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NEW ALGORITHM FOR NUMERICAL SIMULATION OF SURFACE WAVES WITHIN THE FRAMEWORK OF THE FULL NONLINEAR DISPERSIVE MODEL

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Abstract. *The paper describes a new numerical method for modeling long waves with dispersion on a rotating attracting sphere. The algorithm is presented in the form of a finite-difference scheme of predictor-corrector, so that at each step the inhomogeneous hyperbolic system similar to classical shallow water equations with a right-hand side, and the uniformly elliptic equation are solved alternately. Necessary conditions for the stability of the method are obtained and the dispersion properties are investigated.*

1 INTRODUCTION

Approximate hydrodynamic models describing long-wave processes in a liquid have been actively developed over several decades. In recent years they enjoy popularity both among applied researchers and theoreticians. The interest of engineers is caused by the growing demands of practice in regard to the protection and development of sea coasts, whereas theoreticians have found a lot of gaps in the justification of both the models and numerical methods for solving the respective differential equations. Among the recently published papers devoted to the development of nonlinear dispersive (NLD) hydrodynamic models, we note studies [1, 2, 3] which take into account Earth's sphericity and its rotation. Such complication of the NLD-models is necessary for adequate description of the surface waves behaviour in the ocean and along the coastal zone.

In [3, 4] with using the unified approach to a construction of long-wave approximations, the hierarchy of hydrodynamic models, which possess the physically meaningful properties inherited, has been built. The hierarchical chains of shallow water models of first- and second-order long-wave approximation enclosed in each other are constructed on a rotating sphere and on a plane. With regard to numerical methods for the NLD-models, in contrast to the classical case of nonlinear shallow water (NLSW) equations, we have to state their lagging behind the development of the models themselves.

On the one hand, today there is urgent necessity in the numerical simulation based on practice needs, which should take into account sharp changes in the bottom surface, the non-linearity, vorticity, and other effects, on the other hand, there are many gaps in the study of fundamental properties of numerical methods, even at the simplest NLD-equations.

The peculiarity of our investigation is a methodical analysis which was initiated in [5]. The studies showed that the hierarchical approach helps to reveal not only the properties inheritance of the chain differential equations, but the same for the corresponding numerical algorithms, as well as describes the additional properties of difference schemes, acquired during the transition from the classical shallow water equations to NLD-equations. In particular, for Boussinesq type equations the improved stability conditions of some popular difference schemes of first- and second-order approximations are obtained. These conditions are characterized by presence of a new parameter defining the grid fineness compared to the characteristic depth, and in the limit of a grid refinement the conditions can be written in the form of the limitation on the time step only. It is also shown that in the domain of stability of some schemes there are the values of Courant numbers for which the influence of "scheme dispersion" is minimal.

The purpose of this paper is to present new numerical method for solving the NLD-equations on a rotating sphere [1]. The main idea lies in special splitting of the problem allowing the successive solution of two less difficult tasks at each time step. The first task is to calculate a dispersive component of the pressure using the uniformly elliptic equation. The second task is to solve the system of hyperbolic type consisting of the equations of continuity and motion in the dispersion-free shallow water model. The decomposition of the original system on two subproblems allows us to solve numerically the NLD-equations using the well known finite difference predictor-corrector type scheme for classical shallow water model modified so that the pressure is calculated additionally at each time step [6]. The corrected stability conditions and new knowledge on the dispersion properties of the modified predictor-corrector scheme are obtained.

2 NONLINEAR DISPERSIVE HYDRODYNAMIC MODELS

The full nonlinear dispersion (FNLD-) model on a rotating sphere based on the depth-averaged velocity was obtained in 2010 [1]. The paper [7] shows that the FNLD-model of [1] can be derived without any assumptions about the potentiality of the original 3D-flow, and that its defining equations, unlike [2], can be written in the quasi conservative form of mass and momentum balances. Additionally, this model has the balance equation of total energy agreed with a similar equation of 3D-model, that confirms not only the physical consistency of the FNLD-model, but also allows an additional control in the calculations. The equations of our FNLD-model on a rotating attracting sphere have the following form [1]:

