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NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS WITH NON-CONSTANT COEFFICIENTS

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Abstract. *Nowadays, new materials like Functionally Graded Material (FGM) are necessary for sophisticated structures like MEMS systems, advanced electronic devices, etc. Computer modelling of such complex systems, like structures with spatial variation of material properties (e.g. FGM) are, using commercial FEM code with classic elements, needs remarkable effort during preparation phase and sufficient computer equipment for solution phase because of necessity the numbers of elements and material models. Therefore new methods for modeling and simulation of FGM beams with spatial variation of material properties are developed.*

In the proposed contribution, semi-analytical method (based on calculation of transfer functions and transfer constants) for solution of differential equation with non-constant polynomial coefficients, is presented. This method is used in derivation process (for setting up the transfer matrix) of our new beam finite elements for modeling and simulation of Functionally Graded Material (FGM) beam structures (e.g. new 3D FGM beam finite element for modal and structural analysis, new FGM beam finite elements for coupled electro-thermo-mechanical analysis). Numerical experiments are made to show the accuracy and effectiveness of this method.

1 INTRODUCTION

Physical processes in materials, such as heat transfer, conduction of electric current, mechanical stress, etc., are often described by partial differential equations or by system of partial differential equations. An exact analytical solution is usually possible only for simplified investigated domains (e.g. one dimensional link parts - bars) and simple boundary conditions of the investigated system, where the geometry allows to reduce the partial differential equations into ordinary differential equations. The geometry of the bar, as well as the boundary conditions fulfil the requirements that are needed to apply analytical methods for solving the differential equations.

One example, let us write general partial differential equation for heat transfer in considered domain:

$$\frac{\partial T(x,y,z,\tau)}{\partial \tau} = \frac{\lambda}{c\rho} \bigtriangledown^2 T(x,y,z,\tau) + \frac{Q}{c\rho} \tag{1}$$

Quantity $T(x,y,z,\tau)$ [K] is the unknown temperature of the system, τ [s] is the time, λ [$\mathrm{Wm^{-1}K^{-1}}$] is the thermal conductivity of considered material, c [$\mathrm{Jkg^{-1}K^{-1}}$] is its specific heat, ρ [$\mathrm{kgm^{-3}}$] is the density of the material and Q [$\mathrm{Wm^{-3}}$] is the volume heat sources in the system. For steady state and for 1D geometry (bar construction) we can simplify this partial differential equation into the form of ordinary differential equation:

$$\frac{d^2T(x)}{dx^2} = -\frac{Q}{c\lambda} \tag{2}$$

with two boundary conditions, e.g.:

$$\begin{aligned} T(0) &= T_0 \\ T(L) &= T_L \end{aligned} \tag{3}$$

It is possible to write similar ordinary differential equations also for other physical fields (e.g. electric field, structural analysis, etc.). When we consider variable material properties of the bar (e.g. Functionally Graded Material - FGM) the ordinary differential equations contain variable (nonconstant) coefficients.

In this contribution, semi-analytical method based on calculation of transfer functions (transfer constants), for solution of differential equation with polynomial coefficients [1] is presented. This method is used in derivation process (for setting up the transfer matrix) of our new beam finite elements for modeling and simulation of FGM beam structures with 3D spatial variation of material properties (e.g. new 3D FGM beam finite element for modal, structural and buckling analysis [2][3], new FGM beam finite elements for coupled electro-thermo-mechanical analysis[4]).

2 SEMI-ANALYTICAL METHOD FOR SOLUTION OF LINEAR DIFFERENTIAL EQUATION WITH NON-CONSTANT COEFFICIENTS

There will be set out a procedure for solving differential equations with variable coefficients and right-hand side, which is taken from Rubins article [1]. These differential equations must fulfil the following requirements:

- differential equation of one independent variable
- polynomial character of variable coefficients and right side of the differential equation

- known interval of the independent variable, where the solution of the differential equation needs to be determined ($x \in \langle 0, L \rangle$)

The order of differential equation and also the order of its right side are arbitrary.

