ECCOMAS Congress 2016
VII European Congress on Computational Methods in Applied Sciences and Engineering
M. Papadrakakis, V. Papadopoulos, G. Stefanou, V. Plevris (eds.)
Crete Island, Greece, 5–10 June 2016

ANALYSIS OF LAMINATED COMPOSITE PLATES USING ISOGEOMETRIC COLLOCATION METHOD

G. S. Pavan [1], and K. S. Nanjunda Rao[2]

[1]Research Scholar
Department of Civil Engineering, Indian Institute of Science, Bangalore, India
e-mail: pavangs@civil.iisc.ernet.in

[2] Prinicipal Research Scientist
Department of Civil Engineering, Indian Institute of Science, Bangalore, India
e-mail: ksn@civil.iisc.ernet.in

Keywords: Isogeometric collocation Method, NURBS, Laminated composite plate, Sandwich plate, Reissner Mindlin plate theory

Abstract. *Isogeometric analysis (IGA) is a relatively new class of computational methods primarily aimed at integrating finite element analysis with Computer Aided Design (CAD). IGA adopts Non - Uniform Rational B Spline (NURBS) functions as the basis functions to represent geometry and approximate the unknown field variable. The standard Galerkin based isogeometric approach has been extensively explored by researchers during the past decade and is being continued. Apart from the standard Galerkin based isogeometric approach, several other numerical schemes exist within the framework of isogeometric methods. Isogeometric collocation method is one such relatively new numerical procedure. Isogeometric collocation methods discretize the governing PDEs in strong form. In this study, the potential of isogeometric collocation methods for the static analysis of laminated composite plates using Reissner - Mindlin plate theory is explored. Results of benchmark problems on bending of rectangular laminated composite plates subjected to sinusoidal and uniform loading are presented. The results obtained from the proposed isogeometric collocation method for the analysis of laminated composite plates have been assessed by comparing them with the solutions available in literature.*

1 INTRODUCTION

Isogeometric analysis (IGA) is a relatively new class of numerical methods proposed by Hughes et al. [1] in 2005 mainly aimed at integrating Computer Aided Design (CAD) and finite element analysis. IGA simplifies the complex procedure of mesh generation and mesh refinement thereby lowering the huge computational cost incurred during the analysis of large systems. IGA achieves this smooth flow of information by adopting the same NURBS functions employed to represent geometry by CAD models to approximate the unknown field variable in an isoparametric fashion. NURBS functions possess high continuity properties which lends IGA a significant edge over traditional finite element methods during discretization of higher order partial differential equations.

Over the last decade, IGA has been successfully applied on wide ranging topics such as solid and structural mechanics [2–10], fluid mechanics [11–17], phase field modeling [18], fluid structure interaction [19–21] etc. One of the important issue related to the efficient implementation of IGA was the development of an efficient integration rule for the Galerkin based IGA. The typical Gauss quadrature rules adopted for element wise integration in Galerkin based finite element methods is sub-optimal for Galerkin-IGA since it does not take into account the higher inter element continuity of NURBS functions.

One of the recent developments in the field of IGA is the emergence of Isogeometric collocation methods (IGA collocation) proposed by Auricchio et al. [22]. Isogeometric collocation method discretizes the governing differential equations in strong form while retaining the inherent IGA feature of adopting NURBS functions in an isoparametric sense. A comparison study between IGA collocation, Galerkin-IGA and finite element method by Schillinger et al. [23] revealed accuracy of the method achieved at a low cost of computation. The advantages offered by IGA collocation method are more evident when higher order approximations are employed.

Studies on IGA collocation have been performed on solid mechanics [24], phase field modeling [25], contact [26]. IGA collocation method has been successfully applied to Euler - Bernoulli beams and Kirchhoff plates [27]. It is worthy to mention that the high continuity properties possessed by NURBS functions enabled the application of IGA collocation method on Euler-Bernoulli beam and Kirchhoff plate theories which are governed by fourth order differential equations. Studies on performance of IGA collocation on shear deformable beams, rods and plates [28–30] has also been conducted. Kiendl et al. [30] showed that mixed formulation within the framework of IGA collocation is successful in alleviating shear locking observed during lower order approximations for plate bending.

In this paper, we propose IGA collocation method for the analysis of laminated composite plates described by Reissner - Mindlin plate theory. Standard formulation with lateral displacement and two rotations are the three unknown field variables within the framework of IGA collocation. Performance of IGA collocation is assessed by solving benchmark problems and comparing the results obtained with the ones existing in literature.

2 REISSNER MINDLIN PLATE THEORY FOR LAMINATED COMPOSITE PLATES

A laminated composite plate of total thickness *t* composed of *N* orthotropic layers is considered. The undeformed midplane of the plate is taken as the reference plane Ω_0. The kinematic assumptions underlying the Reissner - Mindlin theory are made, namely, cross sections remain straight during deformation but may not remain normal to the reference plane. The displacement field according to Reissner Mindlin plate theory is

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\phi_x(x,y) \\ v(x,y,z) &= v_0(x,y) + z\phi_y(x,y) \\ w(x,y,z) &= w_0(x,y) \end{aligned} \tag{1}$$

where u_0,v_0, w_0 are the displacement components for the midplane in x, y and z directions respectively. The rotations in y-z plane and x-z plane are denoted as ϕ_x and ϕ_y respectively. The strains associated with the displacement field mentioned in equation (1) are

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial u_0}{\partial x} + z\frac{\partial \phi_x}{x} & \gamma_{xy} &= \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \\ \epsilon_{yy} &= \frac{\partial v_0}{\partial y} + z\frac{\partial \phi_y}{\partial y} & \gamma_{yz} &= \frac{\partial w_0}{\partial y} + \phi_y \\ \gamma_{xz} &= \frac{\partial w_0}{\partial x} + \phi_x & \epsilon_{zz} &= 0 \end{aligned}$$

The stresses at the k^{th} layer in local coordinates are given by the following constitutive relationship.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^k = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{44} \end{bmatrix}^k \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^k \tag{2}$$

where

$$\begin{gathered} Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} \qquad Q_{12} = \frac{E_2\nu_{12}}{1-\nu_{12}\nu_{21}} \qquad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} \\ Q_{66} = G_{12} \qquad Q_{55} = G_{13} \qquad Q_{44} = G_{23} \end{gathered} \tag{3}$$

where E_1 and E_2 are the elastic modulus in directions 1 and 2 respectively. G_{12}, G_{13} and G_{23} are the shear modulus in x-y, x-z and y-z planes respectively. ν_{12} and ν_{21} are the two Poisson ratios for orthotropic materials. The material constants in equation (2) v in local coordinates can be expressed in the global coordinates by the following set of relations