$$(HR\sin\theta)_t + (Hu)_\lambda + (Hv\sin\theta)_\theta = 0, \tag{1}$$

$$\begin{aligned}(HuR\sin\theta)_t + (Hu^2 + g\frac{H^2}{2})_\lambda + (Huv\sin\theta)_\theta = gHh_\lambda - Huv\cos\theta -\\ - fvHR\sin\theta + \varphi_\lambda - \psi h_\lambda,\end{aligned} \tag{2}$$

$$\begin{aligned}(HvR\sin\theta)_t + (Huv)_\lambda + \Big[(Hv^2 + g\frac{H^2}{2})\sin\theta\Big]_\theta = gHh_\theta\sin\theta + g\frac{H^2}{2}\cos\theta +\\ + Hu^2\cos\theta + fuHR\sin\theta + (\varphi_\theta - \psi h_\theta)\sin\theta + \Omega^2 HR^2\sin^2\theta\cos\theta.\end{aligned} \tag{3}$$

Here R is a radius of a sphere rotating with a constant velocity Ω around the axis Oz of the fixed coordinate system $Oxyz$, with coordinate plane Oxy coinciding with the equatorial plane of the sphere.

To describe the flow of water a rotating together with sphere coordinate system $O\lambda\theta r$ is used, the beginning of which is located in the center of the sphere. Here λ is the longitude counted in the direction of a rotation from a certain meridian ($0 \le \lambda < 2\pi$), $\theta = \pi/2 - \varphi$ is the addition to the latitude ($-\pi/2 < \varphi < \pi/2$), r is the radial coordinate measured from the center of the sphere.

The Newtonian attractive force **g**, acting on a liquid particle of unit mass, is directed to the center of the Earth. The thickness of the water layer $H = \eta + h > 0$ is assumed to be small compared with the radius of the sphere, so the values of $g = |\mathbf{g}|$, and of water density ρ are assumed to be constant throughout the liquid layer bounded below by the impermeable moving bottom and above by a free surface:

$$r = R - h(\lambda, \theta, t), \quad r = R + \eta(\lambda, \theta, t). \tag{4}$$

The symbols u and v denote the physical components of the velocity vector ($u = Rc^1\sin\theta$, $v = Rc^2$, $c^1 = \dot{\lambda}$, $c^2 = \dot{\theta}$), $f = 2\Omega\cos\theta$ is the Coriolis parameter, expressed in terms of latitude's addition θ, here it is assumed that

$$\theta_0 \le \theta \le \pi - \theta_0, \tag{5}$$

where $\theta_0 = \text{const} > 0$ is a small angle (poles are excluded from the consideration).

The values of φ and ψ, included in the right sides of the equations of motion (2), (3) of the FNLD-model, are the dispersive components, respectively, of depth-integrated pressure p and the pressure p_0 at the bottom:

$$p = \frac{gH^2}{2} - \varphi, \quad p_0 = gH - \psi. \tag{6}$$

Dispersive additives are expressed by the following formulas [8]:

$$\varphi = \frac{H^3}{3}Q_1 + \frac{H^2}{2}Q_2, \quad \psi = \frac{H^2}{2}Q_1 + HQ_2, \tag{7}$$

where $Q_1 = D(\nabla\cdot\mathbf{c}) - (\nabla\cdot\mathbf{c})^2,\ Q_2 = D^2h,\ \mathbf{c} = (c^1, c^2)$,

$$D = \frac{\partial}{\partial t} + \mathbf{c}\cdot\nabla, \quad \nabla = \Big(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\theta}\Big), \quad \mathbf{c}\cdot\nabla = c^1\frac{\partial}{\partial\lambda} + c^2\frac{\partial}{\partial\theta},$$

$$\nabla\cdot\mathbf{c} = c^1_\lambda + \frac{1}{J}\left(Jc^2\right)_\theta, \quad J = -R^2\sin\theta\ .$$