2.1 Solving of the differential equation with non-constant coefficients

Let us consider 1D differential equation with non-constant coefficients and the right side in the form:

$$\sum_{u=0}^{m} \eta_u(x)\, y^{(u)}(x) = \sum_{j=0}^{g} q_j\, a_j(x) \tag{4}$$

where m is a order of the differential equation, $y(x)$ is an unknown function of independent variable x, $y^{(u)}(x)$ is u^{th} derivative of the unknown function $y(x)$ $\left(y^{(u)}(x) = d^u y(x)/dx^u\right)$, $\eta_u(x)$ is a polynomial variable coefficient for u^{th} derivation of the differential equation, g is the order of a polynomial on the right side of the differential equation, q_j is a constant coefficient for j^{th} power of the right side polynomial. Function $a_j(x)$ is an auxiliary function for polynomial formulation

$$\begin{aligned} a_j(x) &= \frac{x^j}{j!} & j > 0 \\ a_j(x) &= 1 & j = 0 \\ a_j(x) &= 0 & j < 0 \end{aligned} \tag{5}$$

Polynomial coefficients $\eta_u(x)$ of the differential equation (4) for $u = \{0; m\}$ can be written as:

$$\begin{aligned} \eta_u(x) &= \sum_{r=0}^{p_u} \rho_{ur}(x) \\ \rho_{ur}(x) &= \eta_{ur}\, a_r(x) \end{aligned} \tag{6}$$

where η_{ur} is a constant coefficient for r^{th} power of polynomial for u^{th} derivation, p_u is an order of polynomial for u^{th} derivation and $a_r(x)$ is an auxiliary function according to (5) for polynomial formulation.
According to this notation, derivation and integration of polynomial can be written in general form:

$$\begin{aligned} a_j^{'}(x) &= a_{j-1}(x) \\ \int_0^x a_j(x) &= a_{j+1}(x) \end{aligned} \tag{7}$$

Then the solution of the differential equation (4) has according to [1] the form:

$$y(x) = \sum_{j=0}^{m-1} y_0^{(j)}\, c_j(x) + \sum_{j=0}^{g} q_j\, b_{j+m}(x) \tag{8}$$

where $y_0^{(j)}$ is j^{th} derivative of the function $y(x)$ at the position $x = 0$ $\left(y_0^{(j)} = y^{(j)}(x)|_{x=0}\right)$, $c_j(x)$ is a transfer function for uniform solution of the differential equation and $b_{j+m}(x)$ is a

transfer function for particular solution of the differential equation (4).
Then the derivative of the solution (8) has a form:

$$y^{(u)}(x) = \sum_{j=0}^{m-1} y_0^{(j)}\, c_j^{(u)}(x) + \sum_{j=0}^{g} q_j\, b_{j+m}^{(u)}(x) \tag{9}$$

For $u = 0$ is Eq. (9) equal to Eq. (8). The solution (8) of the differential equation (4) and its derivatives (9) can be written in the matrix form:

$$\begin{aligned}\begin{bmatrix} y(x) \\ y'(x) \\ y''(x) \\ \vdots \\ y^m(x) \end{bmatrix} &= \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{m-1} \\ c'_0 & c'_1 & c'_2 & \dots & c'_{m-1} \\ c''_0 & c''_1 & c''_2 & \dots & c''_{m-1} \\ \vdots & \vdots & \vdots & \ddots & \\ c_0^m & c_1^m & c_2^m & \dots & c_{m-1}^m \end{bmatrix} \cdot \begin{bmatrix} y_0 \\ y'_0 \\ y''_0 \\ \vdots \\ y_0^m \end{bmatrix} \\ &+ \begin{bmatrix} b_m & b_{m+1} & b_{m+2} & \dots & b_{j+m} \\ b'_m & b'_{m+1} & b'_{m+2} & \dots & b'_{j+m} \\ b''_m & b''_{m+1} & b''_{m+2} & \dots & b''_{j+m} \\ \vdots & \vdots & \vdots & \ddots & \\ b_m^m & b_{m+1}^m & b_{m+2}^m & \dots & b_{j+m}^m \end{bmatrix} \cdot \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ \vdots \\ q_g \end{bmatrix}\end{aligned} \tag{10}$$

