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# BEARING CAPACITY OF STRIP FOOTINGS ON SPATIALLY RANDOM SOILS USING KRIGING AND MONTE CARLO SIMULATION

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***Abstract.*** *The probabilistic analysis of geotechnical structures presenting spatial variability in the soil properties is generally performed using Monte Carlo simulation (MCS) methodology. This methodology is not suitable for the computation of a small failure probability because it becomes very time-expensive in such a case due to the large number of simulations required to calculate the failure probability. For this reason, one needs to keep to a minimum the number of calls to the deterministic model when performing the probabilistic analyses. In order to overcome the shortcoming related to the excessive number of calls of the deterministic model when performing Monte Carlo simulations, Echard et al. (2011) proposed an Active learning reliability method (called AK-MCS) combining Kriging and Monte Carlo Simulation. The method was shown to be very efficient as the obtained probability of failure is very accurate needing only a small number of calls to the deterministic model. The probabilistic analyses presented in this paper were performed using AK-MCS methodology by Echard et al. (2011). The present study involves a probabilistic analysis at the ultimate limit state of a strip footing resting on a spatially varying soil using AK-MCS approach. The objective is the computation of the probability $P_f$ of exceeding the ultimate bearing capacity of the footing under a prescribed footing load. The soil cohesion and angle of internal friction were considered as two non-isotropic non-Gaussian random fields. The deterministic model was based on numerical simulations using the finite difference code FLAC$^{3D}$. Some probabilistic results are presented and discussed.*

## 1 INTRODUCTION

The spatial variability of the soil properties affects the behavior of geotechnical structures (bearing capacity, foundation settlement, slope stability, etc.). Several authors have considered the effect of the spatial variability of the soil properties in their calculation models. For the bearing capacity of foundations which is the subject of the present paper, one may cite among others [1, 2].

It should be mentioned that when dealing with probabilistic studies that involve spatially varying soil properties, the classical Monte Carlo Simulation (MCS) methodology is generally used to determine the probability of failure $P_f$. This method is known to be very time-consuming. This is because (i) it generally makes use of finite element or finite difference models which are generally time-expensive and (ii) it requires a great number of calls of the deterministic model for the computation of the small failure probabilities encountered in practice. Thus, one needs to keep to a minimum the number of calls to the deterministic model when performing the probabilistic analyses.

In order to overcome the shortcoming related to the excessive number of calls of the deterministic model when performing Monte Carlo simulations, Echard et al. [3] proposed an active learning reliability method (called AK-MCS) combining Kriging and Monte Carlo Simulation. The method was shown to be very efficient as the obtained probability of failure is very accurate needing only a small number of calls to the deterministic model. The probabilistic analyses presented in this paper were performed using AK-MCS methodology by [3]. They involve the probabilistic analysis at the ultimate limit state of a strip footing resting on a spatially varying soil using AK-MCS approach. The objective is the computation of the probability $P_f$ of exceeding the ultimate bearing capacity of the footing under a prescribed footing load. The soil cohesion and angle of internal friction were considered as two non-isotropic non-Gaussian random fields. They are characterized by two specified marginal distribution functions and a common autocorrelation function. The methodology proposed by [4] is used to generate these two random fields. The deterministic model is based on numerical simulations using FLAC$^{3D}$ software. Some probabilistic results are presented and discussed.

The paper is organized as follows: The next section aims at presenting the probabilistic method used for the computation of the failure probability of a strip footing resting on a spatially varying soil. It is followed by some numerical results. The paper ends with a conclusion.

## 2 PROBABILISTIC MODELS

The aim of this section is to present the probabilistic model used in the present paper. The impact of the soil spatial variability on the ultimate bearing capacity was considered. It should be remembered here that the system response involves the ultimate bearing capacity ($P_u$) of a strip footing resting on a spatially varying soil. The soil cohesion c and friction angle φ were modeled as two non-isotropic non-Gaussian random fields. The EOLE methodology [4] was used to discretize the two random fields (i.e. to obtain realizations of the soil cohesion c and friction angle φ that respect the correlation structure of those fields). The performance function used to calculate the probability $P_f$ of exceeding the ultimate bearing capacity was defined as follow:

$$G = \frac{P_u}{P_s} - 1 \tag{1}$$

Where $P_s$ is the prescribed footing load. As for the probabilistic method used in this paper, the active learning reliability method combining kriging and Monte Carlo Simulation (called AK-MCS) was employed. This method combines both the classical crude Monte Carlo Simulation (MCS) methodology and the Kriging meta-model technique.

In this section, the EOLE method of discretization of random fields was firstly presented. It is followed by a brief presentation of the crude Monte Carlo method, the kriging metamodeling technique and the combined use of the kriging and the Monte Carlo Simulation (i.e. AK-MCS methodology) used for the probabilistic analysis.

