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ENERGETIC BEM FOR THE NUMERICAL SOLUTION OF DAMPED WAVE PROPAGATION EXTERIOR PROBLEMS

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Abstract. *The analysis of damping phenomena that occur in many physics and engineering problems, such as fluid dynamics, kinetic theory and semiconductors, is of particular interest. For this kind of problems, one needs accurate and stable approximate solutions even on large time intervals. These latter can be obtained reformulating time-dependent problems modeled by partial differential equations (PDEs) of hyperbolic type in terms of boundary integral equations (BIEs) solved via boundary element methods (BEMs).*
In this context, starting from a recently developed energetic weak formulation of the space-time BIE modeling, in particular, classical wave propagation exterior problems [2, 3], we consider here an extension for the damped wave equation in 2D space dimension, based on successful simulations for the 1D case [6, 7].
In fact, the related energetic BEM reveals a robust time stability property, which is crucial in guaranteeing accurate numerical solutions on large time intervals. Several benchmarks will be presented and discussed.

1 INTRODUCTION

A variety of engineering and physical applications, such as the propagation or the scattering of acoustic or electromagnetic waves, leads to the problem of solving linear hyperbolic partial differential equations in two or three dimensional space. These problems are normally considered in an unbounded homogeneous domain and a method to tackle them is to reformulate the partial differential equation as an integral equation on the, usually bounded, boundary of the domain (BIE) which can then be solved using the boundary element method (BEM) ([11], [19]). The BIE-BEM combination is especially useful for solving problems of practical importance with irregular geometries. In some applications, the physically relevant data are given not by the solution in the interior of the domain but rather by the boundary values of the solution or its derivatives. These data can be obtained directly from the solution of boundary integral equations, whereas boundary values obtained from finite element method (FEM) solutions are in general not very accurate. Sometimes, however, a coupling of FEM and BEM proves to be useful. Also the analysis of damping phenomena that occur, for example, in fluid dynamics and in kinetic theory is of particular interest [8, 16, 23]. The use of advanced numerical techniques to solve PDEs, such as the finite element method (FEM) and the finite difference (FD) methods, for instance in structural mechanics and for fluid flow calculations, is now well established.
On the other side, both frequency-domain and time-domain boundary element method (BEM) can be used for hyperbolic boundary value problems [9, 10].
The most frequently adopted discretization scheme is the collocation technique [12, 15] with direct step-by-step evaluation of the time convolution, even if this approach has never reached the same level of maturity as in the frequency domain: the high computing costs of the time convolution and the limited robustness associated to instability phenomena have hindered considerably its diffusion. This is partly because accuracy and stability of the solution are affected by the time step size. Properties of the numerical solution by this method have been already studied in [14]. The former issue has been addressed in several ways. The time convolution can be evaluated applying the convolution quadrature method which has been developed initially in [20, 21] and has since then been successfully applied to many applications (see e.g.[22]). This method has the fundamental property of not using explicitly the expression of the kernel which is instead replaced by its Laplace transform.
On the contrary, the latter problem has been recently tackled in [1, 5, 4] where an energetic direct space-time Galerkin BEM for the discretization of retarded potential BIEs related to 1D, 2D and 3D wave propagation problems has been put forward and compared with the weak formulation due to Bamberger and Ha-Duong [9, 10, 17, 18]. The proposed technique is based on a natural energy identity satisfied by the solution of the corresponding differential problem, which leads to a space-time weak formulation of the BIEs with precise stability properties. Consequently, the integral problem can be discretized by unconditionally stable schemes via the so-called energetic BEM.
In this context, starting from the application of energetic BEM to classical wave propagation exterior problems [2, 3], we consider here an extension for the damped wave equation in 2D space dimension, based on successful simulations for the 1D case [6, 7]. Several benchmarks will be presented and discussed.

2 MODEL PROBLEM and its BIE ENERGETIC WEAK FORMULATION

We will analyze the following initial-boundary value problem for damped wave equation exterior to an open arc $\Gamma \subset \mathbf{R}^2$

$$\Delta u(x,t) - \frac{1}{c^2}\ddot{u}(x,t) - \frac{2D}{c^2}\dot{u}(x,t) - \frac{P}{c^2}u(x,t) = 0 \qquad x \in \mathrm{R}^2 \setminus \Gamma,\ t \in [0,T] \tag{1}$$

$$u(x,0) = 0 \qquad x \in \Gamma \tag{2}$$

$$\dot{u}(x,0) = 0 \qquad x \in \Gamma \tag{3}$$

$$u(x,t) = \bar{u}(x,t) \qquad x \in \Gamma,\ t \in (0,T) \tag{4}$$

where overhead dots indicate derivatives with respect to time, c is the propagation velocity of a perturbation in the domain, $\bar{u}(x,t)$ is the Dirichlet boundary datum; the viscous damping term is characterized by the coefficient D, while the material damping is modeled by the coefficient P.