$$
\begin{aligned}
Q^{'}_{11} &= Q_{11}cos^4\theta + 2\left(Q_{12} + 2Q_{66}\right)sin^2\theta cos^2\theta + Q_{22}sin^4\theta \\
Q^{'}_{12} &= \left(Q_{11} + Q_{22} - 4Q_{66}\right)sin^2\theta cos^2\theta + Q_{12}\left(sin^4\theta + cos^4\theta\right) \\
Q^{'}_{22} &= Q_{11}sin^4\theta + 2\left(Q_{12} + 2Q_{66}\right)sin^2\theta cos^2\theta + Q_{22}cos^4\theta \\
Q^{'}_{16} &= \left(Q_{11} - Q_{12} - 2Q_{66}\right)sin\theta cos^3\theta + \left(Q_{12} - Q_{22} + 2Q_{66}\right)sin^3\theta cos\theta \\
Q_{26} &= \left(Q_{11} - Q_{12} - 2Q_{66}\right)sin^3\theta cos\theta + \left(Q_{12} - Q_{22} + 2Q_{66}\right)sin\theta cos^3\theta \\
Q^{'}_{66} &= \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)sin^2\theta cos^2\theta + Q_{66}\left(sin^4\theta + cos^4\theta\right) \\
Q^{'}_{44} &= Q_{44}cos^2\theta + Q_{55}sin^2\theta \\
Q^{'}_{45} &= \left(Q_{55} - Q_{44}\right)cos\theta sin\theta \\
Q^{'}_{55} &= Q_{44}sin^2\theta + Q_{55}cos^2\theta
\end{aligned}
\tag{4}
$$

Here, θ is the inclination between the local co-ordinate system at the k^{th} layer and the global co-ordinate axes.

2.1 Equations of equilibrium

The equations of equilibrium expressed in terms of the bending moments M_{xx}, M_{yy}, the twisting moments M_{xy}, shear forces Q_x, Q_y and the in - plane forces N_{xx}, N_{yy} and N_{xy} are given below

$$N_{xx,x} + N_{xy,y} = 0 \tag{5a}$$
$$N_{xy,x} + N_{yy,y} = 0 \tag{5b}$$
$$Q_{x,x} + Q_{y,y} = -f \tag{5c}$$
$$M_{xx,x} + M_{xy_y} - Q_x = 0 \tag{5d}$$
$$M_{yy,y} + M_{yx,x} - Q_y = 0 \tag{5e}$$

The stress resultants in equations (5a) are related to displacement variables by the following relations

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{Bmatrix} \tag{6a}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{Bmatrix} \tag{6b}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial w_0}{\partial y} + \phi_y \\ \dfrac{\partial w_0}{\partial x} + \phi_x \end{Bmatrix} \tag{6c}$$

The A_{ij}, B_{ij} and the D_{ij} present in the above set of equations (6) are termed as extensional stiffness, bending stiffness and bending - extensional stiffness respectively and are defined as following

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-t/2}^{t/2} Q_{ij}^{'} \left(1, z, z^2\right) dz \tag{7}$$

Substituting the stress resultant - displacement variable relationships mentioned in equation (6) into (5a), we obtain the equilibrium equations in terms of displacement variables u_0,v_0, w_0, ϕ_x and ϕ_y as shown below

$$A_{11}\frac{\partial^2 u_0}{\partial x^2}+2A_{16}\frac{\partial^2 u_0}{\partial x\partial y}+A_{66}\frac{\partial^2 u_0}{\partial y^2}+A_{12}\frac{\partial^2 v_0}{\partial x\partial y}+A_{16}\frac{\partial^2 v_0}{\partial x^2}+A_{26}\frac{\partial^2 v_0}{\partial y^2}+A_{66}\frac{\partial^2 v_0}{\partial x\partial y}+$$
$$B_{11}\frac{\partial^2 \phi_x}{\partial x^2}+2B_{16}\frac{\partial^2 \phi_x}{\partial x\partial y}+B_{66}\frac{\partial^2 \phi_x}{\partial y^2}+B_{12}\frac{\partial^2 \phi_y^h}{\partial x\partial y}+B_{16}\frac{\partial^2 \phi_y^h}{\partial x^2}+B_{26}\frac{\partial^2 \phi_y^h}{\partial y^2}+B_{66}\frac{\partial^2 \phi_y^h}{\partial x\partial y}=0 \tag{8a}$$

$$A_{16}\frac{\partial^2 u_0}{\partial x^2}+A_{66}\frac{\partial^2 u_0}{\partial x\partial y}+A_{12}\frac{\partial^2 u_0}{\partial x\partial y}+A_{26}\frac{\partial^2 u_0}{\partial y^2}+A_{26}\frac{\partial^2 v_0}{\partial x\partial y}+A_{66}\frac{\partial^2 v_0}{\partial x^2}+A_{22}\frac{\partial^2 v_0}{\partial y^2}+$$
$$A_{26}\frac{\partial^2 v_0}{\partial x\partial y}+B_{16}\frac{\partial^2 \phi_x}{\partial x^2}+B_{66}\frac{\partial^2 \phi_x}{\partial x\partial y}+B_{12}\frac{\partial^2 \phi_x}{\partial y\partial x}+B_{26}\frac{\partial^2 \phi_x}{\partial y^2}+B_{26}\frac{\partial^2 \phi_y^h}{\partial x\partial y}+B_{66}\frac{\partial^2 \phi_y^h}{\partial x^2}+$$
$$B_{22}\frac{\partial^2 \phi_y^h}{\partial y^2}+B_{26}\frac{\partial^2 \phi_y^h}{\partial x\partial y}=0 \tag{8b}$$

$$KA_{55}\frac{\partial \phi_x}{\partial x}+KA_{45}\frac{\partial \phi_x}{\partial y}+KA_{45}\frac{\partial \phi_y^h}{\partial x}+KA_{44}\frac{\partial \phi_y^h}{\partial y}+KA_{55}\frac{\partial^2 w_0}{\partial x^2}+2KA_{45}\frac{\partial^2 w_0}{\partial x\partial y}$$
$$+KA_{44}\frac{\partial^2 w_0}{\partial y^2}+f=0 \tag{8c}$$

$$B_{11}\frac{\partial^2 u_0}{\partial x^2}+2B_{16}\frac{\partial^2 u_0}{\partial x\partial y}+B_{66}\frac{\partial^2 u_0}{\partial y^2}+B_{12}\frac{\partial^2 v_0}{\partial x\partial y}+B_{16}\frac{\partial^2 v_0}{\partial x^2}+B_{26}\frac{\partial^2 v_0}{\partial y^2}+B_{66}\frac{\partial^2 v_0}{\partial x\partial y}+$$
$$D_{11}\frac{\partial^2 \phi_x}{\partial x^2}+2D_{16}\frac{\partial^2 \phi_x}{\partial x\partial y}+D_{66}\frac{\partial^2 \phi_x}{\partial y^2}+D_{12}\frac{\partial^2 \phi_y^h}{\partial x\partial y}+D_{16}\frac{\partial^2 \phi_y^h}{\partial x^2}+D_{26}\frac{\partial^2 \phi_y^h}{\partial y^2}+D_{66}\frac{\partial^2 \phi_y^h}{\partial x\partial y}-$$
$$KA_{55}\left(\frac{\partial w_0}{\partial x}+\phi_x\right)-KA_{45}\left(\frac{\partial w_0}{\partial y}+\phi_y\right)=0 \tag{8d}$$