## 2.1 Method of generation of anisotropic non-Gaussian random fields

Let us consider two anisotropic non-Gaussian random fields $Z_i^{NG}(x, y)(i = c, \varphi)$ described by: (i) constant means and standard deviations ($\mu_i$, $\sigma_i$; $i = c, \varphi$), (ii) non-Gaussian marginal cumulative distribution functions $G_i$ ($i = c, \varphi$), and (iii) a common square exponential autocorrelation function $\rho_Z^{LN}$ [(x, y), (x', y')] which gives the values of the correlation between two arbitrary points (x, y) and (x', y'). This autocorrelation function is given as follows:

$$\rho_z^{LN}[(x,\ y),\ (x\ ',\ y\ ')] = \exp\left( -\left( \frac{|x - x\ '|}{a_x} \right)^2 - \left( \frac{|y - y\ '|}{a_y} \right)^2 \right) \tag{2}$$

where $a_x$ and $a_y$ are the autocorrelation distances along x and y respectively. The Expansion Optimal Linear Estimation method (EOLE) and its extension to cover the case of non-Gaussian random fields are used herein to generate the two random fields of c and φ. Notice that EOLE was first proposed by [4] for the case of uncorrelated Gaussian fields, and then extended by [5] to cover the case of correlated and uncorrelated non-Gaussian fields. In this method, one should first define a stochastic grid composed of q grid points (or nodes) $\left\{ (x_1,\ y_1),\ ...,\ (x_q,\ y_q) \right\}$ for which the values of the field are assembled in a vector $\chi = \left\{ Z(x_1,\ y_1),\ ...,\ Z(x_q,\ y_q) \right\}$. The number of grid points within the stochastic mesh is determined in such a manner that in each direction (x or y), there are five grid points within each autocorraltion distance. Secondly, one should determine the correlation matrix for which each element $\left( \Sigma_{\chi;\chi} \right)_{i,j}^{NG}$ is calculated as follows:

$$\left( \Sigma_{\chi;\chi} \right)_{i,j}^{NG} = \rho_z^{NG}\left[ (x_i,\ y_i),\ (x_j,\ y_j) \right] \tag{3}$$

The common non-Gaussian autocorrelation matrix $\Sigma_{\chi;\chi}^{NG}$ should be transformed into the Gaussian space using Nataf model [6] since the discretization of the random fields using EOLE is done in the Gaussian space. As a result, one obtains two Gaussian autocorrelation matrices $\Sigma_{\chi;\chi}^{c}$ and $\Sigma_{\chi;\chi}^{\varphi}$ that can be used to discretize the two random fields as follows:

$$\tilde{Z}_i(x,\ y) = \mu_i + \sigma_i \sum_{j=1}^{N} \frac{\xi_{i,\ j}}{\sqrt{\lambda_j^i}} .\phi_j^i .\Sigma_{Z(x,y);\chi}^i \qquad i = c,\ \varphi \tag{4}$$

where $(\xi_{i,j}; i = c, \varphi)$ are two non-correlated blocks of independent standard normal random variables. Notice finally that $(\lambda_j^i, \phi_j^i; i = c, \varphi)$ in equation (4) are the eigenvalues and eigenvectors of the two Gaussian autocorrelation matrices $\Sigma_{\chi;\chi}^{c}$ and $\Sigma_{\chi;\chi}^{\varphi}$ respectively, and $\Sigma_{Z(x,y);\chi}$ is the correlation vector between the random vector χ and the value of the field at an

arbitrary point (x, y). Once the two Gaussian random fields are obtained, they should be transformed into the non-Gaussian space by applying the following formula:

$$\tilde{Z}_i^{NG}(x, y) = G_i^{-1}\left\{\Phi\left[\tilde{Z}_i(x, y)\right]\right\} \qquad i = c, \varphi \tag{5}$$

where $\Phi(.)$ is the standard normal cumulative density function.

### 2.2 Monte Carlo method

The Monte Carlo simulation method consists in generating *N* samples which respect the joint probability density function $f_X(X)$ of the *M* random variables ($X_1, \ldots, X_M$) gathered in a vector *X*. For each sample, the system response is calculated. Thus; for the *N* samples, one obtains *N* values of the system response gathered in a vector $\boldsymbol{G} = [G(\boldsymbol{x}_1), G(\boldsymbol{x}_2), \ldots, G(\boldsymbol{x}_N)]$ which may be used to determine the failure probability for a prescribed threshold of this system response. A very large number of realizations is required to obtain a rigorous value of the failure probability especially when computing small failure probabilities. It should be noted herein that the random variables considered in the present paper are the standard normal variables $\xi_j$ ($j=1, \ldots, N$) that appear in Eq. (4) for the computation of a given realization of the cohesion c and the friction angle random fields.