More detailed formulas for Q_1 and Q_2 have the form

$$Q_1 = (\nabla\cdot\mathbf{c})_t + \frac{1}{R\sin\theta}\Big(u\,(\nabla\cdot\mathbf{c})_\lambda + v\,(\nabla\cdot\mathbf{c})_\theta\sin\theta\Big) - (\nabla\cdot\mathbf{c})^2\,,$$

$$Q_2 = (Dh)_t + \frac{1}{R\sin\theta}\Big(u\,(Dh)_\lambda + v\,(Dh)_\theta\sin\theta\Big),$$

where

$$\nabla\cdot\mathbf{c} = \frac{1}{R\sin\theta}\Big(u_\lambda + (v\sin\theta)_\theta\Big), \quad Dh = h_t + \frac{1}{R\sin\theta}\Big(uh_\lambda + vh_\theta\sin\theta\Big).$$

The FNLD-model (1)–(3) with φ and ψ as in (7) is called “full” [8] for the reason that it is derived without the assumption on the smallness of wave amplitudes and in the equations are stored all the nonlinear terms associated with dispersion. So, it can be used to calculate the surface waves propagation over the uneven bottom both in deep water and in the coastal zone. Additionally, the accounting for the waves dispersion can give more accurate results than dispersionless shallow water model. The FNLD-model described above can also simulate the waves generated by the long-time shifts of bottom fragments, that extends the range of problems that can be solved within the framework of the famous [9, 10, 11] weakly nonlinear dispersive (WNLD-) models of shallow water on the sphere.

Moreover, the structure of the FNLD-equations and the presence of the important properties such as conservatism and existence of the total energy balance equation agreed with 3D-model, is stored during the special simplification of the dispersion component and under transition to a flat geometry. Thus, the equations of the weakly nonlinear dispersion model obtained in [7] and the classical (nondispersive) shallow water model on a sphere have the same form as the equations of the FNLD-model on a sphere. The differences appear in the expression for the kinetic energy and in the calculations of the pressure terms (6). We provide the single form of the equations of the entire hierarchical chain of long-wave models in spherical and planar geometries [3], that made it possible to construct the uniform numerical algorithm for them.

3 FINITE-DIFFERENCE METHODS FOR SHALLOW WATER EQUATIONS WITH DISPERSION

Since NLD-equations and finite-difference approximations can’t be studied analytically in the total statement, it is reasonably to consider the simplified statement (cases of plane geometry, a flat bottom $h \equiv h_0 = \text{const}$, linear analogues of equations and difference schemes). This makes it possible to investigate the properties in particular cases (for example, the dispersion properties of linear difference schemes) and leads to a set of necessary restrictions on the use of the total algorithm (for example, a necessary condition for stability, preservation by the algorithm of some properties of the original differential problem, etc.). Here we consider two finite-difference algorithms for solving a system of linear equations of one-dimensional flows

$$\eta_t + h_0 u_x = 0, \quad u_t + g\eta_x = \nu u_{xxt}, \qquad \nu = \frac{h_0^2}{3}, \tag{8}$$

into which the NLD-models from the papers [12, 13, 14] and some other NLD-models can be transformed after linearization. Finite-difference approximations of these linear equations are used to obtain necessary stability conditions of nonlinear difference schemes, and also to research their dissipative and dispersive properties.

NLD-equations together with its linear variant (8) contain third order mixed derivatives with respect to time and space, that requires specific approaches to their numerical solution. The main technique to ensure inheritance of existing computational methods and development of new effective numerical algorithms for NLD-models, is in follows. By introducing new variables NLD-equations can be splitted in such way that one component is either a system of ordinary differential equations (ODE's) or inhomogeneous system of hyperbolic equations, and the other component does not include derivatives with respect to time and in the simplest case looks like as an elliptical equation. This opens up opportunities for using methods of ODE solutions and finite difference/finite-volume schemes, and the finite element method, as well as various combinations of these and other suitable methods.