$$B_{16}\frac{\partial^2 u_0}{\partial x^2}+B_{66}\frac{\partial^2 u_0}{\partial x\partial y}+B_{12}\frac{\partial^2 u_0}{\partial x\partial y}+B_{26}\frac{\partial^2 u_0}{\partial y^2}+B_{26}\frac{\partial^2 v_0}{\partial x\partial y}+B_{66}\frac{\partial^2 v_0}{\partial x^2}+B_{22}\frac{\partial^2 v_0}{\partial y^2}+$$
$$B_{26}\frac{\partial^2 v_0}{\partial x\partial y}+D_{16}\frac{\partial^2 \phi_x}{\partial x^2}+D_{66}\frac{\partial^2 \phi_x}{\partial x\partial y}+D_{12}\frac{\partial^2 \phi_x}{\partial y\partial x}+D_{26}\frac{\partial^2 \phi_x}{\partial y^2}+D_{26}\frac{\partial^2 \phi_y^h}{\partial x\partial y}+D_{66}\frac{\partial^2 \phi_y^h}{\partial x^2}+$$
$$D_{22}\frac{\partial^2 \phi_y^h}{\partial y^2}+D_{26}\frac{\partial^2 \phi_y^h}{\partial x\partial y}-KA_{45}\left(\frac{\partial w_0}{\partial x}+\phi_x\right)-KA_{44}\left(\frac{\partial w_0}{\partial y}+\phi_y\right)=0 \tag{8e}$$

The above equilibrium equations eq. (8) are valid in the interior of the domain. The equilibrium equations at the Neumann boundary are given by:

$$\begin{bmatrix} n_x^2 & n_y^2 & 2n_xn_y \\ -n_xn_y & n_xn_y & n_x^2-n_y^2 \end{bmatrix}\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix}=\begin{Bmatrix} \widehat{N}_{nn} \\ \widehat{N}_{ns} \end{Bmatrix} \tag{9a}$$

$$\begin{bmatrix} n_x^2 & n_y^2 & 2n_xn_y \\ -n_xn_y & n_xn_y & n_x^2-n_y^2 \end{bmatrix}\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}=\begin{Bmatrix} \widehat{M}_{nn} \\ \widehat{M}_{ns} \end{Bmatrix} \tag{9b}$$

$$\begin{bmatrix} n_x & n_y \end{bmatrix}\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix}=\begin{Bmatrix} \widehat{Q}_n \end{Bmatrix} \tag{9c}$$

where $\widehat{N}_{nn}$ & $\widehat{N}_{ns}$ are the in - plane axial forces specified at the Neumann boundary. Similary, $\widehat{M}_{nn}$ & $\widehat{M}_{ns}$ are the bending moments and, $\widehat{Q}_n$ are the shear forces specified at Neumann boundary.

Using the stress resultant - displacement relationship in eq. (6), we obtain eqs. (9) in terms of displacement variables as given below:

$$\begin{aligned}
\widehat{N}_{nn} = {} & \left(n_x^2 A_{11} + n_y^2 A_{12} + 2n_x n_y A_{16}\right)\frac{\partial u_0}{\partial x} + \left(n_x^2 A_{16} + n_y^2 A_{26} + 2n_x n_y A_{66}\right)\frac{\partial u_0}{\partial y} + \\
& \left(n_x^2 A_{16} + n_y^2 A_{26} + 2n_x n_y A_{66}\right)\frac{\partial v_0}{\partial x} + \left(n_x^2 A_{12} + n_y^2 A_{22} + 2n_x n_y A_{26}\right)\frac{\partial v_0}{\partial y} + \\
& \left(n_x^2 B_{11} + n_y^2 B_{12} + 2n_x n_y B_{16}\right)\frac{\partial \phi_x}{\partial x} + \left(n_x^2 B_{16} + n_y^2 B_{26} + 2n_x n_y B_{66}\right)\frac{\partial \phi_x}{\partial y} + \\
& \left(n_x^2 B_{16} + n_y^2 B_{26} + 2n_x n_y B_{66}\right)\frac{\partial \phi_y}{\partial x} + \left(n_x^2 B_{12} + n_y^2 B_{22} + 2n_x n_y B_{26}\right)\frac{\partial \phi_y}{\partial y}
\end{aligned} \tag{10a}$$

$$\begin{aligned}
\widehat{N}_{ns} = {} & \left(-n_x n_y A_{11} + n_x n_y A_{12} + \left(n_x^2 - n_y^2\right) A_{16}\right)\frac{\partial u_0}{\partial x} + \left(-n_x n_y A_{16} + n_x n_y A_{26} + \left(n_x^2 - n_y^2\right) A_{66}\right)\frac{\partial u_0}{\partial y} \\
& + \left(-n_x n_y A_{16} + n_x n_y A_{26} + \left(n_x^2 - n_y^2\right) A_{66}\right)\frac{\partial v_0}{\partial x} + \left(-n_x n_y A_{12} + n_x n_y A_{22} + \left(n_x^2 - n_y^2\right) A_{26}\right)\frac{\partial v_0}{\partial y} \\
& + \left(-n_x n_y b_{11} + n_x n_y B_{12} + \left(n_x^2 - n_y^2\right) B_{16}\right)\frac{\partial \phi_x}{\partial x} + \left(-n_x n_y B_{16} + n_x n_y B_{22} + \left(n_x^2 - n_y^2\right) B_{66}\right)\frac{\partial \phi_x}{\partial y} \\
& + \left(-n_x n_y B_{16} + n_x n_y B_{26} + \left(n_x^2 - n_y^2\right) B_{66}\right)\frac{\partial \phi_y}{\partial x} + \left(-n_x n_y B_{12} + n_x n_y B_{22} + \left(n_x^2 - n_y^2\right) B_{26}\right)\frac{\partial \phi_y}{\partial y}
\end{aligned} \tag{10b}$$

$$\begin{aligned}
\widehat{Q}_n = {} & K\left[A_{45}\frac{\partial w_0}{\partial y}n_x + A_{55}\frac{\partial w_0}{\partial x}n_x + A_{44}\frac{\partial w_0}{\partial y}n_y + A_{45}\frac{\partial w_0}{\partial x}n_y\right] + K\left[A_{55}\phi_x n_x + A_{45}\phi_x n_y\right] \\
& + K\left[A_{45}\phi_y n_x + A_{44} n_y \phi_y\right]
\end{aligned} \tag{10c}$$