### 2.3 Kriging metamodeling technique

The metamodeling technique aims at replacing the response (or the performance function in this paper) of a computationally-expensive mechanical model by a metamodel (i.e. a simple analytical equation). The kriging metamodeling technique is based on the idea that the performance function $G(\boldsymbol{x})$ is seen as a realization of a stochastic field $\hat{G}(\boldsymbol{x})$ [12]. The kriging metamodeling needs a design of experiments DoE of length *N* to define the stochastic parameters of this field and then, predictions of the response can be obtained on any unknown point. Let us define the design of experiments by the vector $\boldsymbol{X} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N]$, with $\boldsymbol{x}_i$ the i[th] experiment, and $\boldsymbol{G} = [G(\boldsymbol{x}_1), G(\boldsymbol{x}_2), \ldots, G(\boldsymbol{x}_N)]$ the corresponding response vector. The approximate relationship between any experiment $\boldsymbol{x}$ and the response G($\boldsymbol{x}$) can be denoted as:

$$\hat{G}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\beta}) + \boldsymbol{Z}(\boldsymbol{x}) \tag{6}$$

Where:

- $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\beta})$ is the deterministic part which gives an approximation of the mean value of the performance function. It corresponds to a regression model that can be written as:

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\beta}) = \boldsymbol{f}^{\boldsymbol{T}}(\boldsymbol{x}).\boldsymbol{\beta} \tag{7}$$

Where $\boldsymbol{\beta}^T = [\beta_1, \ldots, \beta_p]$ is the vector of coefficients to be determined, $\boldsymbol{f}^T(\boldsymbol{x}) = [f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_p(\boldsymbol{x})]^T$ is a collection of regression functions (or regressors). The number *p* of regression functions is assumed to be less than or equal to the number *N* of observations so that the problem is not under-determined (i.e. it does not lack equations to compute the unknowns). In this paper, ordinary kriging is selected which means that $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\beta})$ is a scalar to be determined [i.e. $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\beta})=\beta$]. So the estimated $\hat{G}(\boldsymbol{x})$ can be simplified as:

$$\hat{G}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\beta}) + \boldsymbol{Z}(\boldsymbol{x}) = \beta + \boldsymbol{Z}(\boldsymbol{x}) \tag{8}$$

- $\boldsymbol{Z(x)}$ represents the fluctuation around the mean value. It is given by a Gaussian random process (or a random field) with zero mean and covariance defined by

$$Cov\left[Z(\boldsymbol{x_i}), Z(\boldsymbol{x_j})\right] = \sigma_z^2\, R\left(\boldsymbol{x_i}, \boldsymbol{x_j}\right) \tag{9}$$

where $\sigma_z^2$ is the process variance; $\boldsymbol{x}_i$, $\boldsymbol{x}_j$ are two points from the whole samples X and $R(\boldsymbol{x}_i, \boldsymbol{x}_j)$ is the correlation function between these two points with a correlation parameter vector $\boldsymbol{\theta}$. Several models exist to define the correlation function R. However, in this paper, the anisotropic squared-exponential function is selected. It is given by:

$$R\left(\boldsymbol{x}_i, \boldsymbol{x}_j\right) = \prod_{i=1}^{N} e^{\left(-\theta_i (x_i - x_j)^2\right)} \tag{10}$$

Where N is the number of random variables, $x_i$ and $x_j$ are the i[th] coordinates of the points $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ and $\theta_i$ is a scalar which is equal to the inverse of the correlation length in the i[th] direction.

As may be seen from equation (6), the kriging metamodel consists of a linear regression model $\boldsymbol{F(x, \beta)}$ and a stochastic process $\boldsymbol{Z(x)}$.

As $\boldsymbol{R}$ is a square matrix of dimension *N*x*N*, i.e. $\boldsymbol{R} = \left[R(\boldsymbol{x}_i, \boldsymbol{x}_j)\right]_{N \times N}$ and **F** is a unit vector of dimension N, then β and $\sigma_z^2$ can be estimated as:

$$\hat{\beta} = (\boldsymbol{F}^T \boldsymbol{R}^{-1} \boldsymbol{F})^{-1}\, \boldsymbol{F}^T\, \boldsymbol{R}^{-1}\, \boldsymbol{G} \tag{11}$$

$$\hat{\sigma}_z^2 = \frac{1}{N}\, (\boldsymbol{G} - \beta \boldsymbol{F})^T\, \boldsymbol{R}^{-1} (\boldsymbol{G} - \beta \boldsymbol{F}) \tag{12}$$