The solution (8) and its derivatives (9) of the differential equation (4) are based on determination of the transfer functions generally labelled $c(x)$ and $b(x)$. The calculation of functions $b_{j+m}^{(u)}(x)$ is based on the use of power series and recursive process, considering $u = \{0; m\}$ and $j = \{0; g\}$. It is necessary to guarantee the convergence of the series for a given interval $x \in \langle 0; L\rangle$ for successful calculation of these functions. It is always fulfil for differential equation with constant coefficients η_u but for differential equation with variable (polynomial) coefficients $\eta_u(x)$ it is often necessary to divide the interval of x into the shorter sections (in our case the independent variable is geometric variable, for example $x = L$ is the length of the bar), and thus determine the solution also for inner region of the bar (where $x \in (0; L)$). This division of the interval is implemented in an algorithm. Calculated functions $b_{j+m}^{(u)}(x)$ are used for next calculation of the functions $c_j^{(u)}(x)$, where $u = \{0; m\}$ and $j = \{0; m-1\}$.

2.2 Recursive calculation of the transfer functions using the power series

We can write (6) again:

$$\rho_{ur}(x) = \eta_{ur}\, a_r(x) \tag{11}$$

with assumption for $u = \{0; m\}$ and $r = \{0; p_u\}$.
Let us introduce the following equation:

$$\widetilde{\rho}_{ur}(x) = \rho_{m-u,r}(x)\frac{x^u}{\eta_{m0}} \tag{12}$$

where $u = \{0; m\}$ and $r = \{0; p_{m-u}\}$.
The calculation itself is based on determining of the power series members $e_s^{(u)}$:

$$b_j^{(u)} = \sum_{s=0}^{\infty} e_s^{(u)}(x) \tag{13}$$

where $j = \{m; \max j\}$ and $u = \{0; m\}$. The value $\max j = \max(j_e, j_g)$ where $j_e = m+g$ ensures correct calculation for the high order of the polynomial on the right-hand side of the differential equation (order of the right-hand side polynomial is higher than order of any polynomial on the left-hand side of the differential equation) and the value $j_g = \max(p_t - t + 2m - 1)$ where $t = \{0; m-1\}$ ensures correct calculation for the high order of the polynomials for derivatives on the left-hand side of the differential equation.
First members of the series $(s = 0)$ are:

$$e_0^{(u)}(x) = \frac{a_{j-u}(x)}{\eta_{m0}} \tag{14}$$

Next members for $s = 0$ are calculated using matrix $\varepsilon_{s,ur}(x)$ where $u = \{0; m\}$ and $r = \{0; p_{m-u}\}$. For the members of matrix $\varepsilon_{0,ur}(x)$ it holds:

$$\begin{aligned} \varepsilon_{0,00}(x) &= e_0^{(m)}(x) \\ \varepsilon_{0,ur}(x) &= 0 \end{aligned} \tag{15}$$

For the rest of series members where $s > 0$ we can write a recursive rule:

$$\begin{aligned} \varepsilon_{s,ur}(x) &= \varepsilon_{s-1,u,r-1}(x); \quad u = \{0; m\}, r = \{1; p_{m-u}\} \\ \varepsilon_{s,u0}(x) &= \frac{\varepsilon_{s-1,u-1,0}(x)}{j-m+s}; \quad u = \{1; m\} \\ \varepsilon_{s,00}(x) &= -\Big(\sum_{u=1}^{m} \varepsilon_{s,u0}(x)\widetilde{\rho}_{u0}(x) + \sum_{u=0}^{m}\sum_{r=1}^{p_{m-u}} \varepsilon_{s,ur}(x)\widetilde{\rho}_{ur}(x)\Big) \end{aligned} \tag{16}$$

Backward recursion has then a form:

$$\begin{aligned} e_s^{(m)}(x) &= \varepsilon_{s,00}(x) \\ e_s^{(m)}(x) &= \frac{x}{j+s-u} e_s^{(u+1)}(x) \end{aligned} \tag{17}$$

where $u = \{m-1; 0\}$.