Since the goal of this paper is to approximate u using a BEM technique, we have to obtain a boundary integral reformulation of the problem (1)-(4) over Γ.

For this purpose, using classical arguments [13], let us consider the single layer space-time integral representation formula of $u(x,t)$:

$$u(x,t) = \int_\Gamma \int_0^t G(r, t-\tau)\phi(\xi,\tau)\,d\tau\, d\gamma_\xi =: \mathcal{V}\phi(x,t), \qquad x \in \mathrm{R}^2 \setminus \Gamma,\ t \in [0,T] \tag{5}$$

where $r = \|x-\xi\|_2$, $\phi(x,t) = \left[\frac{\partial u}{\partial \mathbf{n}_x}(x,t)\right]_\Gamma$ is the jump of $u(x,t)$ along Γ and $G(x,t)$ is the forward fundamental solution of the two-dimensional damped wave operator, that is

$$G(r,t) = \begin{cases} \frac{c}{2\pi}\, e^{-Dt}\, \frac{\cos\left(\frac{\sqrt{P-D^2}}{c}\sqrt{c^2t^2-r^2}\right)}{\sqrt{c^2t^2-r^2}} H[ct-r]\,, \quad P \geq D^2 \\[2ex] \frac{c}{2\pi}\, e^{-Dt}\, \frac{\cosh\left(\frac{\sqrt{D^2-P}}{c}\sqrt{c^2t^2-r^2}\right)}{\sqrt{c^2t^2-r^2}} H[ct-r]\,, \quad P < D^2 \end{cases} \tag{6}$$

with $H[\cdot]$ the Heaviside distribution.

Now, with a limiting process for x tending to Γ and using the Dirichlet datum (4), one obtains the following BIE

$$\mathcal{V}\phi(x,t) = \overline{u}(x,t)\,, \quad x \in \Gamma,\ t \in [0,T] \tag{7}$$

whose energetic weak formulation reads (see [2]):

find $\phi \in H^0([0,T], H^{-1/2}(\Gamma))$ such that

$$\langle (\dot{\mathcal{V}\phi}), \psi \rangle_{\Gamma \times L^2([0,T])} = \langle \dot{\overline{u}}, \psi \rangle_{\Gamma \times L^2([0,T])}\,, \tag{8}$$

where ψ is a suitable test function belonging to the same functional space of ϕ.

Note that, performing a time integration by parts in the sense of distributions, the equation (8) can be equivalently rewritten as

$$\langle (\mathcal{V}\phi), \dot{\psi} \rangle_{\Gamma \times L^2([0,T])} = \langle \overline{u}, \dot{\psi} \rangle_{\Gamma \times L^2([0,T])}\,. \tag{9}$$

3 SPACE-TIME GALERKIN APPROXIMATION

For time discretization we consider a uniform decomposition of the time interval $[0,T]$ with time step $\Delta t = T/N_{\Delta t}$, $N_{\Delta t} \in \mathrm{N}^+$, generated by the $N_{\Delta t}+1$ time instants: $t_k = k\Delta t,\ k = 0,\ldots,N_{\Delta t}$ and we choose piecewise constant shape functions for the time approximation of ϕ.

In particular, for $k = 0, \ldots, N_{\Delta t} - 1$, time shape functions for the approximation of ϕ will be defined as

$$\overline{\varphi}_k(t) = H[t - t_k] - H[t - t_{k+1}] \,. \tag{10}$$

For the space discretization we consider a polygonal approximation of boundary Γ denoted by $\widehat{\Gamma}$ and constituted by $N_{\Delta x}$ straight elements $e_i, i = 1, \ldots, N_{\Delta x}$, with $length(e_i) \leq \Delta x$, $e_i \cap e_j = \emptyset$ if $i \neq j$ and such that $\cup_{i=1}^{N_{\Delta x}} \bar{e}_i = \widehat{\Gamma}$.