The correlation parameter θ can be obtained through the maximum likelihood estimation:

$$\theta = arg_\theta \min(N \ln \sigma_z^2 + \ln|\boldsymbol{R}|) \tag{13}$$

Once the parameters $\hat{\beta}$, $\hat{\sigma}_z^2$ and $\theta$ are determined, the Best Linear Unbiased Predictor (BLUP) of the response $\hat{G}(\boldsymbol{x})$ at an unknown point $\boldsymbol{x}$ is shown to be a Gaussian random variate $\hat{G}(\boldsymbol{x}) \sim N\,(\mu_{\hat{G}(x)}, \sigma_{\hat{G}(x)})$ where

$$\mu_{\hat{G}(x)} = \beta + \boldsymbol{r}(\boldsymbol{x}) \boldsymbol{R}^{-1} (\boldsymbol{G} - \beta \boldsymbol{F}) \tag{14}$$

$$\sigma^2_{\hat{G}\,(x)} = \sigma_z^2(\boldsymbol{x})(\boldsymbol{1} + u^T(\boldsymbol{x})(\boldsymbol{F}^T \boldsymbol{R}^{-1} \boldsymbol{F})^{-1} u^T(\boldsymbol{x}) - \boldsymbol{r}^T(\boldsymbol{x}) R^{-1} \boldsymbol{r}(\boldsymbol{x})) \tag{15}$$

Where $\boldsymbol{r(x)}$= [**R**(**x, x**$_1$), **R**(**x, x**$_2$), . . . , **R**(**x, x**$_N$ )], u(**x**) = $\boldsymbol{F}^T \boldsymbol{R}^{-1} \boldsymbol{r}(\boldsymbol{x}) - 1$. Notice that $\mu_{\hat{G}(x)}$ is usually taken as the estimated $\hat{G}(\boldsymbol{x})$ at point $\boldsymbol{x}$. The computation of $\mu_{\hat{G}(x)}$ and $\sigma^2_{\hat{G}\,(x)}$ can be obtained by MATLAB toolbox DACE [7].

Notice that the variance $\sigma^2_{\hat{G}\,(x)}$ is defined as the minimum of the mean squared error between $\hat{G}(\boldsymbol{x})$ and G(x). The variances of points in the initial DoE are zero, but the variances of the

other points are always not zero. A large value of $\sigma^2_{\hat{G}\ (x)}$ means that the prediction is not exact. Therefore, the variance prediction $\sigma^2_{\hat{G}\ (x)}$ is important in the unexplored areas and presents a good index to improve the initial DoE. This property is interesting and is used in the following paragraph.

## 2.4 AK-MCS methodology

In a first stage, a significant number of realizations (about 1 million) of the spatially varying soil medium were generated by MCS using the Expansion Optimal Linear Estimation (EOLE) method. The response of most of these realizations will not be computed from the original deterministic model; only a few numbers of realizations will be computed using this model depending on their corresponding learning function values. In a second stage, an initial small Design of Experiments (DoE) based on a random selection of some realizations among the MCS population was determined and employed to compute the Kriging model (based on the values of the responses obtained using the original deterministic model). In a third stage, the adopted learning function U (which makes use of the already-obtained kriging model) is employed to choose the best next realization (among all the realizations of the MCS population) to be used for the computation of the system response from the original deterministic model. This realization is the one that leads to the minimal value of the learning function U. In a final stage (stage 4), the response obtained from this realization is employed to update the kriging model with this new response value (only in case where the adopted stopping condition has not been reached). Notice that stages 3-4 should be repeated several times until reaching the adopted stopping condition. At the end, the surrogate kriging model is considered to be enough accurate for the estimation of the failure probability and the corresponding value of the coefficient of variation, for the prescribed number of simulations. If the coefficient of variation of the failure probability is found to be too high, the initial MCS population should be increased. A more comprehensive step-by-step procedure describing the implementation of the AK-MCS methodology for the computation of the failure probability may be given as follows:

1. Generate a population S of $n_{MC}$ (say 1,000,000) realizations of the spatially varying soil parameters c and φ using Monte Carlo Simulation. It should be emphasized here that the computation of the performance function for these realizations (as defined in equation 1 based on the computationally-expensive $FLAC^{3D}$ model) is not required at this stage.
2. Randomly select from the S population a small number of realizations (called initial Design of Experiments DoE) containing $N_1$ realizations (say 20). For those $N_1$ realizations, one should evaluate the performance function given by equation (1) based on the computationally expensive $FLAC^{3D}$ model. The required initial DOE may be greater than 20 for a great number of random variables (i.e. for a spatially varying soil property with small values of the autocorrelation distances). This is because for high dimensional problems (i.e. when a large number of random variables is needed to discretize the c and φ random fields), the kriging model needs more points (i.e. realizations) in order to compute correlation parameters $\theta_i$ given by equation (10).
3. Compute the kriging model according to the small initial DoE containing the $N_1$ realizations. This kriging model is given by equation (14). It should be mentioned here that this stage was performed using the Matlab toolbox DACE [7]. In this paper,

ordinary kriging model was used (i.e. the regression model is considered to be constant) and a square exponential correlation function was adopted in the analysis.

