calabria,
via Melissari, I-89124 Reggio Calabria, Italy
e-mail: aurora.pisano@unirc.it; paolo.fuschi@unirc.it

Keywords: Structural Symmetry, Nonlocal Elasticity of Integral Type, Nonlocal Finite Element Method.

Abstract. *The paper deals with symmetric structures made of nonlocal elastic materials. In particular, it refers to a nonlocal elastic strain integral constitutive model. In this context, in contrast to what happens in local elasticity, the solution obtained by analysing a standard symmetric portion of the structure leads to incorrect results. This drawback is due to the loss of nonlocal effects induced on the selected symmetric portion by the removed portion. To recover the mechanical equivalence with the original (entire) structure an enlarged symmetric model has to be considered together with the application of appropriate boundary conditions, hereafter discussed with the aid of a numerical example.*

1 INTRODUCTION

Many structural/mechanical problems are characterized by being symmetric in terms of geometry, material properties and boundary conditions. As well known, the presence of symmetries can drastically simplify the solution of such problems, halving or more the computational effort related to their analysis.

In the context of simple (local) materials the symmetric portions of a structure to be analyzed and the appropriate boundary conditions that have to be applied along the lines of symmetry are clearly identified. The solution computed on a reduced symmetric portion, once mirrored, gives the correct solution for the entire structure. The same cannot be asserted for symmetric structures made of nonlocal materials for which the constitutive law at a given point involves weighted averages of a state variable over a certain neighbor of the point [1]. The wideness of this zone is related to an internal length material scale parameter. For such structures the removal of one or more symmetric portions implies the loss of the nonlocal effects exerted by the removed portions on the one selected for the analysis.

The present paper suggests two possible remedies in order to obtain the expected (correct) solution of a nonlocal symmetric problem by recovering the missing nonlocal effects; precisely:

i) an enlarged symmetric model, whose dimension is related to the internal length material scale, has to be considered;

ii) appropriate (smeared) boundary conditions have to be applied to the above mentioned enlarged model.

Although the arguments developed seem to be of general validity in the context of nonlocal elasticity, in the present paper they are referred to a nonlocal approach of integral type known as strain-difference-based model [2] and the presented numerical example is solved by means of a nonlocal finite element method promoted by the authors [3].

2 THE THEORETICAL FRAMEWORK

2.1 The constitutive model

This paper considers the nonlocal integral elasticity model theorized by Polizzotto et *al.* in [2]. The quoted model, known as strain-difference-based model, proposes the following stress-strain constitutive relation:

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{D}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{x}) - \alpha \int_V \boldsymbol{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{x}') : [\boldsymbol{\varepsilon}(\boldsymbol{x}') - \boldsymbol{\varepsilon}(\boldsymbol{x})] \, \mathrm{d}\, \boldsymbol{x}' \quad \forall\, (\boldsymbol{x}, \boldsymbol{x}') \in V, \tag{1}$$

in which the stress is conceived as the sum of two contributions, the first is the one of the classical elasticity, while the second (integral) contribution is of nonlocal nature involving the nonlocal tensor $\boldsymbol{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{x}')$ and the strain difference field $\boldsymbol{\varepsilon}(\boldsymbol{x}') - \boldsymbol{\varepsilon}(\boldsymbol{x})$. The nonlocal tensor $\boldsymbol{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{x}')$ is defined as:

$$\boldsymbol{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{x}') := [\gamma(\boldsymbol{x})\boldsymbol{D}(\boldsymbol{x}) + \gamma(\boldsymbol{x}')\boldsymbol{D}(\boldsymbol{x}')]\, g(\boldsymbol{x}, \boldsymbol{x}') - \boldsymbol{q}(\boldsymbol{x}, \boldsymbol{x}') \quad \forall\, (\boldsymbol{x}, \boldsymbol{x}') \in V, \tag{2}$$

with:

$$\gamma(\boldsymbol{x}) := \int_V g(\boldsymbol{x}, \boldsymbol{x}')\, \mathrm{d}\, V'; \tag{3}$$

$$\boldsymbol{q}(\boldsymbol{x}, \boldsymbol{x}') := \int_V g(\boldsymbol{x}, \boldsymbol{z}) g(\boldsymbol{x}', \boldsymbol{z}) \boldsymbol{D}(\boldsymbol{z})\, \mathrm{d}\, V^z. \tag{4}$$

