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DELAYED FEEDBACK CONTROL METHOD FOR CALCULATING SPACE-TIME PERIODIC SOLUTIONS OF VISCOELASTIC PROBLEMS

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Abstract. *We are interested in fast techniques for calculating a periodic solution to viscoelastic evolution problems with a space-time periodic condition of "rolling" type. Such a solution is usually computed as an asymptotic limit of the initial value problem with arbitrary initial data. We want to invent a control method, accelerating the convergence. The main idea is to modify our problem by introducing a feedback control term, based on a periodicity error, in order to accelerate the convergence to the desired periodic solution of the problem.*

First, an abstract evolution problem has been studied. From the analytic solution of the modified (controlled) problem, an efficient control has been found, optimizing the spectrum of the problem. The proposed control term can be mechanically interpreted, and its efficiency increases with the relaxation time.

In order to confirm numerically the theoretical results, a finite element simulation has been carried out on a full 2D model for a steady rolling of a viscoelastic tyre with periodic sculpture. It has demonstrated that the controlled solution converges indeed faster than the non-controlled one, and that the efficiency of the method increases with the problem's relaxation time, that is when the memory of the underlying problem is large.

1 INTRODUCTION

In industrial applications, in order to avoid the inversion of very large matrices, dynamic periodic state are often computed as the asymptotic limit solution of an initial value evolution problem with arbitrarily chosen initial data. In such cases, one is not really interested in the evolution history, but only in convergence time. Computations can take a lot of time for ”viscous” problems, when memory effects are very large. Then the convergence time becomes a significant parameter for industrial implementations, and developing methods accelerating convergence is of current interest.

For such problems, even if the asymptotic limit is periodic, the solution of the initial value evolution problem is not. Thus the periodicity of the desired solution can be considered as an extra information (observation), which makes possible to apply control techniques. In other words, we want to use this information to construct a filter for the evolution problem. This modifies the original evolution problem through a control term based on the observation error. In this framework, the present work is dedicated to the development of acceleration methods for the solution of periodic problems. This kind of problems can be faced, for instance, in the cardiac contractions modeling [2]. Another example is a steady rolling of a viscoelastic tyre [1] with a periodic sculpture. In this case, the steady state satisfies a ”rolling” periodicity condition: the state at any point is the same that at the corresponding point of the next sculpture motive but one time period ago. It means the periodicity condition includes shift both in time and space. For such problems, we consider such a space-time periodicity as an observation and we use it for developing an accelerating method. The main idea is to modify the original problem by introducing a feedback control term [2, 3, 4] based on a periodicity error,which accelerates convergence to the stable periodic solution of the problem.

In the following section, we present an analytical analysis of an abstract problem, in order to find out an optimal form of the control. After introducing a delayed feedback control term based on the observation error (in an abstract form), we find the solution to the modified problem, in order study the influence of the control on the convergence rate. Then, we propose the optimal control, optimizing the spectrum of the problem and minimizing the convergence time. The proposed control can be mechanically interpreted. And the efficiency of the method increase with the convergence time of the problem.

The last section applies the technique to a full viscoelastic problem. There we consider the steady rolling of the viscoelastic tyre with a periodic sculpture. The numerical simulation results are presented, comparing both controlled and non-controlled solutions in a simplified 2D framework.

2 ANALYTICAL ANALYSIS

2.1 General problem

Let us consider a general evolution problem with space-time periodic condition:

$$\partial_t u + Au = f \tag{1}$$

$$u(t) = Hu(t-T) \tag{2}$$

defined on a Hilbert space V with scalar product $\langle\cdot,\cdot\rangle$. We make two fundamental assumptions which are usually satisfied for time periodic viscoelastic problems:

- the operator A is V-elliptic, namely there exists a constant $C > 0$ such that $\forall u \in V$ $\langle Au, u\rangle \geq C\|u\|^2$
- the operators A and the space shift H can be diagonalized in the same basis.

In practice, the solution of such problems is usually computed as an asymptotic limit of the solution to the initial value evolution problem (1) with an arbitrary initial condition. Due to the ellipticity condition, this asymptotic limit exists. In the frame of this work, we are going to propose an optimal control method, accelerating the convergence of the initial value problem solution to the solution of (1)-(2).

2.2 Controlled problem

We can consider the condition (2) as an observation and use it to construct a control method. So we modify the initial value problem by introducing a feedback control term [3, 4], based on the observation error, as follows:

$$\partial_t u(t) + Au(t) + \overbrace{G\Big(u(t) - Hu(t-T)\Big)}^{\text{control term}} = f, \tag{3}$$

$$u(t) = \phi(t), \quad t \leq 0. \tag{4}$$

Above $\phi(t)$ represents the initial data, and $u(t) - Hu(t-T)$ is the observation error. Moreover, the gain operator G which acts on this observation error has to be properly identified. It is clear that the solution $u(t,x)$ to the modified problem (3) converges to the solution of the original problem (1). The question is whether there exists a gain operator G, providing the fastest convergence to the time periodic asymptotic limit. This amounts to analyze the influence of the gain operator G on the (vanishing) asymptotic behavior of the homogeneous solution of (3). Thus we have to find the solution of the homogeneous controlled problem as a function of G, which will yield the optimal choice of G by minimizing the real part of the spectrum of the problem. We assume that the gain operator can be expressed as an analytical function $G(A,H)$ of the evolution and shift operators. Then if A and H are diagonalizable in the same base, G is also diagonalizable in this base.

Theorem 1. *Since A, G and H are diagonalizable in the same base $\{v_p(x)\}_{p\in\mathbb{Z}}$, the homogeneous solution to the controlled problem (3) is simply given by*

$$u(t,x) = \sum_{p,k\in\mathbb{Z}} c_{p,k}\, e^{\lambda_{p,k}t} v_p(x) \tag{5}$$

where the coefficients $c_{p,k}$ are defined from the initial data, and the inverse relaxation time $\lambda_{p,k}$ can be expressed in terms of eigenvalues of A, G and H.