The numerical algorithms based on splitting of NLD-equations on the system of ODE's and the elliptic equation, was first proposed in [15] for the unidirectional scalar equation, and then was extended to systems of WNLD- and FNLD-equations [16, 17, 18, 19, 20]. The feature of solving systems of NLD-equations is that the group of terms describing the dispersion contains the time derivative of the velocity u (in one-dimensional case). The easiest way to solve this task lies in the possibility to represent all terms with the time derivative as an expression Q_t with some Q, that allows to bring the implementation of the corresponding numerical algorithm as a step-by-step solving ODE's and the elliptic equation. With regard to the system of equations (8) this method leads to the following system:

$$\eta_t + h_0 u_x = 0, \quad Q_t = g\eta_x, \quad \nu u_{xx} - u = Q.$$

One of the most common ways of constructing a numerical algorithm is as follows. η^{n+1} is first calculated from the continuity equation by the two-layer explicit scheme (or implicit with respect to η). Then Q^{n+1} is calculated from ODE using η^{n+1} obtained. After that the elliptical (in one dimension – ordinary differential) equation for u^{n+1} is solved (in one dimension – by a scalar sweep).

The disadvantage of this approach is the non-divergent form of the system in the nonlinear case. It should also be noted that in two-dimensional case the system of two elliptic equations for the components of the velocity vector is got, which complicates the computational algorithm. But the most significant drawback is missing the opportunity of using methods for solving hyperbolic equations, well developed for NLSW-shallow water model of the first hydrodynamic approximation.

In this paper we propose the different way of splitting. The initial system of NLD-equations is replaced on the extended system consisting of the system of hyperbolic equations similar to NLSW-equations and differing from them by the right side, and of the scalar equation of elliptic type for the dispersive pressure component. This approach to the construction of numerical algorithms can be used for all FNLD- and WNLD-models described above.

Here we look at the essence of this approach on the example of the linear system (8). In this case, the extended system takes the form

$$\eta_t + h_0 u_x = 0, \quad u_t + g\eta_x = \frac{\varphi_x}{h_0}, \quad \varphi_{xx} - \frac{\varphi}{\nu} = c_0^2\eta_{xx}, \qquad c_0 = \sqrt{gh_0}. \tag{9}$$

It contains the new equation for the dependent variable $\varphi = \nu u_{xt}$, which is a linear analog of the dispersion component of the pressure φ from (7). Let Δx and τ be steps of the uniform grid in the plane of variables x and t, $(x_j,\, t^n)$ are grid nodes, $x_j = j\Delta x$, $t^n = n\tau$, η_j^n, u_j^n, φ_j^n are the values of grid functions in these nodes. For difference derivatives the following abbreviations are used:

$$\eta_{t,j}^n = \frac{\eta_j^{n+1} - \eta_j^n}{\tau}, \quad \eta_{x,j}^n = \frac{\eta_{j+1}^n - \eta_j^n}{\Delta x}, \quad \eta_{\bar{x}x,j}^n = \frac{\eta_{j+1}^n - 2\eta_j^n + \eta_{j-1}^n}{\Delta x^2}.$$

We use the predictor-corrector method as the difference scheme and solve the equations of hyperbolic and elliptic parts one after another on each computational time step. In the predictor step the auxiliary quantities $\eta_{j+1/2}^*$, $u_{j+1/2}^*$, $\varphi_{j+1/2}^*$ defined in cell centers are calculated. To do that we first solve the explicit difference equations corresponding to a hyperbolic part

$$\frac{\eta_{j+1/2}^* - \frac{1}{2}(\eta_{j+1}^n + \eta_j^n)}{\tau_{j+1/2}^*} + h_0 u_{x,j}^n = 0, \quad \frac{u_{j+1/2}^* - \frac{1}{2}(u_{j+1}^n + u_j^n)}{\tau_{j+1/2}^*} + g\eta_{x,j}^n = \frac{1}{h_0}\varphi_{x,j}^n, \tag{10}$$

and then calculate $\varphi_{j+1/2}^*$ from

$$\frac{\varphi_{j+3/2}^* - 2\varphi_{j+1/2}^* + \varphi_{j-1/2}^*}{\Delta x^2} - \frac{\varphi_{j+1/2}^*}{\nu} = c_0^2 \frac{\eta_{j+3/2}^* - 2\eta_{j+1/2}^* + \eta_{j-1/2}^*}{\Delta x^2}, \tag{11}$$

where $\tau_{j+1/2}^* = \dfrac{\tau}{2}(1 + \theta_{j+1/2}^n)$, $\theta_{j+1/2}^n$ is the parameter responsible for the TVD-properties.