4. By using DACE toolbox, compute (for the whole population S containing the $n_{MC}$ realizations of the random field) both the kriging predictor values $\mu_{\hat{G}}$ of the performance function (based on equation 14) and their corresponding kriging variance values $\sigma_{\hat{G}}^2$ (based on equation 15). From the obtained values of the kriging predictors $\mu_{\hat{G}}$, obtain an estimation of the probability of failure $\hat{P}_f$ by counting the number of negative predictors $N_{\hat{G}\leq 0}$ and dividing it by the total number of samples in S as follows:

$$\hat{P}_f = \frac{N_{\hat{G}\leq 0}}{n_{MC}} \tag{16}$$

Also compute the coefficient of variation of $\hat{P}_f$ as follows:

$$\hat{COV}(\hat{P}_f) = \sqrt{\frac{1-\hat{P}_f}{\hat{P}_f n_{MC}}} \tag{17}$$

5. Identify the best next realization in S for which one will compute the performance function using FLAC$^{3D}$. This is performed by evaluating a learning function U for each realization in S. The learning function U is given by:

$$U(x_i) = \frac{\left|\mu_{\hat{G}(x_i)}\right|}{\sigma_{\hat{G}(x_i)}} \qquad i = 1, \ldots, n_{MC} \tag{18}$$

The best next candidate realization is the one with minimum value of U. If this minimum value of U is smaller than 2, the performance function based on FLAC$^{3D}$ is evaluated for this best candidate and the initial DoE is updated. Thus one should go back to step 3 and evaluate a new kriging model based on the updated DoE. Steps 3, 4 and 5 are repeated until the minimum value of U becomes larger than 2. At this stage the learning stops and the metamodel is considered accurate enough based on the $n_{MC}$ realizations.

6. When the learning stops, one must compute the estimated values of both the probability of failure $\hat{P}_f$ and the coefficient of variation $\hat{COV}\left(\hat{P}_f\right)$. The obtained value of $\hat{P}_f$ is considered to be accurate if $\hat{COV}\left(\hat{P}_f\right) \leq 5\%$. If the estimated coefficient of variation is larger than 5%, one must increase the population S in step 1 and repeat the procedure.

It should be emphasized herein that a small initial DoE is chosen within the present approach (see step 2) in order to keep to a minimum the number of calls to the deterministic model. This initial DoE is successively increased by a single realization at each time (see step 5). The chosen realization is the one that is improving the most the metamodel because equation (18) searches for the realization that has a small kriging predictor (i.e. a realization that is close to the limit state surface) and/or a high kriging variance (i.e. a high uncertainty in the sign of its performance function value). Notice that the realizations with high uncertainties in the sign of their performance function values (positive or negative) are those that are close to the limit state surface. Finally, notice that the stopping criterion min(U)>2 corresponds to a maximal probability of making a mistake

on the sign of the performance function of Φ(-2)=0.023. This means that the stopping criterion is relevant making use of the realizations with a small uncertainty in making a mistake in their performance functions' signs, the probability of making a mistake in the signs of their performance function values being negligible.

### 2.5 Probabilistic numerical results

The aim of this section is to present the probabilistic numerical results of a strip footing resting on a spatially varying soil and subjected to a vertical loading. The soil shear strength parameters (c and φ) were considered as anisotropic non-Gaussian random fields. The soil cohesion c was modeled by a lognormal distribution. Its mean value and its coefficient of variation (referred to in this paper as reference values) were taken as follows: $\mu_c = 20\ kPa$, $COV_c = 25\%$. On the other hand, the soil friction angle φ was assumed to be bounded (i.e $0 \leq \varphi \leq 45^{\circ}$). A beta distribution was selected for this parameter with a mean value and a coefficient of variation given as follows: $\mu_\varphi = 30^{\circ}$ , $COV_\varphi = 10\%$. **.** The soil dilation angle ψ was considered to be related to the soil friction angle φ by $\psi = 2\varphi / 3$. This means that the soil dilation angle was implicitly assumed as a random field that is perfectly correlated to the soil friction angle random field. Notice that the same autocorrelation function (square exponential) was used for both c and φ.

As for the autocorrelation distances ax and ay of the two random fields c and φ, both cases of isotropic (i.e. ax=ay) and anisotropic (i.e. ax>ay) random fields will be treated although the soil is rarely isotropic in reality. In our study, the reference values adopted for ax and ay were ax=10m and ay=1m. For the considered soil domain and for the different values of the autocorrelation distances (ax, ay) used in the analysis, the total number N of random variables (or eigenmodes) that is used to discretize the two random fields c and φ is determined based on a variance of the error ≤10%. It should be mentioned here that the initial design of experiment (DoE) contain 20 samples.