The functional background compels one to choose spatial shape functions belonging to $L^2(\hat{\Gamma})$ for the approximation of ϕ. Hence, having defined $\mathcal{P}_{d_i}$ the space of algebraic polynomials of degree d_i on the element e_i of $\widehat{\Gamma}$, we consider the space of piecewise polynomial functions

$$X_{\Delta x}^{-1} := \{\widetilde{\varphi}(x) \in L^2(\widehat{\Gamma}) \; : \; \widetilde{\varphi}_{|e_i} \in \mathcal{P}_{d_i} \,, \, \forall \, e_i \subset \widehat{\Gamma} \,\} \,.$$

Hence, denoting with M, the number of unknowns on $\widehat{\Gamma}$, and having introduced in $X_{\Delta x}^{-1}$ the standard piecewise polynomial boundary element basis functions $\varphi_j(x),\ j = 1, \cdots, M$, the approximate solutions of the problem at hand will be expressed as

$$\widehat{\phi}(x,t) := \sum_{k=0}^{N_{\Delta t}-1} \sum_{i=1}^{M} \alpha_i^k \, \widetilde{\varphi}_i(x) \, \overline{\varphi}_k(t) \,. \tag{11}$$

Energetic Galerkin BEM is obtained inserting the introduced discretization into the weak problem (9) and its algebraic reformulation consists in a linear system: $A\Phi = b$ of order $(M \cdot N_{\Delta t})$, whose matrix elements are linear combination of integrals of the form

$$\int_{\widehat{\Gamma}} \widetilde{\varphi}_j(x) \int_0^T \dot{\overline{\varphi}}_h(t) \int_{\widehat{\Gamma}} \widetilde{\varphi}_i(\xi) \int_0^t G(r, t - \tau) \overline{\varphi}_k(\tau) d\tau d\gamma_\xi dt d\gamma_x \,. \tag{12}$$

Specifying the choice made for time basis function, from (12) one obtains a combination of integrals of the form

$$\int_{\widehat{\Gamma}} \widetilde{\varphi}_j(x) \int_{\widehat{\Gamma}} \widetilde{\varphi}_i(\xi) \int_0^{t_h} G(r, t_h - \tau) H[\tau - t_k] d\tau d\gamma_\xi d\gamma_x \,. \tag{13}$$

The analysis of kernel singularities has been performed for the case of the wave equation without damping, i.e. for $P = D = 0$, in [2]; in particular the presence of the Heaviside distribution $H[ct - r]$ and of the square root $\sqrt{c^2t^2 - r^2}$ can cause a lot of numerical troubles that in [2] have been solved by suitable splitting of the outer integral over $\widehat{\Gamma}$ and using quadrature schemes which regularize integrand functions with mild singularities for the second nested integral in space variable. Since we expect a similar behavior for the damped kernel, having set

$$G_0(r,t) = \frac{c}{2\pi} \frac{1}{\sqrt{c^2t^2 - r^2}} H[c\,t - r] \tag{14}$$

the fundamental solution related to the classical wave operator, we consider the expansion of $G(r,t)$ with respect to damping parameters, centered in $P = D = 0$, and we rewrite (13) as

$$\begin{aligned} &\int_{\widehat{\Gamma}} \widetilde{\varphi}_j(x) \int_{\widehat{\Gamma}} \widetilde{\varphi}_i(\xi) \int_0^{t_h} [G(r, t_h - \tau) - G_0(r, t_h - \tau)] H[\tau - t_k] d\tau d\gamma_\xi d\gamma_x \\ &+ \int_{\widehat{\Gamma}} \widetilde{\varphi}_j(x) \int_{\widehat{\Gamma}} \widetilde{\varphi}_i(\xi) \int_0^{t_h} G_0(r, t_h - \tau) H[\tau - t_k] d\tau d\gamma_\xi d\gamma_x \,, \end{aligned} \tag{15}$$

in such a way that the problematic issues described above are confined in the second term, whose time integral can be evaluated analytically. The time integral of the first term is instead performed numerically. At last, the outer integrals over $\widehat{\Gamma}$ are numerically treated by suitable quadrature schemes as in [2].

4 Numerical Results

In the following, we will present some numerical results obtained for 2D exterior problems starting from the proposed energetic weak formulation. We consider the problem (1)-(4) with $\Gamma = \{(x,0)\,|\,x \in [-1,1]\}$ and the Dirichlet boundary datum

$$g(x,t) = -H[t-kx]f(t-kx)x, \quad \text{where} \quad f(z) = \begin{cases} \sin^2\left(\dfrac{\omega z}{2}\right), & \text{if} \quad 0 \leq z \leq \dfrac{\pi}{\omega} \\ 1, & \text{if} \quad z \geq \dfrac{\pi}{\omega} \end{cases}$$

with $\omega = 8\pi$, $k = \cos\theta$ and $\theta = \pi/2$ as suggested in [2]. Further the velocity is fixed as $c = 1$. For the energetic BEM, constant shape and test functions in space and time are here always adopted.