Equation (13) then can be used for calculation the functions $b_j^{(u)}(x)$. But we can see that $s \in \langle 0; \infty \rangle$ so it is necessary to choose maximum permissible limit for s. The fact of lack of convergence or divergence of the series is accepted when this limit is reached. In that case it is necessary to determine the functions $b_j^{(u)}(x)$ using shorter interval, so primary interval $x \in \langle 0; L \rangle$ needs to be divided.

The functions of uniform solution of the differential equation are calculated as follows:

$$c_j^{(u)}(x) = a_{j-u}(x) - \sum_{t=0}^{j}\sum_{r=0}^{p_t} \binom{j-t+r}{r} \eta_{tr} b_{j-t+r+m}^{(u)}(x) \tag{18}$$

where $u = \{0; m-1\}$ and $j = \{0; m-1\}$.

2.3 Characteristics of the algorithm for solving of the differential equations

The result of the calculation of the differential equation with variable coefficients and the right-hand side is the solution according to the equation (8). It should be noted that this is a solution for selected point x of the considered interval of the independent variable, so the program is designed for calculation of values $c_j(x)$ in given point x for $j = \{0; m-1\}$ where m is the order of the differential equation. The values $b_{j+m}(x)$ for selected point x are also calculated for $j = \{0; g\}$ where g is order of the right-hand side polynomial of the differential equation. It means that functions $c(x)$ and $b(x)$ cannot be calculated analytically but only at discrete points x, where x is from interval $\langle 0; L\rangle$.

The algorithm calculates matrices of discrete values $c_j^{(u)}(x)$ and $b_{j+m}^{(u)}(x)$ in defined point x where $u = \{0; m-1\}$ represents derivative for calculations according to (9). It always holds:

$$\begin{aligned} c_j^{(u)}(x)|_{x=0} &= 1; \quad \text{if } i = u \\ c_j^{(u)}(x)|_{x=0} &= 0; \quad \text{if } i \neq u \\ c_{j+m}^{(u)}(x)|_{x=0} &= 0; \quad \text{forall } j = \{0; g\} \end{aligned} \tag{19}$$

For the calculation of $c_j^{(u)}(x)$ and $b_{j+m}^{(u)}(x)$ we do not need to know the the polynomial functions of the right side (does not enter into calculation) of the differential equation but only the degree of the right side polynomial. Its coefficients q_j are used only in the solution (8) and its derivatives (9). So we can use calcualted $c_j^{(u)}(x)$ and $b_{j+m}^{(u)}(x)$ for different coefficients q_j of the right side polynomial if its degree does not change.

Using power series gives us exact solution and because advanced numerical operations as numerical integration are not required in the calculation of the transfer functions $c_j^{(u)}(x)$ and $b_{j+m}^{(u)}(x)$ this procedure is fast and can be easy implemented into the FEM code. Only the differential equations with one independent variable can be solved and the variable coefficients and the term on the right side of the differential equation has to be polynomial.

The whole procedure of calculation transfer functions $c_j^{(u)}(x)$ and $b_{j+m}^{(u)}(x)$ is described in [1] in detail - the block diagram of this procedure is shown in Figure 1. This approach for calculation differential equation with non-constant parameters was implemented into the software Mathematica [5].