$$\begin{aligned}
\widehat{M}_{nn} = {} & \left(n_x^2 B_{11} + n_y^2 B_{12} + 2n_x n_y B_{16}\right)\frac{\partial u_0}{\partial x} + \left(n_x^2 B_{16} + n_y^2 B_{26} + 2n_x n_y B_{66}\right)\frac{\partial u_0}{\partial y} + \\
& \left(n_x^2 B_{16} + n_y^2 B_{26} + 2n_x n_y B_{66}\right)\frac{\partial v_0}{\partial x} + \left(n_x^2 B_{12} + n_y^2 B_{22} + 2n_x n_y B_{26}\right)\frac{\partial v_0}{\partial y} + \\
& \left(n_x^2 D_{11} + n_y^2 D_{12} + 2n_x n_y d_{16}\right)\frac{\partial \phi_x}{\partial x} + \left(n_x^2 B_{16} + n_y^2 B_{26} + 2n_x n_y B_{66}\right)\frac{\partial \phi_x}{\partial y} + \\
& \left(n_x^2 B_{16} + n_y^2 B_{26} + 2n_x n_y B_{66}\right)\frac{\partial \phi_y}{\partial x} + \left(n_x^2 B_{12} + n_y^2 B_{22} + 2n_x n_y B_{26}\right)\frac{\partial \phi_y}{\partial y}
\end{aligned} \tag{10d}$$

$$\begin{aligned}
\widehat{M}_{ns} = {} & \left(-n_x n_y B_{11} + n_x n_y B_{12} + \left(n_x^2 - n_y^2\right) B_{16}\right)\frac{\partial u_0}{\partial x} + \left(-n_x n_y B_{16} + n_x n_y B_{26} + \left(n_x^2 - n_y^2\right) B_{66}\right)\frac{\partial u_0}{\partial y} \\
& + \left(-n_x n_y B_{16} + n_x n_y B_{26} + \left(n_x^2 - n_y^2\right) B_{66}\right)\frac{\partial v_0}{\partial x} + \left(-n_x n_y B_{12} + n_x n_y B_{22} + \left(n_x^2 - n_y^2\right) B_{26}\right)\frac{\partial v_0}{\partial y} \\
& + \left(-n_x n_y D_{11} + n_x n_y D_{12} + \left(n_x^2 - n_y^2\right) D_{16}\right)\frac{\partial \phi_x}{\partial x} + \left(-n_x n_y D_{16} + n_x n_y D_{22} + \left(n_x^2 - n_y^2\right) D_{66}\right)\frac{\partial \phi_x}{\partial y} \\
& + \left(-n_x n_y D_{16} + n_x n_y D_{26} + \left(n_x^2 - n_y^2\right) D_{66}\right)\frac{\partial \phi_y}{\partial x} + \left(-n_x n_y D_{12} + n_x n_y D_{22} + \left(n_x^2 - n_y^2\right) D_{26}\right)\frac{\partial \phi_y}{\partial y}
\end{aligned} \tag{10e}$$

3 Numerical formulation

In order to solve the equations presented in the previous section using Isogeometric collocation method, the unknown field variables are expressed as a linear combination of Non - Uniform Rational B - Splines (NURBS).

3.1 B - Splines and NURBS

B-spline functions are polynomials, say of degree p, are defined piecewise on the parametric space. The parametric space is given by a knot vector. A knot vector Ξ, is a set of non - decreasing coordinates ξ_i in the parametric space, $\Xi = \{\xi_1, \xi_2, \xi_3, ..., \xi_{n+p+1}\}$

, where n is the number of basis functions. The space between two consecutive knots in the parametric space is defined as a knot span. Knot values can be repeated at a particular location in the parametric space and are called as multiple knots. The B-splines are C^{p-1} continuous at a particular knot and C^{p-k} continuous at multiple knot locations with multiplicity k.

The i^{th} B-spline basis function of degree p for a given knot vector Ξ is defined recursively as shown below

$$N_{i,0}(\xi) = \begin{cases} 1, & \xi_i \leq \xi < \xi_{i+1} \\ 0, & otherwise \end{cases} \tag{11a}$$

$$N_{i,p} = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \tag{11b}$$

B-Splines in two dimensions are obtained by taking the tensor product of their univariate counterparts as shown below

$$B_{ij,pq}(\xi.\eta) = N_{i,p}(\xi) M_{j,q}(\eta) \qquad i = 1, ..., n \qquad j = 1, ..., m. \tag{12}$$

where ξ and η are the parametric coordinates in two directions.

A bivariate NURBS function R_{ij} is defined as the weighted bivariate B-Spline B_{ij} function,

$$R_{ij}(\xi, \eta) = \frac{B_{ij,pq}(\xi, \eta)\,\omega_{ij}}{\sum_{k=1}^{n} \sum_{l=1}^{m} B_{ij,pq}(\xi, \eta)\,\omega_{ij}} \tag{13}$$

ω_{ij} are weights adopted.

The isogeometric class of numerical methods adopt NURBS functions to discretize the unknown field variables and geometry. Likewise, isogeometric collocation methods adopt NURBS to approximate the unknown field variables, namely, u, v, w, ϕ_x & ϕ_y as shown below:

$$u_0^h(\xi,\eta) = \sum_{i=1}^{n}\sum_{i=1}^{m} R_{ij}^{w}(\xi,\eta)\,\widehat{u}^{ij} \tag{14a}$$

$$v_0^h(\xi,\eta) = \sum_{i=1}^{n}\sum_{i=1}^{m} R_{ij}^{w}(\xi,\eta)\,\widehat{v}^{ij} \tag{14b}$$

$$w_0^h(\xi,\eta) = \sum_{i=1}^{n}\sum_{i=1}^{m} R_{ij}^{w}(\xi,\eta)\,\widehat{w}^{ij} \tag{14c}$$

$$\phi_x^h(\xi,\eta) = \sum_{i=1}^{n}\sum_{i=1}^{m} R_{ij}^{w}(\xi,\eta)\,\hat{\phi_x}^{ij} \tag{14d}$$

$$\phi_y^h(\xi,\eta) = \sum_{i=1}^{n}\sum_{i=1}^{m} R_{ij}^{w}(\xi,\eta)\,\hat{\phi_y}^{ij} \tag{14e}$$

3.2 Collocation using NURBS functions

The discretized strong form of the equilibrium equations are collocated on the images of greville points in physical space. The greville points are defined in the parametric space. The greville points $\hat{\xi_i}$ defined for a spline space of degree p and knot vector $\{\xi_i, ..., \xi_{i+p+1}\}$ are given by

$$\hat{\xi_i} = \frac{\xi_{i+1} + \xi_{i+2} + ... + \xi_{i+p}}{p} \tag{15}$$

In two dimensional problems like plate problems, a greville point is defined by $\left(\hat{\xi_i}, \hat{\eta_i}\right)$, which are the greville coordinates in two parametric directions. From eq. (15), it can be observed that some of the greville points are located within the domain and others are located at the boundary. Since the equations of equilibrium are different for the domain interior and boundary, the equations to be collocated need to be chosen suitably depending on the location of the greville point.