In the above equations $\boldsymbol{D}(\boldsymbol{x})$ is the symmetric and positive definite elastic moduli tensor, while $g(\boldsymbol{x}, \boldsymbol{x}')$ is a positive scalar attenuation function depending on the internal length material scale, say ℓ, as well as on the Euclidean distance between points $\boldsymbol{x}$ and $\boldsymbol{x}'$ in V. The attenuation function assigns a *weight* to the nonlocal effects induced at the field point $\boldsymbol{x}$ by a phenomena acting at the source point $\boldsymbol{x}'$. This function has a peak at $\boldsymbol{x} \equiv \boldsymbol{x}'$ and rapidly decreases with increasing distance, becoming practically null beyond the so-called *influence distance*, say L_R which is a multiple of ℓ. It is worth noting that the model of Eq.(1) possesses, beside ℓ, a second material model parameter, α, which controls the proportion of the nonlocal contribution. Both material parameters ℓ and α should be experimentally determined.

2.2 The NL-FEM formulation

Let us consider a nonlocal elastic body occupying a volume V whose boundary surface is S. The body is subjected to body forces $\boldsymbol{b}(\boldsymbol{x})$ in V and surface tractions $\boldsymbol{t}(\boldsymbol{x})$ on S_t. Kinematic boundary conditions $\boldsymbol{u}(\boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x})$ are also specified on $S_u = S - S_t$. Moreover, the governing constitutive relation is the one given by Eq.(1). The pertinent boundary value problem is governed by the standard equilibrium and compatibility equations together with the stress strain relation (1). Following Polizzotto et *al.* [2] and Fuschi et *al.* [3], it can be shown that the nonlocal total potential energy functional associated to such nonlocal boundary value problem can be written as:

$$
\begin{aligned}
\Pi[\boldsymbol{u}(\boldsymbol{x})] := & \frac{1}{2}\int_V \nabla\boldsymbol{u}(\boldsymbol{x}) : \boldsymbol{D}(\boldsymbol{x}) : \nabla\boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}V + \\
& + \frac{\alpha}{2}\int_V \nabla\boldsymbol{u}(\boldsymbol{x}) : \gamma^2(\boldsymbol{x})\boldsymbol{D}(\boldsymbol{x}) : \nabla\boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}V + \\
& - \frac{\alpha}{2}\int_V\int_V \nabla\boldsymbol{u}(\boldsymbol{x}) : \boldsymbol{\mathcal{J}}(\boldsymbol{x},\boldsymbol{x}') : \nabla\boldsymbol{u}(\boldsymbol{x}')\,\mathrm{d}V'\,\mathrm{d}V + \\
& - \int_V \boldsymbol{b}(\boldsymbol{x})\cdot\boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}V - \int_{S_t} \boldsymbol{t}(\boldsymbol{x})\cdot\boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}S.
\end{aligned}
\tag{5}
$$

Grounding on Eq.(5) a nonlocal finite element formulation can be obtained from the following discretized form of functional Π:

$$
\begin{aligned}
\Pi[\boldsymbol{d_n}] = & \frac{1}{2}\sum_{n=1}^{N_e} \boldsymbol{d}_n^T\,\boldsymbol{k}_n^{loc}\,\boldsymbol{d}_n + \frac{\alpha}{2}\sum_{n=1}^{N_e} \boldsymbol{d}_n^T\,\boldsymbol{k}_n^{nonloc}\,\boldsymbol{d}_n + \\
& - \frac{\alpha}{2}\sum_{n=1}^{N_e}\sum_{m=1}^{N_e} \boldsymbol{d}_n^T\,\boldsymbol{k}_{nm}^{nonloc}\,\boldsymbol{d}_m - \sum_{n=1}^{N_e} \boldsymbol{d}_n^T\,\boldsymbol{f}_n,
\end{aligned}
\tag{6}
$$