#### 2.5.1 Deterministic model

The deterministic model was based on numerical simulations using the finite difference code FLAC$^{3D}$. The soil behavior was modeled using a conventional elastic-perfectly plastic model based on Mohr-Coulomb failure criterion. Notice that the soil Young modulus E and Poisson ratio υ were assumed to be deterministic since the ultimate bearing capacity is not sensitive to these variables. Their corresponding values are respectively $E = 60MPa$ and $\nu = 0.3$. Finally, concerning the footing, a weightless strip foundation of 1m width and 0.25m height is used. It is assumed to follow an elastic linear model ( $E = 25GPa, \nu = 0.4$ ). The connection between the footing and the soil mass is modeled by interface elements having the same mean values of the soil shear strength parameters in order to simulate a perfectly rough soil-footing interface. These parameters have been considered as deterministic in this study. Concerning the elastic properties of the interface, they also have been considered as deterministic and their values are as follows: $K_s = 1GPa$ , $K_n = 1GPa$ where $K_s$ and $K_n$ are respectively the shear and normal stiffness of the interface.

As shown in Figure 1, a strip footing of width B=1m that rests on a soil domain of width 13B and depth 5B was considered in the analysis. For the displacement boundary conditions, the bottom boundary was assumed to be fixed and the vertical boundaries were constrained in motion in the horizontal direction. For the computation of the bearing capacity of the rigid rough strip footing subjected to a central vertical load using FLAC$^{3D}$, the following method is adopted: an optimal controlled downward vertical velocity of $5 \times 10^{-6}$ m/timestep (i.e. displacement per timestep) was applied to the bottom central node of the footing. Damping of

the system is introduced by running several cycles until a steady state of plastic flow is developed in the soil underneath the footing. At each cycle, the vertical footing load is obtained by using a FISH function that calculates the integral of the normal stress components for all elements in contact with the footing. The value of the vertical footing load at the plastic steady state is the ultimate footing load.

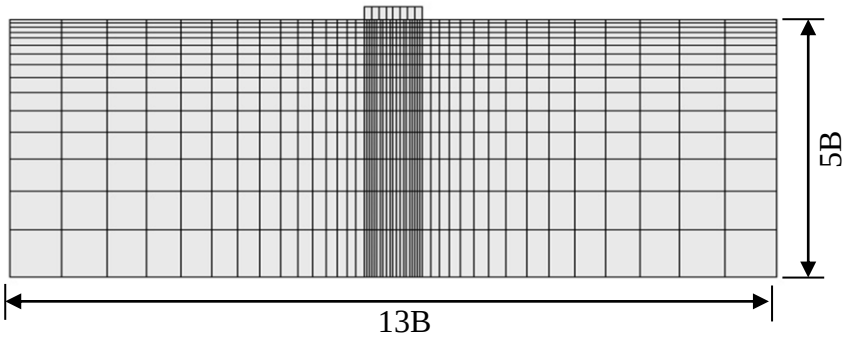


**Figure 1.** Soil domain and mesh in the FLAC$^{3D}$

### 2.5.2 Probabilistic results for the reference case

Figure (2) presents for the reference case ($a_x$=10m, $a_y$=1m) the failure probability $P_f$, the coefficient of variation $COV_{Pf}$ and the learning function values U as function of the added points (i.e. added realizations) for a number of realizations $n_{MC}$ equal to 500000. It should be mentioned here that 491 additional points were added to the initial DoE before the algorithm stops (U>2). The final obtained values of $P_f$ and $COV_{Pf}$ are respectively $1.408x10^{-3}$ and 3.766%. Figure (2) shows that the probability of failure starts to converge at about 480 calls to the deterministic model.