Calculated predictor values are then used in the second stage – corrector – to calculate values η_j^{n+1}, u_j^{n+1} and φ_j^{n+1}:

$$\eta_{t,j}^n + h_0 \frac{u_{j+1/2}^* - u_{j-1/2}^*}{\Delta x} = 0, \quad u_{t,j}^n + g\frac{\eta_{j+1/2}^* - \eta_{j-1/2}^*}{\Delta x} = \frac{1}{h_0}\frac{\varphi_{j+1/2}^* - \varphi_{j-1/2}^*}{\Delta x}, \tag{12}$$

$$\varphi_{\bar{x}x,j}^{n+1} - \frac{\varphi_j^{n+1}}{\nu} = c_0^2 \eta_{\bar{x}x,j}^{n+1}. \tag{13}$$

It is easy to show that for $\theta \equiv \text{const} \neq 0$ the scheme has the first-order approximation, for $\theta \equiv 0$ it has the second. It is an interesting case of quasi constant values of $\theta = O(\tau, \Delta x)$, when the scheme parameter depends on the mesh size, and decreases in proportion to the decrease of a step, but does not depend on the coordinates of nodes, i.e. behaves as a constant parameter. In the case of quasi-constant parameter θ the scheme is second-order approximation. The necessary condition for stability of the written scheme is as follows:

$$c_0 æ \leq \frac{\sqrt{1 + 4\delta^2\theta/3}}{1 + \theta}, \tag{14}$$

where $0 \leq \theta \leq 1$, $æ = \tau/\Delta x$ and the parameter

$$\delta = \frac{h_0}{\Delta x}$$

characterizes the size of the spatial step Δx in relation to the characteristic depth h_0. The presence of this parameter is the peculiarity of the finite difference schemes for the NLD-equations since it together with the Courant number determines the stability condition and occurs in the formulas describing the dissipative and dispersion properties of numerical schemes. This is one of the differences from the dispersionless case where under similar conditions only one parameter (the Courant number) appears. Increasing δ leads to the fact that the stability condition (14) becomes more and more weak and under $\delta \gg 1$ can be transformed into a time step limitation:

$$\tau \leq \frac{2\sqrt{\theta}}{\sqrt{3}(1+\theta)}\tau_0 < \frac{1}{\sqrt{3}}\tau_0 \approx 0.58\tau_0, \tag{15}$$

where another new parameter $\tau_0 = h_0/c_0$ is introduced, denoting the characteristic time which it takes to wave propagating with the velocity c_0 to pass the distance equal to the characteristic depth h_0.

This fact can be interpreted in such a way that for sufficiently fine grid the stability condition can be replaced by (15), i.e. we get a restriction on the time step τ independent from Δx. Such a situation does not occur in the theory of difference schemes for classical shallow water. Really, for dispersionless linear shallow water equations, i.e. (8) with $\nu = 0$, the predictor-corrector scheme becomes easier due to the fact that there is no need to calculate the dispersion additive φ. The stability condition of such scheme has the form

$$c_0 æ \leq \frac{1}{\sqrt{1+\theta}}. \tag{16}$$

Here we have the stability condition in the form of restrictions on the Courant number, which is the standard for explicit schemes approximating hyperbolic equations. Grinding steps of the spatial grid requires proportional refinement of time steps.

We emphasize the importance of the observed feature consisting in the fact that the stability condition (14) of the difference scheme for dispersion equations is less stringent than (16) for dispersionless ones. This fact also holds for other known schemes [5]. However, it seems that in earlier studies this fact has not been discovered, and the previous developers of numerical algorithms for the NLD-equations chose, as a rule, time steps on the basis of more stringent restrictions: of NLSW-equations, expressed in Courant numbers. The discovered fact was confirmed in the course of experiments conducted for nonlinear case: for NLD-equations a stability remained at bigger time steps than it required for NLSW-equations.