where $\boldsymbol{d}_n$ and $\boldsymbol{d}_m$ are the nodal displacements vectors of finite elements $\#n$ and $\#m$ respectively, $\boldsymbol{f}_n$ is the equivalent nodal forces vector, $\boldsymbol{k}_n^{loc}$ denotes the local element stiffness matrix, while $\boldsymbol{k}_n^{nonloc}$ and $\boldsymbol{k}_{nm}^{nonloc}$ are the element nonlocal stiffness matrices. Finally, N_e is the number of finite elements in which V has been subdivided.
With reference to Eq.(6) the following positions hold true:

$$
\boldsymbol{k}_n^{loc} := \int_{V_n} \boldsymbol{B}_n^T(\boldsymbol{x})\,\boldsymbol{D}(\boldsymbol{x})\,\boldsymbol{B}_n(\boldsymbol{x})\,\mathrm{d}V_n; \tag{7}
$$

$$
\boldsymbol{k}_n^{nonloc} := \int_{V_n} \boldsymbol{B}_n^T(\boldsymbol{x})\,\gamma^2(\boldsymbol{x})\,\boldsymbol{D}(\boldsymbol{x})\,\boldsymbol{B}_n(\boldsymbol{x})\,\mathrm{d}V_n; \tag{8}
$$

$$\boldsymbol{k}_{nm}^{nonloc} := \int_{V_n}\int_{V_m} \boldsymbol{B}_n^T(\boldsymbol{x})\, \boldsymbol{\mathcal{J}}\,(\boldsymbol{x},\boldsymbol{x}')\, \boldsymbol{B}_m(\boldsymbol{x}')\, \mathrm{d}\, V_m\, \mathrm{d}\, V_n; \tag{9}$$

$$\boldsymbol{f}_n := \int_{V_n} \boldsymbol{N}_n^T(\boldsymbol{x})\, \boldsymbol{b}(\boldsymbol{x})\, \mathrm{d}\, V_n + \int_{S_{t(n)}} \boldsymbol{N}_n^T(\boldsymbol{x})\, \boldsymbol{t}(\boldsymbol{x})\, \mathrm{d}\, S_n. \tag{10}$$

In the above expressions $\boldsymbol{N}_n(\boldsymbol{x})$ and $\boldsymbol{B}_n(\boldsymbol{x})$ denote the matrices of the *n-th* element shape functions and their Cartesian derivatives, respectively.
It is worth noting that matrix $\boldsymbol{k}_n^{nonloc}$ accounts for the influence exerted on the *n-th* element by the nonlocal diffusive processes over the whole domain and this by the presence of $\gamma^2(\boldsymbol{x})$; matrix $\boldsymbol{k}_{nm}^{nonloc}$ accounts explicitly for the nonlocal effects exerted by the *m-th* element on the *n-th* one and this by the presence of $\boldsymbol{B}_n(\boldsymbol{x})$ and $\boldsymbol{B}_m(\boldsymbol{x}')$ related to the elements $\#n$ and $\#m$, respectively. $\boldsymbol{k}_{nm}^{nonloc}$ is a *set of nonlocal matrices* pertaining to element $\#n$, precisely: a self-stiffness matrix, obtained for $m = n$, plus all the cross-stiffness matrices given by $m = 1, 2, ..., N_e$ with $m \neq n$.
Following the standard rationale of the FEM, Eq.(6) can be rewritten in terms of global DOFs and by minimization would furnish the solving global linear equation system, (see [3] for more details).