Proof. We write the solution $u(t,x)$ in the base $\{v_p(x)\}_{p\in\mathbb{Z}}$, where A, G and H are diagonalizable:

$$u(t,x) = \sum_{p\in\mathbb{Z}} u_p(t) v_p(x) \tag{6}$$

where $u_p(t)$ are the coordinates of $u(t,x)$ in this base. Let α_p, γ_p and σ_p be the eigenvalues of A, G and H respectively, associated to the same eigenfunction $v_p(x)$. Then $u_p(t)$ satisfies

$$\partial_t u_p(t) + (\alpha_p + \gamma_p)\, u_p(t) - (\gamma_p \sigma_p) u_p(t-T) = 0, \qquad \forall p \in \mathbb{Z} \tag{7}$$

This is a simple delay-differential equation [5, 6, 7]. We are looking for its solution in the exponential form $u_p(t) = e^{\lambda_p t}$. The characteristic equation for λ_p is

$$\lambda_p + (\alpha_p + \gamma_p) - (\gamma_p \sigma_p)\, e^{-T\lambda_p} = 0. \tag{8}$$

Multiplying this by $Te^{T(\lambda_p+\alpha_p+\gamma_p)}$, this characteristic equation takes the form:

$$T\,(\lambda_p + \alpha_p + \gamma_p)\, e^{T(\lambda_p+\alpha_p+\gamma_p)} = T\,(\gamma_p \sigma_p)\, e^{T(\alpha_p+\gamma_p)} \tag{9}$$

To represent a solution of this equation, one have to introduce a Lambert W function [8, 10, 11, 7]. It is a complex-valued function denoted by $W[z]$ and satisfying $W[z]\, e^{W[z]} = z$ $(\forall z \in \mathbb{C})$. The Lambert W function will be described in more details in the next paragraph. For the time being, simply note that the Lambert function W is multivalued and has an infinite number of branches $W_k,\ k \in \mathbb{Z}$. Then, we can rewrite the equation (9) in terms of W_k as :

$$T\,(\lambda_p + \alpha_p + \gamma_p) = W_k\left[T\,(\gamma_p \sigma_p)\, e^{T(\alpha_p+\gamma_p)}\right], \qquad k \in \mathbb{Z}. \tag{10}$$

So the characteristic equation (8) has an infinite number of roots, corresponding to different branches W_k of the Lambert function :

$$\lambda_{p,k} = -\,(\alpha_p + \gamma_p) + \frac{1}{T} W_k\left[T\,(\gamma_p \sigma_p)\, e^{T(\alpha_p+\gamma_p)}\right], \qquad k \in \mathbb{Z}, \quad p \in \mathbb{Z}. \tag{11}$$

Thus, the general solution to (7) is

$$u_p(t) = \sum_{k\in\mathbb{Z}} c_{p,k} e^{\lambda_{p,k} t}, \qquad \forall p \in \mathbb{Z} \tag{12}$$

Assembling all components of the eigenbase, we then obtain the solution (6) to the controlled problem (3):

$$u(t,x) = \sum_{p\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} e^{\lambda_{p,k} t} c_{p,k} v_p(x) \tag{13}$$

with $\lambda_{p,k}$ as defined by (11). Coefficients $c_{p,k}$ are to be found from the initial data.

□

2.3 A few properties of the Lambert W function

Let us take a brief overview on the Lambert W function [8, 9, 10, 11, 7] with its infinite number of branches $W_k,\ k \in \mathbb{Z}$. For any branch, its first derivative is given by

$$\tfrac{d}{dz} W_k\left[z\right] = \frac{W_k\left[z\right]}{z(W_k\left[z\right]+1)}, \qquad k \in \mathbb{Z} \tag{14}$$

Their real values have some order [10, 11]. The rightmost value corresponds to the so called principal branch W_0:

$$\max_{k\in\mathbb{Z}} \Re W_k\left[z\right] = \Re W_0\left[z\right], \quad \forall z \in \mathbb{C}. \tag{15}$$

This real value of this principal branch is bounded from below by -1 and reaches the minimum at $z = -\frac{1}{e}$, i.e.

$$\Re W_0\left[z\right] \geq -1, \quad \forall z \in \mathbb{C}, \tag{16}$$

$$W_0\left[-\tfrac{1}{e}\right] = -1. \tag{17}$$

Inside the circle $|z| < \frac{1}{e}$, the principal branch can be decomposed in power series [9]

$$W_0\left[z\right] = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n, \qquad |z| < \frac{1}{e} \tag{18}$$

The real part of the principal branch $\Re W_0$ is plotted on Figure 1 (in this section, all plots are extracted from Mathematica 10.0).

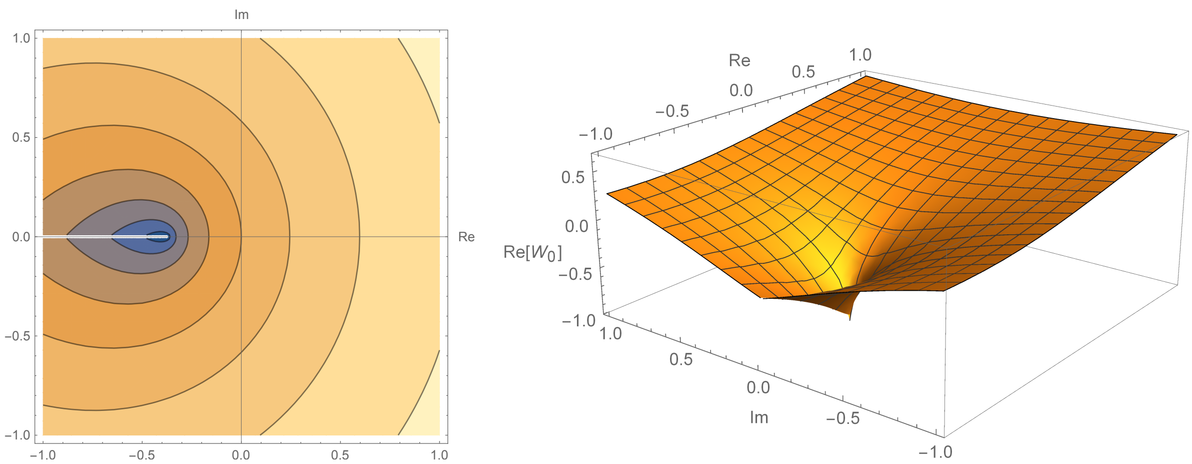

Figure 1: Real part of the principal branch of Lambert W function.