At first, we chose a uniform decomposition of Γ in 80 straight elements ($\Delta x = 0.025$) and we set $\Delta t = 0.025$. We show in Figure 1, the time history of the solutions $\phi(x,t)$ on the straight element e_{20} and on the time interval $[0,4]$, for $P = 0$ varying $D = 0, 1, 10, 20$ on the left and for $D = 0$ varying $P = 0, 10, 20$ on the right. Note the effects of increasing viscous and material damping which substantially change the aspect of the solution related to the classical wave equation, visible in the graphs for trivial parameters.

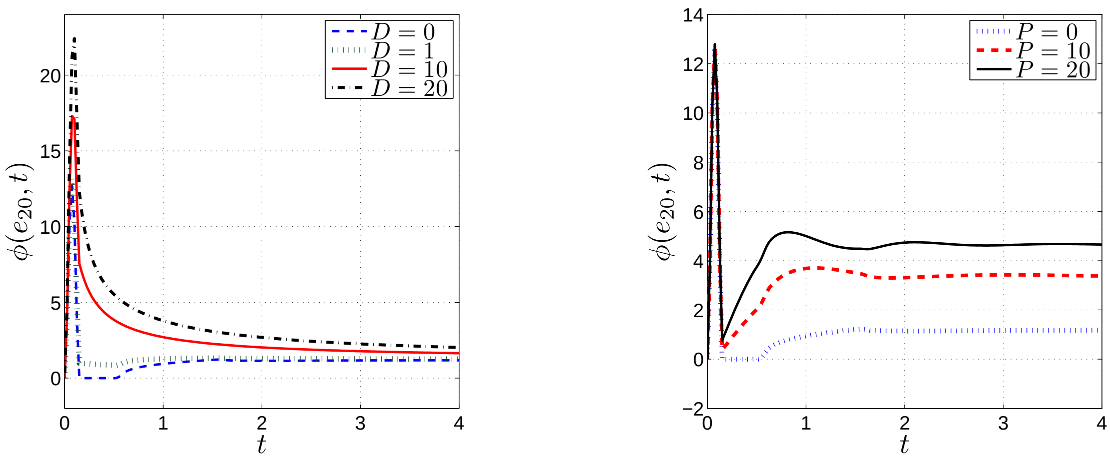

Figure 1: Time history of $\phi(x,t)$ on straight element e_{20}.

Then, we chose a uniform decomposition of Γ in 40 subintervals ($\Delta\, x = 0.05$) and enlarge the observation time interval, fixing $T = 10$ and $\Delta t = 0.1$. Since in this benchmark the Dirichlet datum becomes independent of time, for $P = 0$ we expect that the BIE transient solution $\phi(x,t)$ on Γ tends to the stationary one $\phi_\infty(x)$ (Fig. 2, left), i.e. the solution of the BIE related to the following Dirichlet problem for the Laplace equation:

$$\begin{array}{ll} \Delta u_\infty(x) = 0 & x \in \mathrm{R}^2 \setminus \Gamma \\ u_\infty(x) = -x & x \in \Gamma \\ u_\infty(x) = O(1) & \|x\| \to \infty \end{array}$$

Looking at the graphs of the time history of $\|\phi(\cdot,t) - \phi_\infty(\cdot)\|_{L^1(\Gamma)}$, shown in Figure 2 (right),