3 A SYMMETRIC NONLOCAL SQUARE PLATE

As already asserted in the introductory section, the analysis of a symmetric structure made of a nonlocal elastic material does not lead to a correct solution when a standard symmetric portion of the structure is considered. This assertion appears now more evident if the implications given by the constitutive assumption of Eq.(1) are considered. The boundary zone adjacent to the symmetry line and belonging to the removed symmetric portion affects the mechanical behavior of its mirror reflection on the portion to be analyzed. The dimension of the quoted boundary zone is related to the influence distance L_R within which all the nonlocal effects are active. With the aid of the following numerical example the effects on the solution due to nonlocality will be highlighted, together with the remedies proposed to recover the correct solution.
Let's consider the nonhomogeneous square plate shown in Fig.1a having side length equal to 5 cm and thickness t equal to 0.5 cm. The plate is made up of a nonlocal elastic material for which the parameter ℓ is set equal to 0.1 cm, while the parameter α is set equal to 50. A central square part of the plate, of sides $a = 1$ cm, has Young's modulus $E_1 = 84$ GPa while the remaining part has Young's modulus $E_2 = 260$ GPa; a Poisson's coefficient $\nu = 0.2$ is assumed for the whole structure. The plate is fixed at the left edge (i.e. at $x = 0$ cm) and it is subjected to a uniform prescribed displacement $\bar{u}_x = 0.001$ cm at the opposite edge ($x = 5$ cm). In the performed analysis a normalized bi-exponential attenuation function is considered in the form:

$$g(\boldsymbol{x},\boldsymbol{x}') := \frac{1}{2\pi\ell^2 t} \exp\left(-|\boldsymbol{x}-\boldsymbol{x}'|/\ell\right). \tag{11}$$

The structure of Fig.1a is symmetric with respect to geometry, materials and boundary conditions and Fig.1c reports the classical symmetric scheme to be utilized in the analysis of the plate. However, if the model of Fig.1c is assumed when the considered structure is made of a nonlocal elastic material, the results obtained from models a) and c) turn out to be more different as much more are the nonlocal effects.
This is shown by the results given in terms of strain component profiles ε_x, ε_y and ε_{xy} plotted at the mid-plate horizontal section, $\bar{y} \backsimeq 2.5$ cm drawn in Figs.2a-c for models b), complete,