We can also define a Lambert operator W as follows : for any linear operator A, the Lambert W operator function of A is a linear operator $W\left[A\right]$, such that $W\left[A\right] e^{W[A]} = A$. Under minor assumptions on the operator A, one can also prove that this operator has an infinite number of branches W_k, $k \in \mathbb{Z}$, with a principal branch W_0.

2.4 Main result: the optimal control

The next theorem presents the main result of the current paper, namely the characterization of the optimal control for the problem (3).

Theorem 2. *Under the assumptions of paragraph 2.1, the optimal gain operator for the problem (3), providing the fastest convergence to the asymptotic solution, is given by*

$$G = \frac{1}{T} W_0 \left[-\tfrac{1}{e} \, e^{-TA} H^{-1} \right], \tag{19}$$

where W_0 denotes the operator form of the principal branch of the Lambert W function.

Proof. According to Theorem 1, the solution of the controlled problem takes the exponential form (5) with inverse (complex) relaxation time defined by (11). When all $\lambda_{k,p}$ have negative real part, the solution converges in time to the asymptotic limit, corresponding to the periodic solution of the problem. By properly constructing the eigenvalues of G in (11), we can control the problem's inverse relaxation times $\lambda_{k,p}$. Decreasing their real part, we accelerate the convergence. The optimal control, providing the fastest convergence, moves all eigenvalues $\lambda_{k,p}$ on the left as much as possible. According to (15), for any $p \in \mathbb{Z}$ we have $\Re\lambda_{0,p} = \max_{k\in\mathbb{Z}} \Re\lambda_{k,p}$. Therefore, we need to optimize the principal branches $\lambda_{0,p}$:

$$\lambda_{0,p} = -\alpha_p + \underbrace{\frac{1}{T} \Big(W_0 \left[T\sigma_p\gamma_p \, e^{T(\gamma_p+\alpha_p)} \right] - T\gamma_p \Big)}_{\text{control term}}, \qquad p \in \mathbb{Z}. \tag{20}$$

The second term in the right part characterizes the convergence rate of the controlled problem. For the non-controlled solution ($\gamma_p = 0, \ \forall p$), this term is equal to zero. Convergence slows down, if it is real value is positive, and accelerates, if it is negative. When its real value is minimal, $\lambda_{0,p}$ is moved to the left as much as possible, and the associated γ_p is optimal. Therefore, for each $p \in \mathbb{Z}$ we need to compute γ_p, which minimizes the control term:

$$\gamma_p = \arg\min_{\gamma} \Re \Big(W_0 \left[T\gamma \, e^{T\gamma} \sigma_p e^{T\alpha_p} \right] - T\gamma \Big), \qquad p \in \mathbb{Z}. \tag{21}$$

Let us fix p and make the change of variable:

$$\gamma_p = \frac{1}{T} W_0 \left[z \right], \text{ yielding } z = T\gamma_p e^{T\gamma_p}. \tag{22}$$

We therefore have to solve the following minimization problem:

$$\begin{gathered} \text{Find } \ z \in \mathbb{C} \ \text{ minimizing } \ \Re\, J(z), \\ J(z) = W_0 \left[\eta z \right] - W_0 \left[z \right], \end{gathered} \tag{23}$$

where $\eta = \sigma_p e^{T\alpha_p}$ is a given complex number constructed from the eigenvalues of A and H. The minimizers cancel the derivatives of the cost function J, which from formula (14) satisfy :

$$\tfrac{d}{dz} J(z) = \frac{W_0 \left[\eta z \right]}{z(W_0 \left[\eta z \right] + 1)} - \frac{W_0 \left[z \right]}{z(W_0 \left[z \right] + 1)} \tag{24}$$

$$= \frac{J(z)}{z(W_0 \left[\eta z \right] + 1)(W_0 \left[z \right] + 1)} \tag{25}$$

There are two candidates, where the derivative takes infinite values:

$$W_0\,[z] = -1 \qquad \Leftrightarrow \qquad z = -\tfrac{1}{e} \tag{26}$$

or

$$W_0\,[\eta z] = -1 \qquad \Leftrightarrow \qquad z = -\tfrac{1}{\eta e} \tag{27}$$

The point $z = 0$ makes no interest, since it corresponds to the non-controlled case. From (16) we also have

$$\Re J(-\tfrac{1}{e}) = \qquad \Re\, W_0\left[-\tfrac{\eta}{e}\right] + 1 \qquad \geq 0 \tag{28}$$

$$\Re J(-\tfrac{1}{\eta e}) = \ -\left(1 + \Re\, W_0\left[-\tfrac{1}{\eta e}\right]\right) \ \leq 0 \tag{29}$$

Hence we conclude that the solution of the minimization problem (23) is given by the critical point $z = -\frac{1}{\eta e}$, that is

$$\gamma_p = \frac{1}{T} W_0\left[-\tfrac{1}{e}\,\sigma_p^{-1} e^{-T\alpha}\right] \tag{30}$$

In operator terms, this yields

$$G = \frac{1}{T}\, W_0\left[-\tfrac{1}{e}\, H^{-1} e^{-TA}\right]. \tag{31}$$

□

Corollary 2.1. *If the inverse of the shift H is weakly contracting $\|H^{-1}\| \leq 1$, the optimal control takes the explicit form*

$$G = -\frac{1}{T}\sum_{n=1}^{\infty} c_n \cdot \left(H^{-1} e^{-TA}\right)^n \quad \textit{with} \ \ c_n = \frac{n^{n-1}}{n!\, e^n}. \tag{32}$$

Proof. Since A is elliptic, then $\Re\alpha_p > 0$ and $|e^{-\alpha_p}| \leq 1$. If $\|H^{-1}\| \leq 1$, then $|\sigma_p^{-1}| \leq 1$. Therefore

$$|-\tfrac{1}{e}\,\sigma_p^{-1} e^{-\alpha_p}| < \tfrac{1}{e}, \qquad \forall p \in \mathbb{Z}. \tag{33}$$