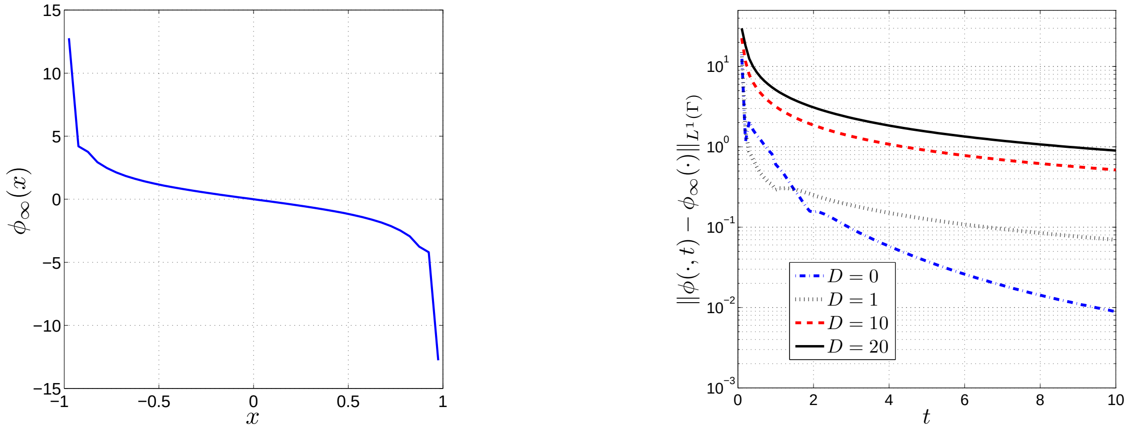

Figure 2: Stationary solution $\phi_\infty(x)$ on Γ (left); convergence of damped BIE solution towards the solution of Laplace problem, when $P = 0$ (right)

we observe the expected convergence, that becomes slower for increasing values of parameter D.
Analogously, for $D = 0$ and $P \neq 0$ we expect that the transient solution $\phi(x,t)$ on Γ tends to the stationary one $\bar{\phi}_\infty(x)$ (Fig. 3), i.e. the solutions of the BIE related to the following Dirichlet problem for the Helmholtz equation:

$$\begin{array}{ll} \Delta v_\infty(x) + k^2 v_\infty(x) = 0 & x \in \mathrm{R}^2 \setminus \Gamma \\ v_\infty(x) = -x & x \in \Gamma \\ v_\infty(x) = O(\|x\|^{-1}) & \|x\| \to \infty \end{array}$$

with wave number $k = \sqrt{-P}/c^2$.

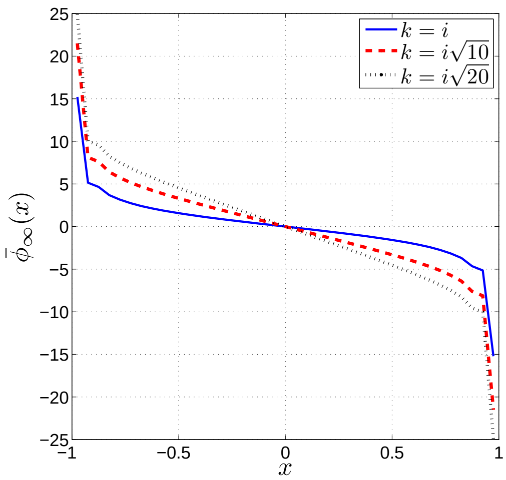

Figure 3: Stationary solutions $\bar{\phi}_\infty(x)$ on Γ.

Looking at the graphs of the time history of $\|\phi(\cdot,t) - \bar{\phi}_\infty(\cdot)\|_{L^1(\Gamma)}$, in Figure 4, we observe the expected convergence, that becomes more oscillating for increasing values of parameter P.

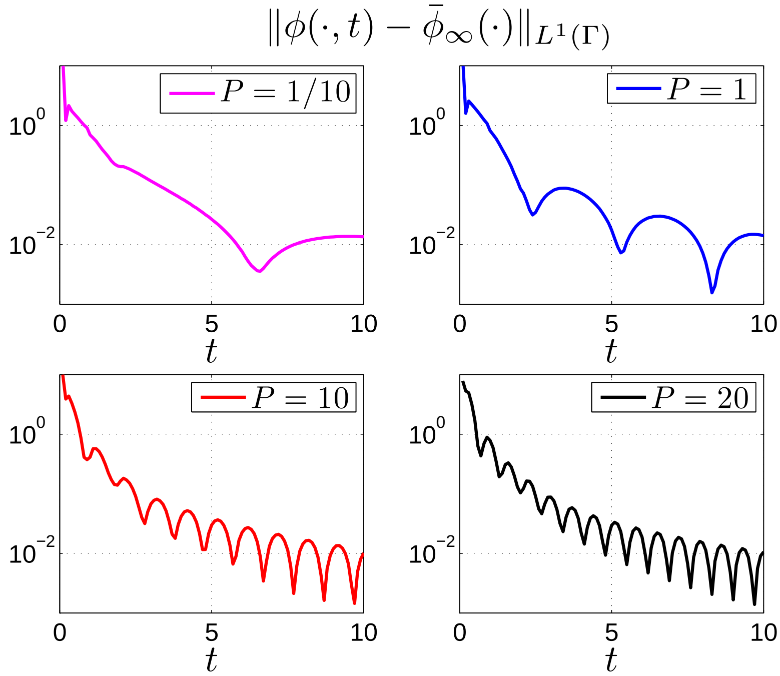

Figure 4: Convergence of damped BIE solution towards the solution of Helmholtz problem, when $D = 0$ and $P \neq 0$.

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