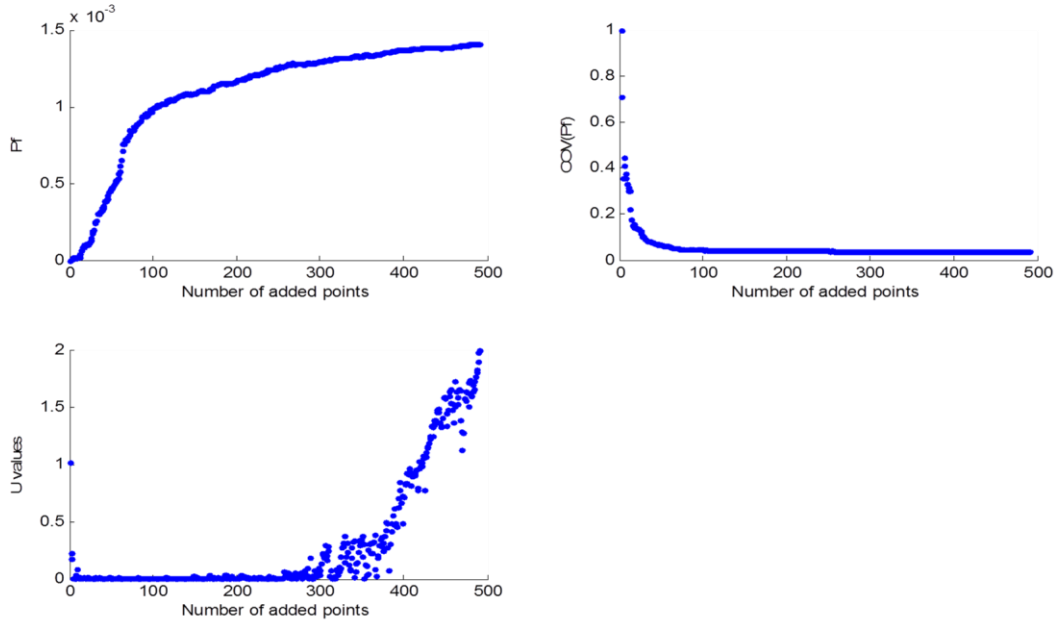


**Figure 2.** $P_f$, cov($P_f$) and U values as function of the added points for the reference case

Figures (3) presents typical realizations and the corresponding velocity field of the random fields for the reference case. As may be seen from this figure, the spatial variability of the soil

properties can produce a non-symmetrical mechanism even though the footing is subjected to a symmetrical vertical load.

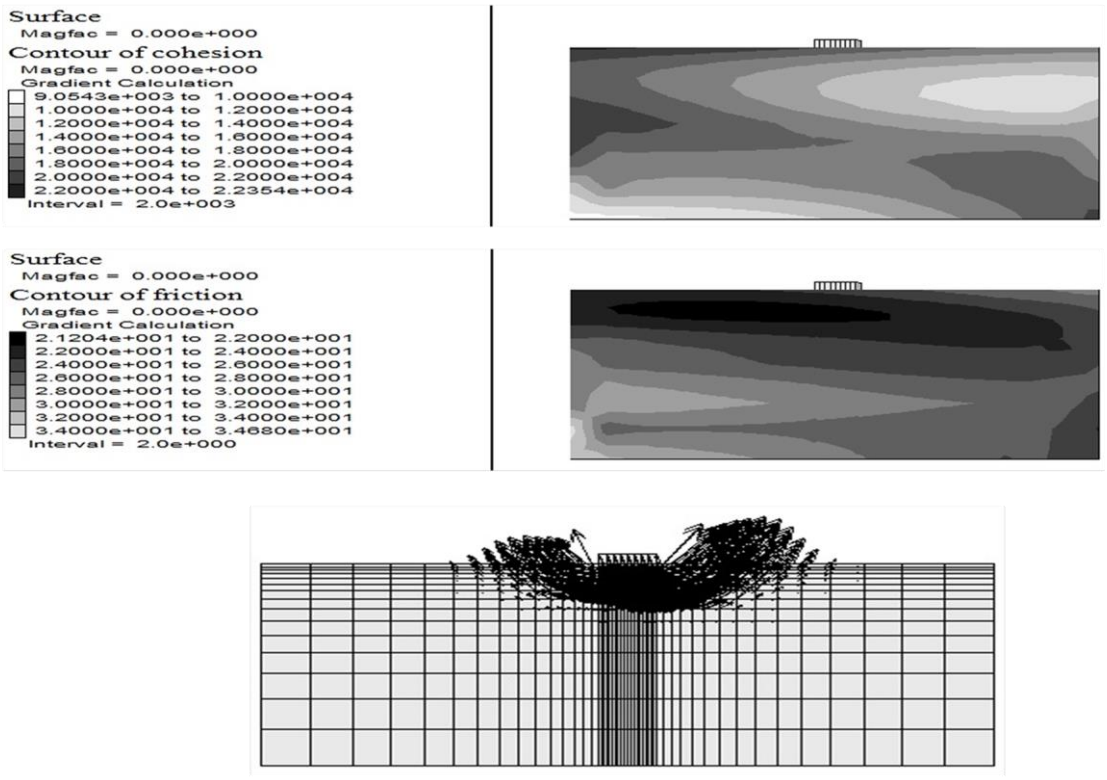


**Figure 3.** Typical realization and the corresponding velocity field of the random fields for the reference case ($a_x$=10m, $a_y$=1m) ($n_{MC}$ = 500000)

### 2.5.3 Effect of the number of simulations on $COV_{Pf}$

To ensure a small uncertainty on the estimation of the failure probability, the number of realizations $n_{MC}$ to be used must be sufficient to provide a small value of the coefficient of variation $COV_{Pf}$ of this failure probability. Figure (4) shows for the reference case the variation of $COV_{Pf}$ (as computed by AK-MCS methodology) with the number of realizations $n_{MC}$. As expected, Figure (4) shows that $COV_{Pf}$ decreases (i.e. the accuracy of the failure probability increases) with the increase of $n_{MC}$.