So γ_p in (30) can be decomposed in the power series (18):

$$\gamma_p = \frac{1}{T} W_0\left[-\tfrac{1}{e}\,\sigma_p^{-1} e^{-T\alpha}\right] \tag{34}$$

$$= \frac{1}{T}\sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!}\left(-\tfrac{1}{e}\,\sigma_p^{-1} e^{-T\alpha}\right)^n \tag{35}$$

$$= -\frac{1}{T}\sum_{n=1}^{\infty} \frac{n^{n-1}}{n!\, e^n}\left(\sigma_p^{-1} e^{-T\alpha}\right)^n \tag{36}$$

yielding the claimed statement. □

Remark 2.1. *In practice, the norm of the shift operator or of its inverse is unity. We can therefore use (32) for calculating G, usually restricting ourselves to the first five terms.*

Remark 2.2. *Note that $e^{-t\mathcal{A}}$ is a fundamental solution of the non-controlled problem, i.e. $u(t,x) = e^{-t\mathcal{A}}u(0,x)$, when $G = 0$. That is e^{-TA} plays role of the T-shift in time for the non-controlled solution. Thus the proposed control can be interpreted as a correction of the present solution by canceling all the future T-shifted periodicity errors :*

$$\mathcal{G}\Delta u(t) \simeq -\frac{1}{T}\sum_{n=1}^{\infty} c_n\, H^{-n}\Delta u(t+nT) \tag{37}$$

where the periodicity error

$$\Delta u(t) = u(t) - Hu(t-T) \tag{38}$$

2.5 Efficiency of the method

We have developed an optimal control for an abstract evolution problem, accelerating convergence to the periodic solution. Now we want to estimate the efficiency of the proposed control, that is how fast does the controlled solution converge in comparison with the non-controlled one. This is based on a direct calculation yielding

$$\begin{aligned} \lambda_{0,p} &= -\alpha_p + \frac{1}{T}\left(W_0\left[T\gamma_p\sigma_p e^{T(\alpha_p+\gamma_p)}\right] - T\gamma_p\right) & (39)\\ &= -\alpha_p - \frac{1}{T}\left(1 + W_0\left[-\tfrac{1}{e}\sigma_p^{-1}e^{-T\alpha_p}\right]\right) & (40)\\ &= -\alpha_p - \frac{1}{T}\, g\left(\log\sigma_p + T\alpha_p\right) & (41)\end{aligned}$$

where we define the spectral control term by

$$g(z) = 1 + W_0\left[-\tfrac{1}{e}\, e^{-z}\right] \tag{42}$$

The real part of $g(z)$, $z \in \mathbb{C}$ on the right half-plane $\Re z \geq 0$ is represented on Figure 2, and its restriction on the real axis is detailed on Figure 3.

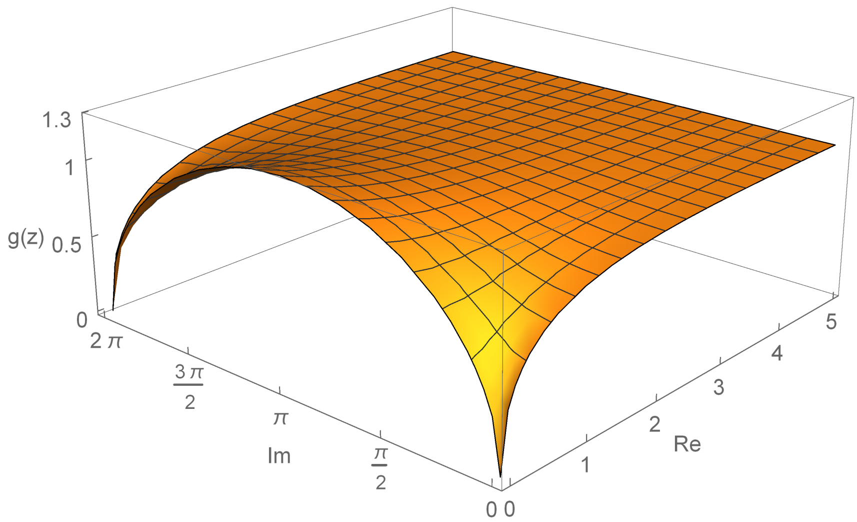

Figure 2: Real part of $g(z)$ on the complex right half-plane $z \in \mathbb{C}$, $\Re z \geq 0$.

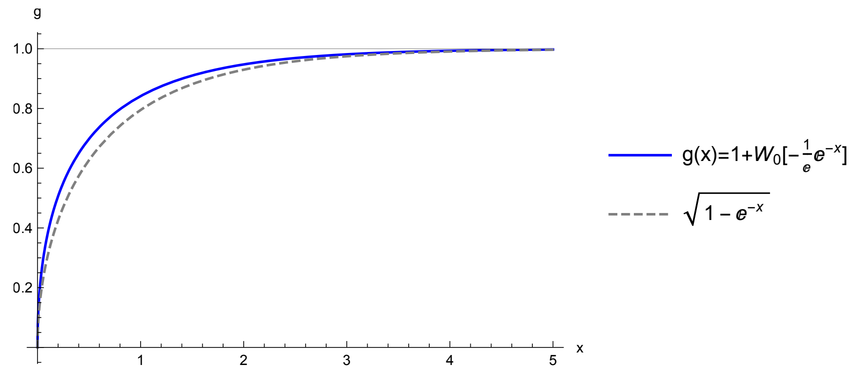

Figure 3: Function $g(x) = 1 + W_0\left[-\frac{1}{e}\,e^{-x}\right]$ on the real half-line $x \geq 0$.

A direct calculation then yields

$$1 + \frac{g(\Re(\log\sigma_p + T\alpha_p))}{T\Re\alpha_p} \leq \left|\frac{\Re\lambda_{0,p}}{\Re\alpha_p}\right| \leq 1 + \frac{g(\Re(\log\sigma_p + T\alpha_p) + i\pi)}{T\Re\alpha_p} \tag{43}$$

For the proposed optimal control, we therefore see that the gain increases when the real part of the spectrum of the operator A goes to zero (see Figure 4). In other words, the method gets more and more efficient when the problem gets more and more viscous, as represented on Figure 4, which computes the approximate minimal gain as the function of the eigenvalues of A, obtained by replacing the control term g by its real approximation $\sqrt{1-e^{-x}}$.