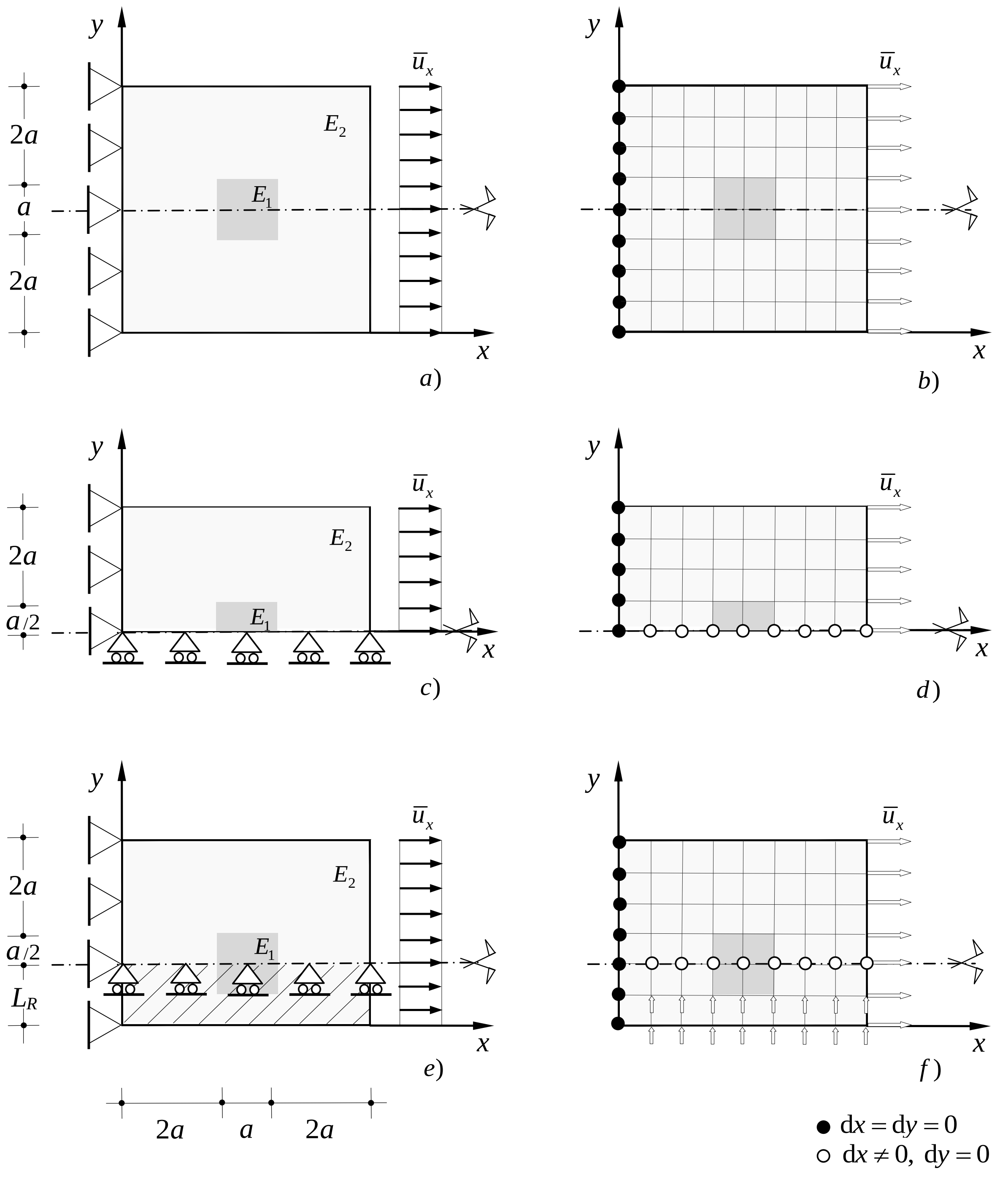

Figure 1: Nonlocal elastic symmetric square plate under tension with piecewise constant Young modulus. Mechanical model of: *a*) whole structure; *c*) symmetric half structure; *e*) enlarged symmetric half structure. *b*), *d*) and *f*) FE models of structures *a*), *c*) and *e*), respectively.

Figure 1: Nonlocal elastic symmetric square plate under tension with piecewise constant Young modulus. Mechanical model of : *a*) whole structure; *c*) symmetric half structure; *e*) enlarged symmetric half structure. *b*), *d*) and *f*) FE models of structures a), c) and e), respectively.