3 NUMERICAL EXAMPLE

Let us consider differential equation of 2^{nd} order on the interval $y \in \langle 0, L\rangle$, $L = 0.1$ with non-constant polynomial coefficients:

$$\eta_2(x)y''(x) + \eta_1(x)y'(x) + \eta_0(x)y(x) = q \tag{20}$$

where

$$\begin{aligned} \eta_0 =& -637\,500\,000\,000x^4 + 146\,625\,000\,000x^3 - 10\,965\,000\,000x^2 \\ &+ 267\,750\,000x + 60\,000 \\ \eta_1 =& \frac{129\,417\,900}{121}x^2 - \frac{12\,941\,790}{121}x \\ \eta_2 =& \frac{43\,139\,300}{121}x^3 - \frac{6\,470\,895}{121}x^2 + \frac{2\,217\,793}{7\,260} \\ q =& -6\,375\,000\,000\,000x^4 + 1\,459\,950\,000\,000x^3 - 109\,650\,000\,000x^2 + \\ &+ 2\,742\,500\,000x + 610\,000 \end{aligned} \tag{21}$$

with boundary conditions:

$$y[0] = y_0 = 10 \qquad y'[L] = y_L = -122.623 \tag{22}$$

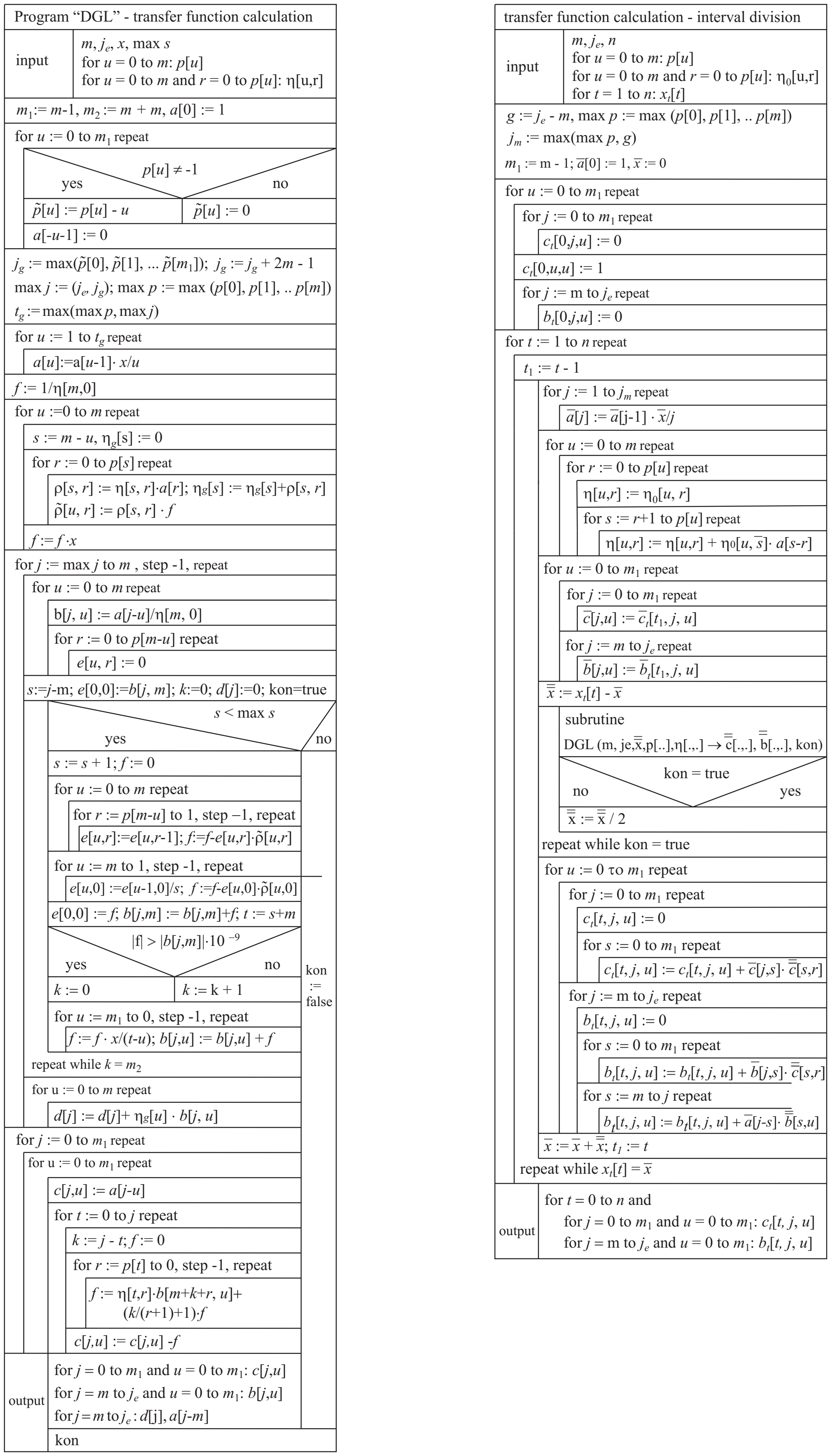

Figure 1: Block diagram of calculation transfer functions [1]

This differential equation can represent differential equation for heat transfer (consideration of the convective effect and internal heat generation) in the form:

$$-\lambda(x)\frac{d^2T(x)}{dx^2} - \frac{d\lambda}{dx}\frac{xT(x)}{dx} + \alpha(x)T(x)\frac{o}{A} = Q(x) + \alpha(x)T_{amb}\frac{o}{A} \tag{23}$$

where we consider that thermal conductivity $\lambda(x)$ [Wm^{-1}K^{-1}], coefficient of convective heat transfer $\alpha(x)$ [Wm^{-2}K^{-1}] and volume heat $Q(x)$ [Wm^{-3}] are polynomial functions. A is a cross-section of the bar, o is a perimeter ot the bar ant T_{amb} is a constant ambient temperature.