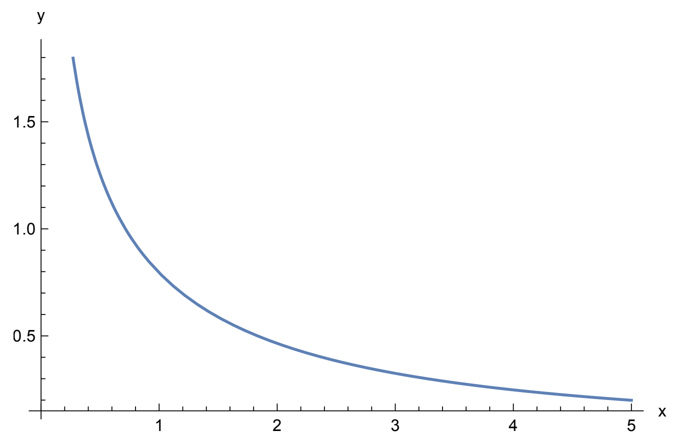

Figure 4: Estimation of the slowest convergence rate: y - gain, x - spectrum of A.

3 APPLICATION: STEADY ROLLING OF A VISCOELASTIC PERIODIC WHEEL

3.1 Model problem

The present section is dedicated to the application of the developed control method to a model problem, in order to justify on practice the theoretical results discussed in the previous section. It considers a steady rolling problem of a 2D viscoelastic tyre [1] presenting a space periodic sculpture. In our approach the periodicity of the sculpture is represented not by a modification of the geometry but by a space periodic modification of the Young's modulus. Thus our model problem considers the steady rolling of a simple viscoelastic ring, with a Young's modulus periodic as a function the angle. We apply a rotating lateral force, representing the contact force applied by the soil. In contrast with the real contact model, we suppose herein that this lateral force is a given function of time and space. Viscoelasticity implies that we deal with an evolution problem in terms of strains. We are interested in the established space-time periodic state, which is the asymptotic limit of the initial value problem with an arbitrary initial state. Applying the optimal control method described in the previous section, we expect to accelerate the convergence to the desired periodic solution.

In more details, let us consider a steady rolling of a wheel (Figure 5, left) with an angular velocity ω. A given force, presenting a ground contact pressure, is applied to the outer border Γ_1 and normal to its surface. In Lagrange (material) configuration, it moves along the exterior boundary with angular velocity $-\omega$. In a Lagrangian frame, its normal component takes the form $f(t,\underline{x}) = f(R_{\omega t}\underline{x})$, with $f(\underline{x})$ quadratic on part of the boundary and zero elsewhere as shown on Figure 5, right. The notation $R_{\omega t}$ represents a rotation of angle ωt around the origin.

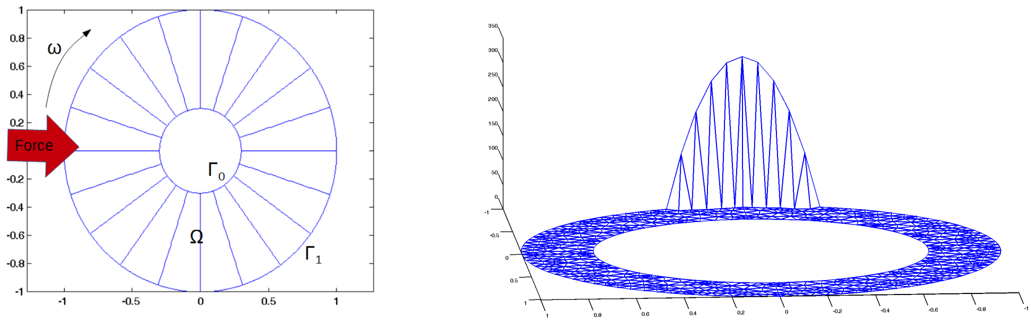

Figure 5: Left: 2D tyre. Right: Normal pressure

On the inner boundary Γ_0, a zero displacement condition is imposed. Thus, in the absence of mass forces, the momentum conservation law writes:

$$\operatorname{div} \underline{\underline{\sigma}} = 0, \qquad \text{in } \Omega \tag{44}$$

$$\underline{\underline{\sigma}}\,\underline{n}\,|_{\Gamma_1} = f\,\underline{n}, \qquad \underline{u}|_{\Gamma_0} = 0, \tag{45}$$

where $\underline{u}$ is the displacement field and $\underline{\underline{\sigma}}$ is a Cauchy stress tensor, $\underline{n}$ denotes a unit normal to the boundary.

The material is considered to be viscoelastic and nearly incompressible. We assume that we are in small strains. The viscoelastic constitutive law is of Kelvin-Voight type :

$$\underline{\underline{\sigma}}\,(t,x) = \eta\,\partial_t \underline{\underline{\varepsilon}}\,(t,\underline{x}) + \underline{\underline{\underline{\underline{K}}}}(\underline{x})\,\underline{\underline{\varepsilon}}\,(t,\underline{x}) \tag{46}$$

where $\underline{\underline{\varepsilon}}$ is the strain tensor

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla\underline{u} + (\nabla\underline{u})^\top\right) = \nabla_s\underline{u}, \tag{47}$$

η is a scalar viscosity coefficient; $\underline{\underline{\underline{\underline{K}}}}$ is the fourth-order elasticity tensor:

$$\underline{\underline{\underline{\underline{K}}}}\,\underline{\underline{\varepsilon}} = \lambda \operatorname{tr}\underline{\underline{\varepsilon}}\,\underline{\underline{\mathbb{1}}} + 2\mu\underline{\underline{\varepsilon}}, \tag{48}$$

where Lamé coefficients λ and μ are expressed via Young's modulus E and Poisson's ratio ν:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \tag{49}$$

Given a nearly incompressible material, the Poisson's ratio is taken close to 0.5. As it was mentioned above, the periodic structure of the wheel is represented by the angular periodicity of the Young's modulus $E(\underline{x})$. Namely, there is an angle $\omega T = 2\pi/M$ (where M is the number of sculpture motives), such that

$$E(\underline{x}) = E(R_{\omega T}\underline{x}). \tag{50}$$

Herein, we have simply taken $E(\underline{x}) = E_0\left(1 + \cos\left[\frac{2\pi}{\omega T}\arg\underline{x}\right]\right)$, where E_0 is the mean of Young's modulus. We are looking for a "rolling" periodic solution, i.e. a solution satisfying the following space-time periodicity condition:

$$\underline{u}(t,\underline{x}) = R_{\omega T}^{-1}\,\underline{u}(t-T, R_{\omega T}\underline{x}). \tag{51}$$