To investigate the dispersion properties of the predictor-corrector scheme let's calculate its phase error

$$\Delta\phi = \mp\frac{c_0 æ}{6}\Big(c_0^2 æ^2(3\theta+1)-1\Big)\xi^3 + O(\xi^4),$$

where $\xi \in [0,\ \pi],\ \xi = k\Delta x,\ k$ is the wave number of a harmonic. We see that the main part of the phase error on long waves has generally the same order of ξ as the "physical" dispersion Φ of the model (8)

$$\Phi = \pm\left[c_0 æ\xi - \frac{c_0 æ}{6}\delta^2\xi^3 + O(\xi^4)\right],$$

that is a lack. If for a given value $\theta \geq 0$ a Courant number is selected by the formula

$$c_0 æ = \frac{1}{\sqrt{1+3\theta}}, \tag{17}$$

the phase error will be at least one order of ξ less than the "physical" dispersion, and herewith the stability condition (14) is performed. Therefore, under the condition (17) the numerical dispersion for the long wave will not suppress the "physical" one.

4 THE NUMERICAL ALGORITHM FOR NONLINEAR DISPERSIVE EQUATIONS ON A ROTATING SPHERE

The system of equations (1)–(3) of the FNLD-model on the sphere, as the linear system (8) discussed above, is not a system of Cauchy–Kovalevskaya type due to the fact that the equations of motion contain the mixed third-order derivatives of the velocity vector components with respect to time and space. The direct approximation of the derivatives leads to a complex problem which is difficult to solve. As shown above with respect to the linear case, it turned out fruitful to split the system (8) on the scalar equation of elliptic type and the system of hyperbolic equations.

The same approach can be implemented for the system of FNLD-equations (1)–(3) on a rotating sphere. As a result of splitting we obtain the extended system of equations consisting of an elliptic equation for the dispersive component φ of the depth-integrated pressure p and the hyperbolic system of equations (1)–(3) with time derivatives only of the first order that differs from the classical shallow water model on a sphere only by additional terms associated with the dispersive additives φ and ψ in the right part of the motion equations (we consider here φ and ψ as variables, but not as expressions).

The equation for φ is as follows [21]:

$$\Big[\frac{1}{\sin\theta}\Big(\frac{\varphi_\lambda}{H}-\frac{\nabla\varphi\cdot\nabla h}{Hr}h_\lambda\Big)\Big]_\lambda+\Big[\Big(\frac{\varphi_\theta}{H}-\frac{\nabla\varphi\cdot\nabla h}{Hr}h_\theta\Big)\sin\theta\Big]_\theta-k_0\varphi=F, \tag{18}$$

where $k_0=k_{00}+(k_{01})_\lambda+(k_{02})_\theta,\ r=4+\nabla h\cdot\nabla h,$

$$k_{00}=\frac{12(r-3)}{H^3r}R^2\sin\theta,\quad k_{01}=\frac{6h_\lambda}{H^2r\sin\theta},\quad k_{02}=\frac{6h_\theta}{H^2r}\sin\theta.$$

A distinctive feature of the equation (18) is the fact that neither the left nor the right part does not contain time derivatives of the dependent variables H, $\mathbf{u}=(u,v)$. In addition, under conditions $H>0$ and (5) the equation (18) is uniformly elliptic. Therefore, it is possible to construct the difference scheme with the positive definite operator and to use a suitable iterative methods for solving scalar elliptic equations.

For the dispersion component ψ an individual equation is not required, since the function ψ is associated with values of H, $\mathbf{u}$ and φ by expression [21]:

$$\psi=\frac{1}{r}\left(\frac{6\varphi}{H}+HQ+\nabla\varphi\cdot\nabla h\right), \tag{19}$$

where

$$\nabla\varphi\cdot\nabla h=\frac{1}{R^2}\left(\frac{\varphi_\lambda\bar h_\lambda}{\sin^2\theta}+\varphi_\theta\left(\bar h_\theta-\eta_{00,\theta}\right)\right)=\nabla\varphi\cdot\nabla\bar h-\varphi_\theta\frac{\Omega^2}{g}\sin\theta\cos\theta,$$