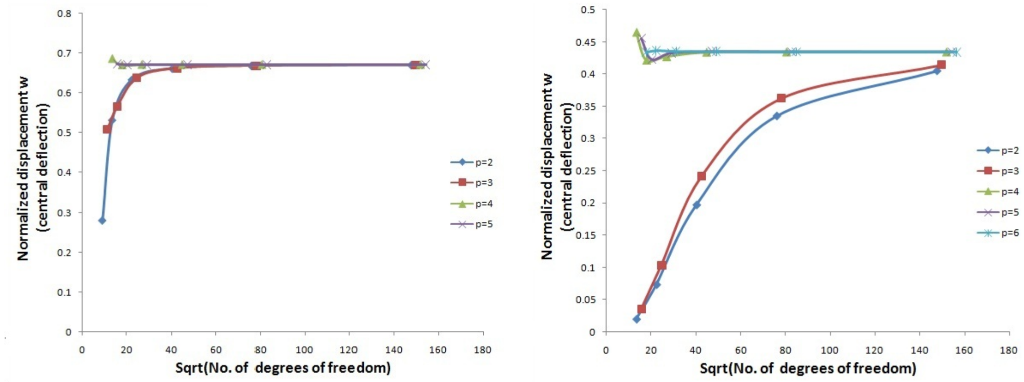
Using eqs.(14), we discretize the equilibrium equations eqs. (8) defined in the interior of the domain. The residual of these equations $\left(R_u^{\Omega}, R_v^{\Omega}, R_w^{\Omega}, R_{\phi_x}^{\Omega} \& R_{\phi_y}^{\Omega}\right)$ are presented as follows:

$$\begin{aligned}
R_u^{\Omega} = \quad & A_{11}\frac{\partial^2 u_0^h}{\partial x^2} + 2A_{16}\frac{\partial^2 u_0^h}{\partial x \partial y} + A_{66}\frac{\partial^2 u_0^h}{\partial y^2} + A_{12}\frac{\partial^2 v_0^h}{\partial x \partial y} + A_{16}\frac{\partial^2 v_0^h}{\partial x^2} + A_{26}\frac{\partial^2 v_0^h}{\partial y^2} + A_{66}\frac{\partial^2 v_0^h}{\partial x \partial y} + \\
& B_{11}\frac{\partial^2 \phi_x^h}{\partial x^2} + 2B_{16}\frac{\partial^2 \phi_x^h}{\partial x \partial y} + B_{66}\frac{\partial^2 \phi_x^h}{\partial y^2} + B_{12}\frac{\partial^2 \phi_y^h}{\partial x \partial y} + B_{16}\frac{\partial^2 \phi_y^h}{\partial x^2} + B_{26}\frac{\partial^2 \phi_y^h}{\partial y^2} + B_{66}\frac{\partial^2 \phi_y^h}{\partial x \partial y} \qquad (16a) \\
R_v^{\Omega} = \quad & A_{16}\frac{\partial^2 u_0^h}{\partial x^2} + A_{66}\frac{\partial^2 u_0^h}{\partial x \partial y} + A_{12}\frac{\partial^2 u_0^h}{\partial x \partial y} + A_{26}\frac{\partial^2 u_0^h}{\partial y^2} + A_{26}\frac{\partial^2 v_0^h}{\partial x \partial y} + A_{66}\frac{\partial^2 v_0^h}{\partial x^2} + A_{22}\frac{\partial^2 v_0^h}{\partial y^2} + \\
& A_{26}\frac{\partial^2 v_0^h}{\partial x \partial y} + B_{16}\frac{\partial^2 \phi_x^h}{\partial x^2} + B_{66}\frac{\partial^2 \phi_x^h}{\partial x \partial y} + B_{12}\frac{\partial^2 \phi_x^h}{\partial y \partial x} + B_{26}\frac{\partial^2 \phi_x^h}{\partial y^2} + B_{26}\frac{\partial^2 \phi_y^h}{\partial x \partial y} + B_{66}\frac{\partial^2 \phi_y^h}{\partial x^2} + \\
& B_{22}\frac{\partial^2 \phi_y^h}{\partial y^2} + B_{26}\frac{\partial^2 \phi_y^h}{\partial x \partial y} \qquad (16b) \\
R_w^{\Omega} = \quad & KA_{55}\frac{\partial \phi_x^h}{\partial x} + KA_{45}\frac{\partial \phi_x^h}{\partial y} + KA_{45}\frac{\partial \phi_y^h}{\partial x} + KA_{44}\frac{\partial \phi_y^h}{\partial y} + KA_{55}\frac{\partial^2 w_0^h}{\partial x^2} + 2KA_{45}\frac{\partial^2 w_0^h}{\partial x \partial y} \\
& + KA_{44}\frac{\partial^2 w_0^h}{\partial y^2} + f \qquad (16c) \\
R_{\phi_x}^{\Omega} = \quad & B_{11}\frac{\partial^2 u_0^h}{\partial x^2} + 2B_{16}\frac{\partial^2 u_0^h}{\partial x \partial y} + B_{66}\frac{\partial^2 u_0^h}{\partial y^2} + B_{12}\frac{\partial^2 v_0^h}{\partial x \partial y} + B_{16}\frac{\partial^2 v_0^h}{\partial x^2} + B_{26}\frac{\partial^2 v_0^h}{\partial y^2} + B_{66}\frac{\partial^2 v_0^h}{\partial x \partial y} + \\
& D_{11}\frac{\partial^2 \phi_x^h}{\partial x^2} + 2D_{16}\frac{\partial^2 \phi_x^h}{\partial x \partial y} + D_{66}\frac{\partial^2 \phi_x^h}{\partial y^2} + D_{12}\frac{\partial^2 \phi_y^h}{\partial x \partial y} + D_{16}\frac{\partial^2 \phi_y^h}{\partial x^2} + D_{26}\frac{\partial^2 \phi_y^h}{\partial y^2} + D_{66}\frac{\partial^2 \phi_y^h}{\partial x \partial y} - \\
& KA_{55}\left(\frac{\partial w_0^h}{\partial x} + \phi_x^h\right) - KA_{45}\left(\frac{\partial w_0^h}{\partial y} + \phi_y^h\right) \qquad (16d) \\
R_{\phi_y}^{\Omega} = \quad & B_{16}\frac{\partial^2 u_0^h}{\partial x^2} + B_{66}\frac{\partial^2 u_0^h}{\partial x \partial y} + B_{12}\frac{\partial^2 u_0^h}{\partial x \partial y} + B_{26}\frac{\partial^2 u_0^h}{\partial y^2} + B_{26}\frac{\partial^2 v_0^h}{\partial x \partial y} + B_{66}\frac{\partial^2 v_0^h}{\partial x^2} + B_{22}\frac{\partial^2 v_0^h}{\partial y^2} + \\
& B_{26}\frac{\partial^2 v_0^h}{\partial x \partial y} + D_{16}\frac{\partial^2 \phi_x^h}{\partial x^2} + D_{66}\frac{\partial^2 \phi_x^h}{\partial x \partial y} + D_{12}\frac{\partial^2 \phi_x^h}{\partial y \partial x} + D_{26}\frac{\partial^2 \phi_x^h}{\partial y^2} + D_{26}\frac{\partial^2 \phi_y^h}{\partial x \partial y} + D_{66}\frac{\partial^2 \phi_y^h}{\partial x^2} + \\
& D_{22}\frac{\partial^2 \phi_y^h}{\partial y^2} + D_{26}\frac{\partial^2 \phi_y^h}{\partial x \partial y} - KA_{45}\left(\frac{\partial w_0^h}{\partial x} + \phi_x^h\right) - KA_{44}\left(\frac{\partial w_0^h}{\partial y} + \phi_y^h\right) \qquad (16e)
\end{aligned}$$