In other words, the state at any point is the same that the one at the corresponding point of the next sculpture motive but one time period ago. Let us therefore define the space shift

$$H = R_{\omega T}^{-1}\,\underline{u}(t-T, R_{\omega T}\underline{x}) \tag{52}$$

Altogether, we are looking for the displacement field $\underline{u}(t,\underline{x})$ which satisfies :

$$\begin{aligned} \operatorname{div}\left[\eta\,\nabla_s\partial_t\underline{u} + \underline{\underline{\underline{\underline{K}}}}\,\nabla_s\underline{u}\right] &= 0, && \text{in } \Omega \\ \left[\eta\,\nabla_s\partial_t\underline{u} + \underline{\underline{\underline{\underline{K}}}}\,\nabla_s\underline{u}\right]\cdot\underline{n} &= f\,\underline{n}, && \text{on } \Gamma_1 \\ \underline{u} &= 0, && \text{on } \Gamma_0 \\ \underline{u}(t) &= H\underline{u}(t-T). \end{aligned} \tag{53}$$

3.2 Controlled problem

Let us introduce the space $V_0 = \{\underline{v} \in H^1(\Omega)^2 \mid \underline{v} = 0 \text{ on } \Gamma_0\}$ and let us define the operators $\mathcal{C}$ and $\mathcal{K}$ as follows: $\forall \underline{v} \in V_0$

$$\int_\Omega \mathcal{C}\underline{u}\cdot\underline{v} = \int_\Omega \eta\,\nabla_s\underline{u} : \nabla_s\underline{v} \tag{54}$$

$$\int_\Omega \mathcal{K}\underline{u}\cdot\underline{v} = \int_\Omega \underline{\underline{\underline{\underline{K}}}}\,\nabla_s\underline{u} : \nabla_s\underline{v} \tag{55}$$

Then the system (53) writes:

$$\int_\Omega (\mathcal{C}\partial_t \underline{u} + \mathcal{K}\underline{u}) \cdot \underline{v} = \int_{\Gamma_1} f\, \underline{n} \cdot \underline{v}, \qquad \forall \underline{v} \in V_0, \tag{56}$$
$$\underline{u}(t) = H\underline{u}(t-T).$$

The associated controlled problem is

$$\int_\Omega \Big(\mathcal{C}\partial_t \underline{u} + \mathcal{K}\underline{u} + \mathcal{G}\left(\underline{u}(t) - H\underline{u}(t-T)\right)\Big) \cdot \underline{v} = \int_{\Gamma_1} f\, \underline{n} \cdot \underline{v}, \qquad \forall \underline{v} \in V_0 \tag{57}$$
$$\underline{u}(t) = \underline{u}_0(t), \qquad t \leq 0$$

with arbitrary initial data $\underline{u}_0$. According to Theorem 2, the optimal gain operator takes the form

$$\mathcal{G} = -\frac{1}{T}\mathcal{C} \sum_{n=1}^{\infty} c_n\, e^{-nT\left(\mathcal{C}^{-1}\mathcal{K}\right)} H^{-n}, \qquad c_n = \frac{n^{n-1}}{n!\, e^n}. \tag{58}$$

If we proceed with the finite element discretization, it can be quite expensive to compute an exponential of the matrix associated to the operator $\mathcal{C}^{-1}\mathcal{K}$. So we need to simplify this expression.

3.3 Simplified controlled problem

Let $\underline{u}_*$ be the periodic solution of the problem (53), that is the true displacement, and $\underline{\underline{\sigma}}_* = \eta\, \nabla_s \partial_t \underline{u}_* + \underline{\underline{\underline{\underline{K}}}}\ \nabla_s \underline{u}_*$ be the true Cauchy stress. Associated to this true stress tensor $\underline{\underline{\sigma}}_*$, the true strain tensor $\underline{\underline{\varepsilon}}_* = \nabla_s \underline{u}_*$ is the solution of the first order differential equation

$$\eta\, \partial_t \underline{\underline{\varepsilon}} + \underline{\underline{\underline{\underline{K}}}}\ \underline{\underline{\varepsilon}} = \underline{\underline{\sigma}}_* \tag{59}$$
$$\underline{\underline{\varepsilon}}\,(t) = H \underline{\underline{\varepsilon}}\,(t-T) \tag{60}$$

If we suppose that we know the true stress tensor $\underline{\underline{\sigma}}_*$, we can apply the optimal feedback control of the previous section to this problem, leading to the associated controlled problem:

$$\eta\, \partial_t \underline{\underline{\varepsilon}} + \underline{\underline{\underline{\underline{K}}}}\ \underline{\underline{\varepsilon}} + \eta\, \underline{\underline{\underline{\underline{G}}}}\left(\underline{\underline{\varepsilon}} - H\underline{\underline{\varepsilon}}(t-T)\right) = \underline{\underline{\sigma}}_* \tag{61}$$
$$\underline{\underline{\varepsilon}}(t) = \nabla_s \underline{u}_0(t), \quad t \leq 0$$

with arbitrary initial data $\underline{u}_0$. According to Theorem 2, the optimal gain operator has the form

$$\underline{\underline{\underline{\underline{G}}}} = -\frac{1}{T} \sum_{n=1}^{\infty} c_n\, e^{-nT\left(\eta^{-1}\underline{\underline{\underline{\underline{K}}}}\right)} H^{-n}, \qquad c_n = \frac{n^{n-1}}{n!\, e^n}, \tag{62}$$