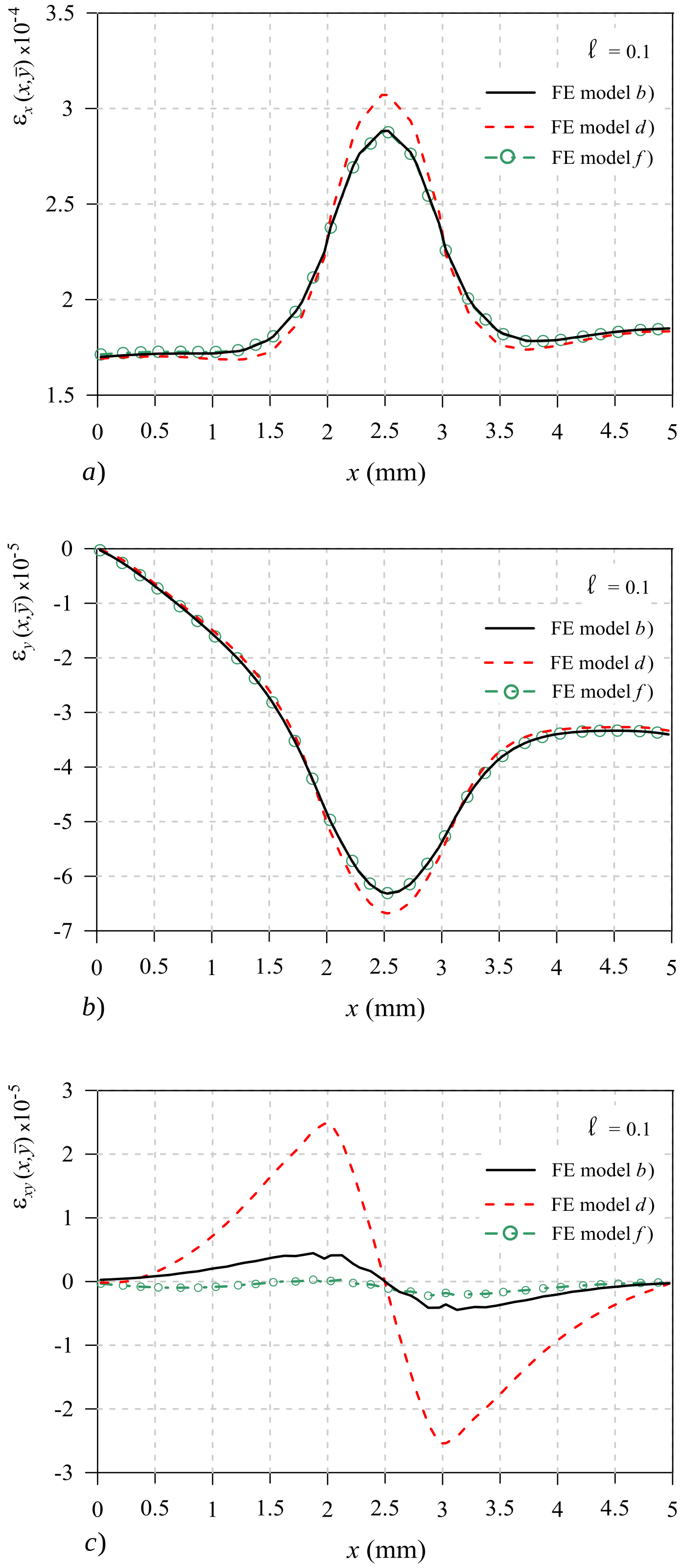

Figure 2: Nonlocal elastic symmetric square plate. Strain components profiles ε_x, ε_y, and ε_{xy} versus x at $y \backsimeq 2.5$. Solution referred to the entire plate (solid lines), solution referred to the half symmetric portion (dashed lines), solution referred to the enlarged half symmetric portion (dashed lines with circles).

and d), reduced. In particular the structure of Fig.1a has been discretized into 400 FEs of equal dimensions and consequently the model of the half classical symmetric structure is made of 200 FEs.

By inspection of Figure 2 appears evident how the half symmetric model furnishes a solution that deviates from the expected one. In order to recover the correct nonlocal solution the symmetric scheme of Fig.1c has to be enlarged, with respect to the symmetry axis, by adding a *symmetrical boundary zone* whose wideness is equal to the influence distance L_R as sketched in Fig.1e by the filled area below the symmetry axis. The related FE model of Fig.1f utilizes 320 FEs. To the enlarged model have also to be applied appropriate kinematic boundary conditions able to mimic the missing nonlocal effects. However, such kinematic conditions are unknown and the question of their specification is a crucial point. The need of them, to attain the exact solution, is here proved by making use of the solution in terms of displacements of the original (entire) structure solved on the scheme of Fig.1a. As in fact, if as nonlocal boundary conditions are assumed the vertical displacements coming from the solution (in the corresponding zone) of the original (entire) structure, the enlarged symmetric model gives exactly the expected solution for the strain profiles ε_x and ε_y and a very good approximation of the strain profile ε_{xy}, as shown in Figures 2.

4 CONCLUSIONS

The paper has pointed out some incoherences arising in the study of a symmetric structure made up of a nonlocal elastic material. In particular, it is highlighted that the physical reason for such incoherences is due to the fact that the removal of a symmetric portion produces the loss of the nonlocal effects exerted by that portion. It is shown that, to re-introduce in the solution the missed nonlocal effects, an enlarged symmetric model, with respect to the standard symmetric one, has to be employed. Appropriate nonlocal boundary conditions have also to be considered.

REFERENCES

[1] Z.P. Bažant, M.Jirásek. Nonlocal integral formulations of plasticity and damage: survey of progress. *Journal of Engineering Mechanics* **11**, 1119–1149, 2002.

[2] Polizzotto C., Fuschi P. and Pisano A.A. “A nonhomogeneous nonlocal elasticity model,” *European Journal of Mechanics A/Solids* **25**, 308–333, 2006.

[3] Fuschi P., Pisano A.A. and De Domenico D. “Plane stress problems in nonlocal elasticity: finite element solutions with a strain-difference-based formulation,” *Journal of Mathematical Analysis and Applications* **431**, 714–736, 2015.