Similarly, residuals of the equilibrium equations valid at Neumann boundaries R_u^{Γ}, R_v^{Γ}, R_w^{Γ}, $R_{\phi_x}^{\Gamma}$ & $R_{\phi_y}^{\Gamma}$ are obtained by substituting eq. (14) into eqs. (10). The order of polynomial and knot vector are the same for the NURBS functions specified in eqs. (14). The collocation scheme adopted is

$$\begin{aligned}
R_u^{\Omega} = 0 \quad R_v^{\Omega} = 0 \quad R_w^{\Omega} = 0 \quad R_{\phi_x}^{\Omega} = 0 \quad R_{\phi_y}^{\Omega} = 0 \quad on\ S\left(\left\{\hat{\xi}\right\}^{\Omega}\right) \\
R_u^{\Gamma} = 0 \quad R_v^{\Gamma} = 0 \quad R_w^{\Gamma} = 0 \quad R_{\phi_x}^{\Gamma} = 0 \quad R_{\phi_y}^{\Gamma} = 0 \quad on\ S\left(\left\{\hat{\xi}\right\}^{\Gamma}\right)
\end{aligned} \qquad (17)$$

where $S\left(\left\{\hat{\xi}\right\}^{\Gamma}\right)$ is the mapping of Greville points from the parametric space to physical space.



(a) h = a/10 (b) h = a/100

Figure 1: Lateral displacement w^* versus $\sqrt{N_f}$

4 NUMERICAL RESULTS

4.1 Bending of all round simply supported square cross ply laminated plates

Consider an all round simply supported rectangular laminated plate. The laminate consists of four plies (0/90/90/0) of equal thickness. Let h be the total thickness of the plate. The material properties are

$$E_1 = 25E_2, \quad G_{12} = G_{13} = 0.5E_2, \quad G_{23} = 0.2E_2, \quad \nu_{12} = 0.25, \& \qquad K = 5/6 \tag{18}$$

where the origin of the co-ordinate system is taken at a corner of the plate, $0 \leq x \leq a \;\&\; 0 \leq y \leq b$. The lateral displacement and the stresses evaluated are non - dimensionalized using the following relationship,

$$\begin{aligned} w^* &= w_0(\frac{a}{2},\frac{b}{2})\frac{E_2h^3}{b^4q_0}, &\qquad \sigma^*_{xx} &= \sigma_{xx}(\frac{a}{2},\frac{b}{2},\frac{h}{2})\frac{h^2}{b^2q_0} \\ \sigma^*_{yy} &= \sigma_{yy}(\frac{a}{2},\frac{b}{2},\frac{h}{4})\frac{h^2}{b^2q_0}, &\qquad \sigma^*_{xy} &= \sigma_{xy}(a,b,-\frac{h}{2})\frac{h^2}{b^2q_0} \\ \sigma^*_{xz} &= \sigma_{xz}(a,\frac{b}{2},k=1,3)\frac{h}{bq_0} \end{aligned} \tag{19}$$

The results obtained are listed in Table 1 and compared with methods existing in literature. Figure 1 shows the plot of normalized central displacement w^* versus N_f (square root of the number of degrees of freedom) for two plate width to thickness ratios of 10 & 100. Convergence of normalized central displacement w^* can be observed.

4.2 Bending of all round clamped square sandwich plates

Consider an all round clamped square sandwich plate. The sandwich plate consists of a central core, embedded with face sheets on top and bottom. The face sheets are

Table 1: Simply supported $[0/90/90/0]$ square laminated plate under sinusoidal load.

a/h	Method	w^*	σ^*_{xx}	σ^*_{yy}	σ^*_{xy}	τ^*_{xz}
100	HSDT [31]	0.4343	0.5507	0.2769	0.0217	0.2948
	HSDT [32]	0.4343	0.5387	0.2708	0.0213	0.2897
	FSDT [33]	0.4337	0.5382	0.2705	0.0213	0.178
	Elasticity ‘ [34]	0.4347	0.539	0.271	0.0214	0.339
	HSDT [35]	0.4365	0.5413	0.3359	0.0215	0.4106
	Layer-wise [36]	0.4374	0.542	0.2697	0.0216	0.3232
	RBF-PS [37]	0.432	0.5387	0.2697	0.0213	0.3154
	Wavelets [38]	0.4335	0.5381	0.2704	0.0213	0.339
	IGA- galerkin [39]	0.4354	0.5376	0.2702	0.0213	0.1907
	Quadratic	0.3339	0.4138	0.2079	0.182	1.5795
	Cubic	0.3622	0.4493	0.2255	0.0206	1.0969
	Quartic	0.4335	0.5379	0.27	0.0235	0.9867
	Quintic	0.4346	0.5392	0.2707	0.0223	0.8195
10	HSDT [31]	0.7149	0.5589	0.3974	0.0273	0.2697
	HSDT [32]	0.7147	0.5456	0.3888	0.0268	0.264
	FSDT [33]	0.6628	0.4989	0.3615	0.0241	0.1667
	Elasticity [34]	0.743	0.559	0.403	0.0276	0.301
	HSDT [35]	0.7153	0.5466	0.4383	0.0267	0.3347
	Layer-wise [36]	0.7309	0.5496	0.3956	0.0273	0.2888
	RBF-PS [37]	0.7204	0.5609	0.3911	0.0273	0.2843
	Wavelets [38]	0.6627	0.4989	0.3614	0.0241	0.3181
	IGA- galerkin [39]	0.6654	0.4983	0.361	0.0242	0.1669
	Quadratic	0.6670	0.5022	0.3623	0.0202	0.1742
	Cubic	0.6688	0.5038	0.3636	0.0204	0.1741
	Quartic	0.6693	0.5043	0.3639	0.0203	0.1744
	Quintic	0.6709	0.5055	0.3648	0.0203	0.1744
20	HSDT [31]	0.5061	0.5523	0.311	0.0233	0.2883
	HSDT [32]	0.506	0.5393	0.3043	0.0228	0.2825
	FSDT [33]	0.4912	0.5273	0.2957	0.0221	0.1749
	Elasticity [34]	0.517	0.543	0.309	0.023	0.328
	HSDT [35]	0.507	0.5405	0.3648	0.0228	0.3818
	Layer-wise [36]	0.5121	0.5417	0.3056	0.023	0.3248
	RBF-PS [37]	0.5078	0.5436	0.3052	0.023	0.3066
	Wavelets [38]	0.4912	0.5273	0.2956	0.0221	0.3332
	IGA- galerkin [39]	0.4931	0.5268	0.2953	0.0221	0.1755
	Quadratic	0.4882	0.5232	0.2923	0.0203	0.1818
	Cubic	0.4907	0.5264	0.2942	0.0204	0.1813
	Quartic	0.4940	0.5299	0.2962	0.0205	0.1827