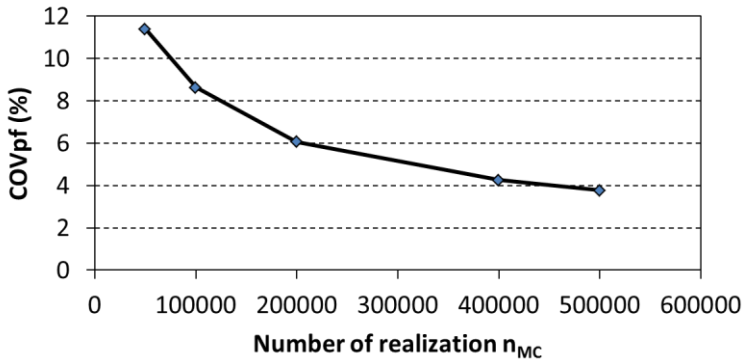


**Figure 4.** Coefficient of variation of $P_f$ versus the number of realizations $n_{MC}$

### 2.5.4 Effect of the autocorrelation distance

Table 1 presents the effect of the isotropic autocorrelation distance ($a$x=ay) on the failure probability ($P_f$) and the corresponding coefficient of variation $COV_{Pf}$ as computed by AK-MCS methodology using $n_{MC}$=1,000,000 realizations of the random field. This table also gives, for each autocorrelation distance, (i) the number of eigenmodes that is needed to satisfy the criterion imposed on the variance of the error of the random field (i.e. variance error<10%), (ii) the number of added points (i.e. the number of added realizations) and (iii) the obtained value of the learning function U for which the learning stops. As may be seen from this table, $P_f$ increases with the increase in the autocorrelation distance. A small value of the coefficient of variation of the failure probability (smaller than 3%) was obtained for the adopted value of $n_{MC}$. The number of added realizations required to lead to a good approximation of the kriging model seems to be greater for the smaller values of the autocorrelation distance, although there is no regular increase in the number of added points with the decrease of the autocorrelation distance. Indeed, this number depends on the evolution of the learning function value to satisfy the stopping conditions.

**Table 1.** Number of random variables used to discretize the random field in EOLE method, $P_f$, $COV_{Pf}$ and number of added points in the case of an isotropic case

| ax=ay (m) | Number of needed random variables for the two random fields ($c$, $\varphi$) | $P_f \times 10^{-3}$ | $Cov_{pf}$ % | Number of added points | U |
|---|---|---|---|---|---|
| 10 | 4 | 2.407 | 2.0358 | 243 | 2.033194 |
| 20 | 4 | 3.509 | 1.685 | 342 | 2.007373 |
| 50 | 4 | 3.824 | 1.614 | 65 | 2.292211 |
| 100 | 4 | 3.87 | 1.604 | 76 | 2.147826 |

## 3 CONCLUSION

The probabilistic analysis of shallow foundations resting on a spatially varying soil was generally performed in literature using MCS methodology. The mean value and the standard deviation of the system response were extensively investigated. This was not the case for the failure probability because MCS methodology requires a large number of calls of the deterministic model to accurately calculate a small failure probability. This paper mainly presented a probabilistic analysis at the ultimate limit state of a strip footing resting on a spatially varying soil using an active learning reliability method combining kriging and Monte Carlo Simulation (called AK-MCS). The method was shown to be very efficient as the obtained probability of failure is very accurate needing only a small number of calls to the deterministic model. Within this method, one performs a Monte Carlo simulation without evaluating the whole population using the original deterministic model. Indeed, the population is predicted using a kriging metamodel which is defined using only a few points of the population that are evaluated employing the original deterministic model. The objective of this paper is the computation of the probability $P_f$ of exceeding the ultimate bearing capacity of the footing under a prescribed footing load. The soil cohesion and angle of internal friction were considered as two non-isotropic non-Gaussian random fields. The deterministic model

was based on numerical simulations using the finite difference code FLAC$^{3D}$. The main findings of this study can be summarized as follows:

1. The probabilistic analysis has shown that the small failure probability $P_f$ was computed accurately (i.e. with a small $COV_{Pf}$) using only a small number of calls of the deterministic model.
2. The failure probability $P_f$ increases with the increase in the autocorrelation distance. The number of added realizations required to lead to a good approximation of the kriging model seems to be greater for the smaller values of the autocorrelation distance.

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