The solution of differential equation (20) according to (8) is:

$$y(x) = \sum_{j=0}^{m-1=1} y_0^{(j)} c_j(x) + \sum_{j=0}^{g=4} \varepsilon_j b_{j+2}(x) = c_0(x)y_0 + c_1(x)y_0' + \sum_{j=0}^{g=4} \varepsilon_j b_{j+2}(x) \tag{24}$$

where $m = 2$ and $g = 4$. The transfer functions $c_j^{(u)}(x = L)$ and $b_{j+m}^{(u)}(x = L)$ for differential equation (20) with non-constant coefficient (21) according the proposed method have a form:

$$c_j^{(u)}(x = L) = \begin{bmatrix} c_0 & c_1 \\ c_0' & c_1' \end{bmatrix} = \begin{bmatrix} 202.258 & 3.33436 \\ 9722.66 & 160.297 \end{bmatrix} \tag{25}$$

$$b_{j+m}^{(u)}(x = L) = \begin{bmatrix} b_2 & b_3 & b_4 & b_5 & b_6 \\ b_2' & b_3' & b_4' & b_5' & b_6' \end{bmatrix} = \tag{26}$$

$$= \begin{bmatrix} -0.000159832 & -2.67325 \times 10^{-6} & -4.87898 \times 10^{-8} & -8.52047 \times 10^{-10} & -1.33152 \times 10^{-11} \\ -0.007846 & -0.000141612 & -2.89679 \times 10^{-6} & -5.68637 \times 10^{-8} & -9.91117 \times 10^{-10} \end{bmatrix}$$

The solution of differential equation (20) according the proposed method for $n = 50$ internal points compared with solution obtained by numerical solution in software Mathematica [5](explicit RungeKutta method) is shown in Figure 2 and its first derivative is shown in Figure 3.