Note that the strain estimator $\underline{\underline{\varepsilon}}$ is not necessarily a displacement symmetric gradient any more and that we do not know the value of the true periodic stress. However, its initial value and its asymptotic limit is a gradient, and we know the divergence of the true stress field. Hence, we decide to decompose the strain estimator into a symmetric gradient and a zero divergence field

$$\underline{\underline{\varepsilon}} = \nabla_s \underline{u} + \tilde{\underline{\underline{\varepsilon}}}\,, \tag{63}$$

and to take the divergence of the strain controlled problem, omitting all terms in $\underline{\underline{\tilde{\varepsilon}}}$, which leads to the new controlled system

$$
\begin{aligned}
\text{div}\,\left[\eta\,\partial_t\nabla_s\underline{u} + \underline{\underline{K}}\;\nabla_s\underline{u} + \eta\,\underline{\underline{G}}\,\Big(\nabla_s\underline{u} - H\nabla_s\underline{u}(t-T)\Big)\right] &= 0 \\
\left[\eta\,\partial_t\nabla_s\underline{u} + \underline{\underline{K}}\;\nabla_s\underline{u} + \eta\,\underline{\underline{G}}\,\Big(\nabla_s\underline{u} - H\nabla_s\underline{u}(t-T)\Big)\right]\cdot\underline{n} &= f\,\underline{n}, \qquad \text{on } \Gamma_1 \\
\underline{u} &= 0, \qquad \text{on } \Gamma_0 \\
\underline{u}(t) &= \underline{u}_0(t), \quad t \le 0
\end{aligned}
\tag{64}
$$

It is seen that $\underline{u}_*$ is the asymptotic limit for this problem, i.e. $\underline{u} \to \underline{u}_*$, and thus (64) is a controlled system associated to the initial model problem (53) with a simplified gain operator, applied to the strain field and computed by the formula

$$
\underline{\underline{G}} = -\frac{1}{T}\sum_n c_n\, e^{-nT\left(\eta^{-1}\underline{\underline{K}}\right)} H^{-n}, \qquad c_n = \frac{n^{n-1}}{n!\,e^n}. \tag{65}
$$

In this way, we have applied the developed optimal control method to the viscoelastic constitutive law. We may have sacrificed a part of the control "optimality" to the simplicity of its form. In practice, for a plane strain problem, introducing the reduced Lamé coefficients $\tilde{\lambda} = -nT\lambda/\eta$ and $\tilde{\mu} = -nT\mu/\eta$, the exponential coefficients in the gain operator (65) are defined by

$$
e^{-nT\eta^{-1}\underline{\underline{K}}}\begin{bmatrix}\varepsilon_{11} & \varepsilon_{12}\\ \varepsilon_{12} & \varepsilon_{22}\end{bmatrix} = e^{\tilde{\lambda}+2\tilde{\mu}}\sinh\tilde{\lambda}(\varepsilon_{11}+\varepsilon_{22})\underline{\underline{\mathbb{1}}} + e^{2\tilde{\mu}}\begin{bmatrix}\varepsilon_{11} & \varepsilon_{12}\\ \varepsilon_{12} & \varepsilon_{22}\end{bmatrix} \tag{66}
$$

3.4 Numerical results

The controlled and non-controlled evolution problems have been solved numerically, in order to compare their rates of convergence and to justify in practice that the controlled one is really faster. The numerical simulation has been carried out in MatLab R2014a by the finite elements method with $P1$ elements on a conforming triangular mesh (Figure 6), ωT-periodic with respect to rotation. It represents a tyre with 15 motives, i.e. $\omega T = 2\pi/15$. The periodic Young's modulus is in the form: $E(\underline{x}) = E_0\,(1+\cos[15\arg\underline{x}])$, where E_0 is a homogeneous stiffness. An undeformed state is taken as initial data, that is $\underline{u}(0,\underline{x}) = 0$.

We carry out several pairs of simulations (controlled and non-controlled cases) with different values of viscosity coefficient, in order to verify if the efficiency estimation from paragraph 2.5 holds. The gain of the control method is represented by ratio n_0/n_c of the iteration numbers for non-controlled and controlled problem respectively. The gain values corresponding to distinct viscosity coefficients are presented in Table 1. These results are depicted on Figure 7. We can see that the efficiency indeed increases with the viscosity coefficient.

On Figure 8 there are plotted some simulation results: l_2-norm evolution of numerical solution (left column) and periodicity error (right column) with distinct viscosity coefficients ($\eta = E_0, 3E_0, 8E_0$ respectively). The abscissa ticks correspond to the number of ωT-periods. Dashed vertical lines mark the convergence time, that is the moment when the periodicity error becomes less than the specified accuracy. The large visible oscillations of the solution are associated with 2π-periodicity of the tyre, while the oscillations associated with ωT-periodicity are of a much smaller scale and aren't quite visible on the plot.

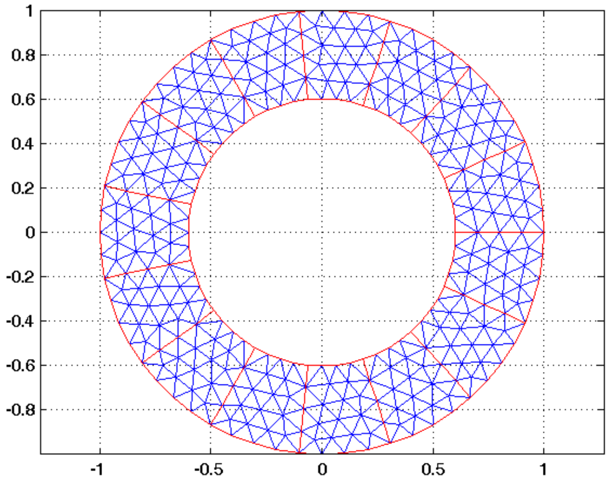

Figure 6: Periodic mesh, representing a tyre with 15 motives

Viscosity η	relax.time^{-1} E_0T/η	Gain n_0/n_c
$0.1E_0$	8.378	1.19
$0.5E_0$	1.676	1.38
E_0	0.838	1.62
$2E_0$	0.419	2.05
$3E_0$	0.279	2.41
$4E_0$	0.209	2.72
$5E_0$	0.168	2.97
$8E_0$	0.105	3.33

Table 1: Convergence time ratio with respect to viscosity coefficient.