Table 2: All round clamped square sandwich plate under uniform load.

a/h	Method	w^*	σ^*_{xx}	σ^*_{yy}	σ^*_{xy}	τ^*_{xz}
100	Reddy J N [40]	0.2785	0.5347	0.0094	0.003	0.24
	Quadratic	0.2350	0.4489	0.009	0.0114	0.3225
	Cubic	0.2482	0.4749	0.0092	0.0123	0.3098
	Quartic	0.2749	0.5264	0.0114	0.0151	0.3368
50	Reddy J N [40]	0.3111	0.5356	0.0108	0.0039	0.2406
	Quadratic	0.3044	0.5044	0.0109	0.0139	0.3342
	cubic	0.3085	0.5113	0.0112	0.0143	0.3299
	Quartic	0.3162	0.5241	0.0120	0.0152	0.3365
10	Reddy J N [40]	1.2654	0.5018	0.0550	0.0120	0.2318
	Quadratic	1.3569	0.4153	0.0361	0.0281	0.2893
	cubic	1.3591	0.4154	0.0362	0.0284	0.2894
	Quartic	1.3619	0.4162	0.0362	0.0285	0.2899

assumed to be orthotropic and the core is assumed to be transversely isotropic. Let h be the total thickness of the plate. Thickness of the core is 0.8 times the total thickness while the face sheets are one - tenth the total thickness of the sandwich plate. Material properties of the core are given below

$$E_1 = 25E_2, \quad E_2 = 10^6 \quad G_{12} = G_{13} = 0.5E_2, \quad G_{23} = 0.2E_2, \quad \nu_{12} = 0.25, \& \qquad K = 5/6 \tag{20}$$

Material properties of the face sheets are provided below

$$E_1 = E_2 = 10^6 \quad G_{13} = G_{23} = 0.06 * 10^6 psi, \quad G_{12} = 0.016 * 10^6 psi, \quad \nu_{12} = 0.2 \tag{21}$$

The transverse load acting on the rectangular plate is given by $q(x, y) = q_0$, where the origin of the co-ordinate system is taken at a corner of the plate, $0 \leq x \leq a \,\&\, 0 \leq y \leq b$. The lateral displacement and the stresses evaluated are non - dimensionalized using the following relations.

$$\begin{aligned} w^* &= w_0(\frac{a}{2}, \frac{b}{2})\frac{E_2 h^3}{b^4 q_0}, \qquad \sigma^*_{xx} = \sigma_{xx}(\frac{a}{2}, \frac{b}{2}, \frac{h}{2})\frac{h^2}{b^2 q_0} \\ \sigma^*_{yy} &= \sigma_{yy}(\frac{a}{2}, \frac{b}{2}, \frac{h}{2})\frac{h^2}{b^2 q_0}, \quad \sigma^*_{xy} = \sigma_{xy}(a, b, -\frac{h}{2})\frac{h^2}{b^2 q_0} \\ \sigma^*_{xz} &= \sigma_{xz}(\frac{a}{2}, b, 0)\frac{h}{b q_0} \end{aligned} \tag{22}$$

The results obtained are listed in Table 2 and compared with solution existing in literature.

5 CONCLUSIONS

In this paper, analysis of laminated composite plates has been performed using Isogeometric collocation method. The method has been assessed for both laminated plate and sandwich plates under all round simply supported and all round clamped conditions respectively. The accuracy obtained by this method is comparable with the methods existing in literature. Shear locking was observed under the thin plate limit when lower order approximations were adopted. Mixed formulation may be considered in order to alleviate shear locking. Further research direction would be to study free vibration analysis, buckling analysis and extension of present formulation under large deformations.

References

[1] T. J. Hughes, J. A. Cottrell, and Y. Bazilevs, "Isogeometric analysis: Cad, finite elements, nurbs, exact geometry and mesh refinement," *Computer methods in applied mechanics and engineering*, vol. 194, no. 39, pp. 4135–4195, 2005.

[2] J. Caseiro, R. Valente, A. Reali, J. Kiendl, F. Auricchio, and R. A. De Sousa, "On the assumed natural strain method to alleviate locking in solid-shell nurbs-based finite elements," *Computational Mechanics*, vol. 53, no. 6, pp. 1341–1353, 2014.

[3] J. Cottrell, T. Hughes, and A. Reali, "Studies of refinement and continuity in isogeometric structural analysis," *Computer methods in applied mechanics and engineering*, vol. 196, no. 41, pp. 4160–4183, 2007.

[4] J. A. Cottrell, A. Reali, Y. Bazilevs, and T. J. Hughes, "Isogeometric analysis of structural vibrations," *Computer methods in applied mechanics and engineering*, vol. 195, no. 41, pp. 5257–5296, 2006.

[5] C. De Falco, A. Reali, and R. Vázquez, "Geopdes: a research tool for isogeometric analysis of pdes," *Advances in Engineering Software*, vol. 42, no. 12, pp. 1020–1034, 2011.

[6] T. Elguedj, Y. Bazilevs, V. M. Calo, and T. J. Hughes, "B and f projection methods for nearly incompressible linear and non-linear elasticity and plasticity using higher-order nurbs elements," *Computer methods in applied mechanics and engineering*, vol. 197, no. 33, pp. 2732–2762, 2008.

[7] T. J. Hughes, J. A. Evans, and A. Reali, "Finite element and nurbs approximations of eigenvalue, boundary-value, and initial-value problems," *Computer Methods in Applied Mechanics and Engineering*, vol. 272, pp. 290–320, 2014.

[8] S. Morganti, F. Auricchio, D. Benson, F. Gambarin, S. Hartmann, T. Hughes, and A. Reali, "Patient-specific isogeometric structural analysis of aortic valve closure,"

Computer Methods in Applied Mechanics and Engineering, vol. 284, pp. 508–520, 2015.

[9] S. Lipton, J. A. Evans, Y. Bazilevs, T. Elguedj, and T. J. Hughes, "Robustness of isogeometric structural discretizations under severe mesh distortion," *Computer Methods in Applied Mechanics and Engineering*, vol. 199, no. 5, pp. 357–373, 2010.

[10] A. Reali, "An isogeometric analysis approach for the study of structural vibrations," *Journal of Earthquake Engineering*, vol. 10, no. spec01, pp. 1–30, 2006.

[11] I. Akkerman, Y. Bazilevs, V. Calo, T. Hughes, and S. Hulshoff, "The role of continuity in residual-based variational multiscale modeling of turbulence," *Computational Mechanics*, vol. 41, no. 3, pp. 371–378, 2008.

[12] Y. Bazilevs, V. Calo, J. Cottrell, T. Hughes, A. Reali, and G. Scovazzi, "Variational multiscale residual-based turbulence modeling for large eddy simulation of incompressible flows," *Computer Methods in Applied Mechanics and Engineering*, vol. 197, no. 1, pp. 173–201, 2007.