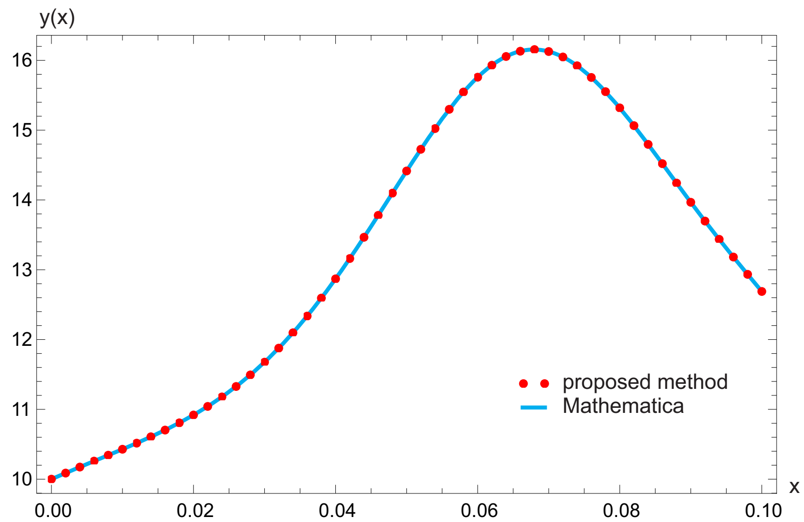

Figure 2: The solution of differential equation

As it can be seen in Figures 2 and 2, a very good agreement of both solution results has been obtained.

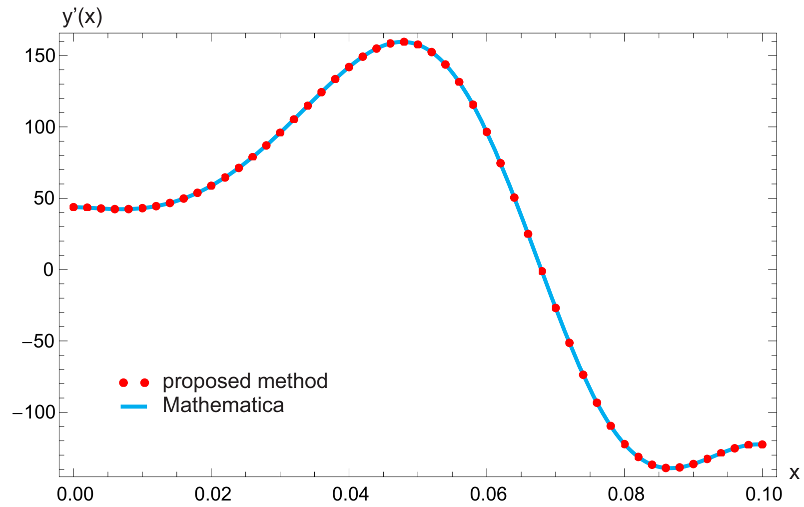

Figure 3: The 1^{st} derivation of the solution of differential equation

4 CONCLUSIONS

In this contribution the approach for solving differential equation with non-constant (polynomial) coefficient has been presented. The general solution of the homogeneous differential equation is formulated with the transfer functions $c_j^{(u)}(x)$ and the particular solution with $b_{j+m}^{(u)}(x)$. The transfer functions $b_{j+m}^{(u)}(x)$ are calculated with the help of series formulas and then the functions $c_j^{(u)}(x)$ may be determined with these $b_{j+m}^{(u)}(x)$.

The numerical example solution of differential equation of $2^n d$ order with non-constant polynomial coefficients using proposed approach and comparison with results obtained in software Mathematica have been . On base of the transfer relations the effective matrix of the 3D beam finite elements for modal, structural and buckling analysis [2][3] or coupled electro-thermo-mechanical analysis [4] of the FGM single beams and beam structures can be established. Material properties of the FGM can vary in all three direction x, y, z. Homogenization of the spatially varying material properties in the real FGM beam (material properties vary in all three direction) and the calculation of effective parameters of the homogenized beam (material properties vary only in longitudinal direction) are done by the extended mixture rules and the multilayer method.

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