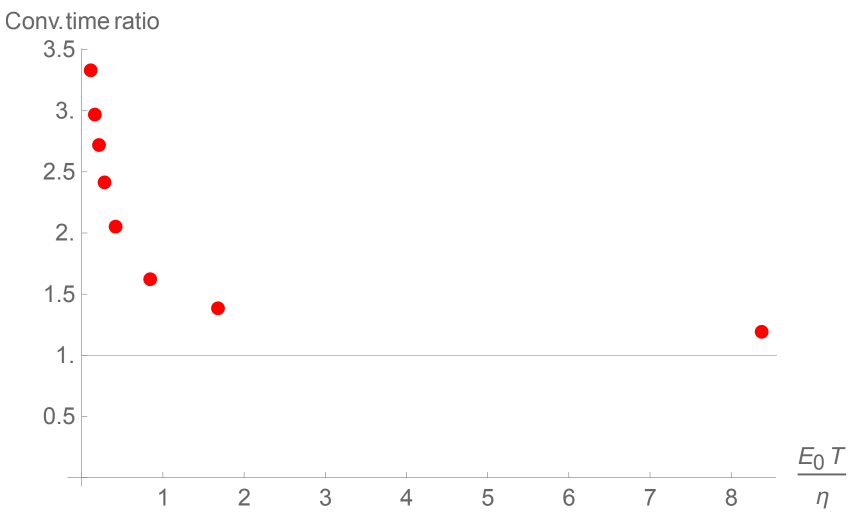

Figure 7: Convergence time ratio (gain) as the function of the inverse relaxation time.

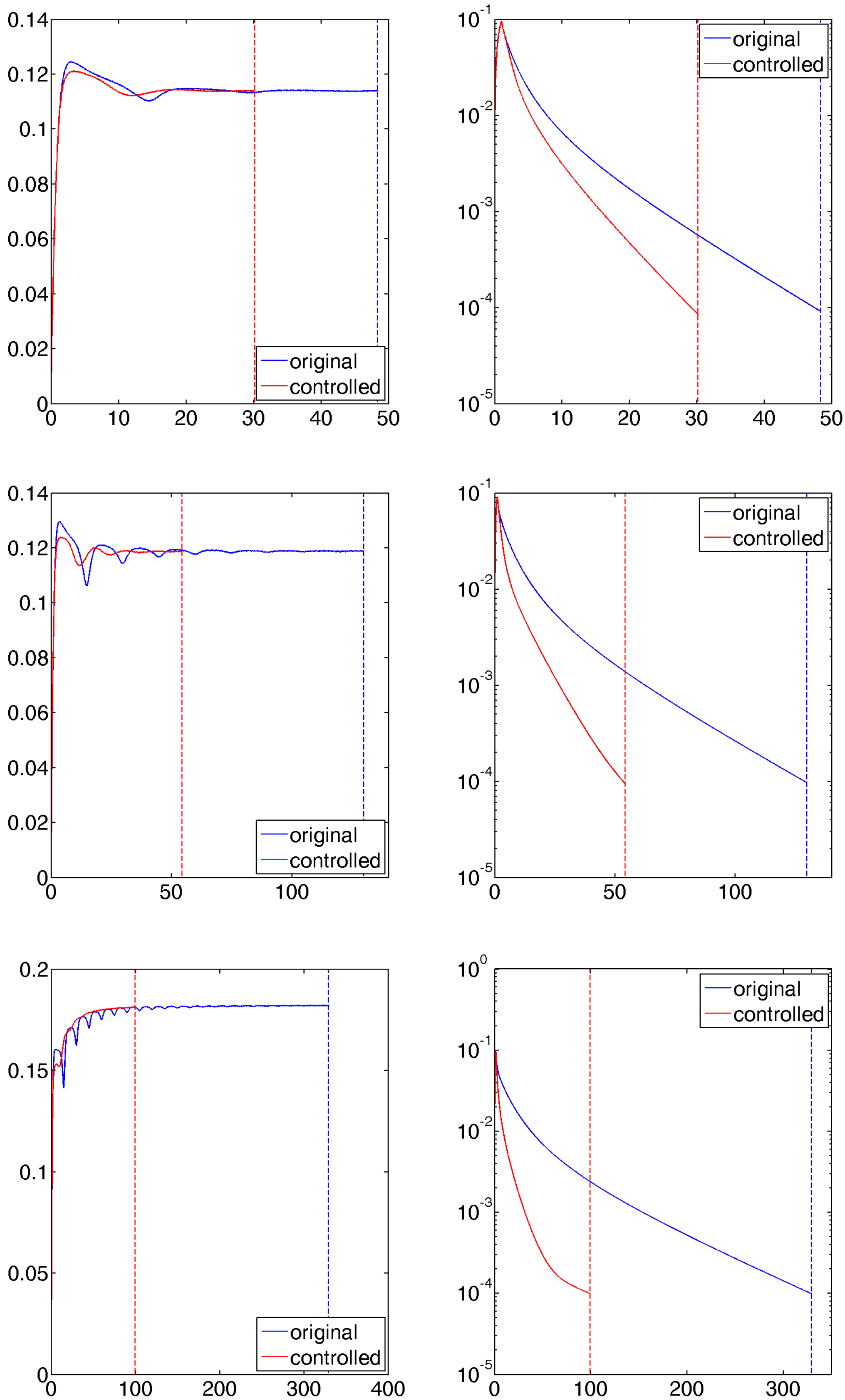

Figure 8: Results: l_2-norm of the numerical solution (left) and of the periodicity error (right) for $\eta = E_0, 3E_0, 8E_0$ respectively.

4 CONCLUSION

In this paper, we have developed a new control method for calculating viscoelastic evolution problems while accelerating the convergence of its solution to the asymptotic limit, which is time periodic. The method is similar to the feedback control methods, using the eigenvalue assignment technique [3, 4]. From another perspective, it can be interpreted as an observer-controller filtering method, where the space-time periodicity condition plays the role of the observation [2]. It turns out that the proposed control term actually acts like the correction of the present solution by the future periodicity errors (37).

As seen from the construction of the optimal control, its mathematical properties use the Lambert W function. In this way, we have estimated the dependence of the convergence acceleration on the relaxation rate. In fact, the acceleration increases with the memory of the problem. So the developed method might not be efficient for fast converging problems (which is not really of interest) but becomes more and more efficient for the slowly converging problems.

The developed control method has been tested on a full 2D model problem. It has been applied to the steady rolling of a viscoelastic tyre with a periodic sculpture. An approach, when the optimal control is applied to the viscoelastic evolution law, has been formulated. In this case, the control term is quite simple and cheap to compute. The numerical solution of the problem has justified the theoretical results: for a very viscous material, we have indeed obtained a significant acceleration of convergence.

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