[13] Y. Bazilevs and T. Hughes, "Nurbs-based isogeometric analysis for the computation of flows about rotating components," *Computational Mechanics*, vol. 43, no. 1, pp. 143–150, 2008.

[14] A. Buffa, C. De Falco, and G. Sangalli, "Isogeometric analysis: Stable elements for the 2d stokes equation," *International Journal for Numerical Methods in Fluids*, vol. 65, no. 11-12, pp. 1407–1422, 2011.

[15] J. Liu, H. Gomez, J. A. Evans, T. J. Hughes, and C. M. Landis, "Functional entropy variables: a new methodology for deriving thermodynamically consistent algorithms for complex fluids, with particular reference to the isothermal navier–stokes–korteweg equations," *Journal of Computational Physics*, vol. 248, pp. 47–86, 2013.

[16] H. Gomez, T. J. Hughes, X. Nogueira, and V. M. Calo, "Isogeometric analysis of the isothermal navier–stokes–korteweg equations," *Computer Methods in Applied Mechanics and Engineering*, vol. 199, no. 25, pp. 1828–1840, 2010.

[17] H. Gomez and T. J. Hughes, "Provably unconditionally stable, second-order time-accurate, mixed variational methods for phase-field models," *Journal of Computational Physics*, vol. 230, no. 13, pp. 5310–5327, 2011.

[18] H. Gómez, V. M. Calo, Y. Bazilevs, and T. J. Hughes, "Isogeometric analysis of the cahn–hilliard phase-field model," *Computer methods in applied mechanics and engineering*, vol. 197, no. 49, pp. 4333–4352, 2008.

[19] Y. Bazilevs, V. Calo, Y. Zhang, and T. J. Hughes, “Isogeometric fluid–structure interaction analysis with applications to arterial blood flow,” *Computational Mechanics*, vol. 38, no. 4-5, pp. 310–322, 2006.

[20] Y. Bazilevs, V. M. Calo, T. J. Hughes, and Y. Zhang, “Isogeometric fluid-structure interaction: theory, algorithms, and computations,” *Computational mechanics*, vol. 43, no. 1, pp. 3–37, 2008.

[21] Y. Zhang, Y. Bazilevs, S. Goswami, C. L. Bajaj, and T. J. Hughes, “Patient-specific vascular nurbs modeling for isogeometric analysis of blood flow,” *Computer methods in applied mechanics and engineering*, vol. 196, no. 29, pp. 2943–2959, 2007.

[22] F. Auricchio, L. B. Da Veiga, T. Hughes, A. Reali, and G. Sangalli, “Isogeometric collocation methods,” *Mathematical Models and Methods in Applied Sciences*, vol. 20, no. 11, pp. 2075–2107, 2010.

[23] D. Schillinger, J. A. Evans, A. Reali, M. A. Scott, and T. J. Hughes, “Isogeometric collocation: Cost comparison with galerkin methods and extension to adaptive hierarchical nurbs discretizations,” *Computer Methods in Applied Mechanics and Engineering*, vol. 267, pp. 170–232, 2013.

[24] F. Auricchio, L. B. da Veiga, T. Hughes, A. Reali, and G. Sangalli, “Isogeometric collocation for elastostatics and explicit dynamics,” *Computer Methods in Applied Mechanics and Engineering*, vol. 249, pp. 2–14, 2012.

[25] H. Gomez, A. Reali, and G. Sangalli, “Accurate, efficient, and (iso) geometrically flexible collocation methods for phase-field models,” *Journal of Computational Physics*, vol. 262, pp. 153–171, 2014.

[26] L. De Lorenzis, J. Evans, T. Hughes, and A. Reali, “Isogeometric collocation: Neumann boundary conditions and contact,” *Computer Methods in Applied Mechanics and Engineering*, vol. 284, pp. 21–54, 2015.

[27] A. Reali and H. Gomez, “An isogeometric collocation approach for bernoulli–euler beams and kirchhoff plates,” *Computer Methods in Applied Mechanics and Engineering*, vol. 284, pp. 623–636, 2015.

[28] L. B. da Veiga, C. Lovadina, and A. Reali, “Avoiding shear locking for the timoshenko beam problem via isogeometric collocation methods,” *Computer Methods in Applied Mechanics and Engineering*, vol. 241, pp. 38–51, 2012.

[29] F. Auricchio, L. B. da Veiga, J. Kiendl, C. Lovadina, and A. Reali, “Locking-free isogeometric collocation methods for spatial timoshenko rods,” *Computer Methods in Applied Mechanics and Engineering*, vol. 263, pp. 113–126, 2013.

[30] J. Kiendl, F. Auricchio, L. B. da Veiga, C. Lovadina, and A. Reali, "Isogeometric collocation methods for the reissner–mindlin plate problem," *Computer Methods in Applied Mechanics and Engineering*, vol. 284, pp. 489–507, 2015.

[31] J. N. Reddy, "A simple higher-order theory for laminated composite plates," *Journal of applied mechanics*, vol. 51, no. 4, pp. 745–752, 1984.

[32] G. Akhras, M. Cheung, and W. Li, "Finite strip analysis of anisotropic laminated composite plates using higher-order shear deformation theory," *Computers & Structures*, vol. 52, no. 3, pp. 471–477, 1994.

[33] G. Akhras, M. Cheung, and W. Li, "Static and vibration analysis of anisotropic composite laminates by finite strip method," *International Journal of Solids and Structures*, vol. 30, no. 22, pp. 3129–3137, 1993.

[34] N. Pagano, "Exact solutions for rectangular bidirectional composites and sandwich plates," *Journal of composite materials*, vol. 4, no. 1, pp. 20–34, 1970.

[35] A. Ferreira, C. Roque, and P. Martins, "Analysis of composite plates using higher-order shear deformation theory and a finite point formulation based on the multiquadric radial basis function method," *Composites Part B: Engineering*, vol. 34, no. 7, pp. 627–636, 2003.

[36] A. Ferreira, "Analysis of composite plates using a layerwise theory and multiquadrics discretization," *Mechanics of Advanced Materials and Structures*, vol. 12, no. 2, pp. 99–112, 2005.

[37] A. Ferreira and G. Fasshauer, "Computation of natural frequencies of shear deformable beams and plates by an rbf-pseudospectral method," *Computer Methods in Applied Mechanics and Engineering*, vol. 196, no. 1, pp. 134–146, 2006.

[38] A. Ferreira, L. M. Castro, and S. Bertoluzza, "A high order collocation method for the static and vibration analysis of composite plates using a first-order theory," *Composite Structures*, vol. 89, no. 3, pp. 424–432, 2009.

[39] C. H. Thai, H. Nguyen-Xuan, N. Nguyen-Thanh, T.-H. Le, T. Nguyen-Thoi, and T. Rabczuk, "Static, free vibration, and buckling analysis of laminated composite reissner–mindlin plates using nurbs-based isogeometric approach," *International Journal for Numerical Methods in Engineering*, vol. 91, no. 6, pp. 571–603, 2012.

[40] J. N. Reddy, *Mechanics of laminated composite plates and shells: theory and analysis*. CRC press, 2004.