$$Q=(-g\nabla\bar\eta+\mathbf{a})\cdot\nabla h+\frac{1}{R^2\sin\theta}\left(\frac{u^2}{\sin\theta}\,h_{\lambda\lambda}+2uv\;h_{\lambda\theta}+v^2h_{\theta\theta}\;\sin\theta\right)+B,$$

$$B=h_{tt}+2\Big(\frac{u}{R\sin\theta}h_{\lambda t}+\frac{v}{R}h_{\theta t}\Big),$$

$$\mathbf{a}=(a_1,a_2),\quad a_1=-(2uv\,\mathrm{ctg}\,\theta+fvR)\sin\theta,\quad a_2=u^2\mathrm{ctg}\,\theta+fuR,$$

$f=2\Omega\cos\theta$ is the Coriolis parameter. Functions $\bar\eta$ and $\bar h$ are connected with η and h by relations

$$-h=\eta_{00}-\bar h,\quad \eta=\eta_{00}+\bar\eta$$

and describe the bottom and the free surface not in the form of deviations (4) from the surface of a sphere of radius R, but the deviations $\bar{h}$ and $\bar{\eta}$ from the undisturbed free boundary, as is usually done for solving shallow water equations in plane geometry. In this case the unperturbed free boundary at rest differs from spherical [7] and is described by the equation $r = R + \eta_{00}(\theta)$, where

$$\eta_{00}(\theta) = \frac{1}{2g}\Omega^2 R^2 \sin^2\theta + \text{const.}$$

The numerical algorithm for the extended system of equations (1)–(3), (18) is a direct generalization of the finite-difference scheme (10)–(13) which was discussed above in the context of a linear case. This method is the predictor-corrector scheme, whose steps consist of two stages. On the predictor step the values H^*, $\mathbf{u}^*$ are first determined in the centers of cells as the solution of the explicit difference equations for the system of hyperbolic equations written in non-divergent form. Then we solve the difference equation for φ^* using the new values H^*, $\mathbf{u}^*$ in coefficients and a right-hand side, and define ψ^* by formula (19).

The values H^*, $\mathbf{u}^*$, φ^* and ψ^* found on the predictor step are used then on the corrector step for computation of final values H^{n+1} and $\mathbf{u}^{n+1}$ with the numerical solution of the system (1)–(3) in the divergent form of the left part. In the last turn values of functions φ^{n+1} and ψ^{n+1} are calculated. The similar algorithm can be applied also to the solution of the FNLD-equations in plane geometry [6].

Using a benchmark about the propagation of waves on transoceanic distances it is shown in [21] that in terrestrial conditions the centrifugal force can be ignored and the accounting effects of sphericity can be critical important. Influence of the Coriolis force increases with the horizontal sizes and with propagation distances. The accounting of the sphere rotation lowers the maximum positive amplitudes of waves. Dispersion is also manifested at long distances of waves traveling and can lead to essential reduction of the maximum amplitude of a head wave which disintegrates into a train of solitary waves of smaller length. But, unlike the Coriolis force, dispersion appears more strongly for the smaller size of initial disturbance, at that the accounting of dispersion leads to the deceleration of the waves advance.

5 CONCLUSIONS

This paper sets forth the numerical algorithm for the simulation of propagation of surface waves in the frame of the full nonlinear dispersion model which takes into account the sphericity of the Earth and its rotation. The algorithm is based on the solution of the extended system of equations consisting of the elliptic equation for the dispersive component of the pressure and the hyperbolic first order equations, which differ from the equations of classical shallow water model only by presence of additional terms related to the dispersive additives. The algorithm is implemented as the explicit two-step predictor-corrector scheme with solving the resulting subproblems at each step.

Since the structure of the equations of the full nonlinear dispersive model holds true for the weakly dispersion model on a sphere, as well as the model of weakly dispersive flows over weakly deformable bottom [4], the algorithm for the numerical solution of the full nonlinear dispersive equations can be transferred unchanged also to this approximate models.

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