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NONLINEAR PATCH METHOD AND APPLICATION

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Abstract. *We have extended the iterative multiscale patch method to a class of nonlinear problems using an adaptative linearization strategy. Theoretical convergence results are provided in this paper, and we showcase this nonlinear patch method in a simulation of a space plasma around a negatively charged array of solar generator interconnect.*

1 INTRODUCTION

Since 2005, the French aerospace laboratory (ONERA) together with the French and European space agencies (CNES and ESA) have been developing an open-source plasma-satellite simulation code called *Spacecraft Plasma Interaction System* (SPIS) [16]. Modern spacecraft technological challenges, such as the use of plasma thrusters and bigger solar generators, require simulation codes to resolve small-scale structures in relatively big computational domains. In order to solve these multiscale problems of interest to the industry, modern numerical methods have to be integrated in plasma simulation codes.

Several numerical methods have been proposed for the simulation of multiscale plasma effects. One important category of methods uses analytical models for the small scales, as is done in the *Variational Multiscale method* [13, 14]. Notably, it has been used to simulate the dynamics of an electric arc in a hot plasma coupled with a flowing gaz [18], and to model the transport of charges in fluids [20, 8]. While these methods are showing promising results, they require the integration of analytical models in the simulation codes. One can use numerical models to solve these scales instead, such as what is done in the case of highly oscillatory problems with the *Finite Element Heterogeneous Multiscale Method* [2, 5, 1]. These methods are effective for solving multiscale problems with non-local small-scale dynamics, such as fast varying medium properties.

In our case however, the small-case effects are due to local singularities, for instance on a boundary condition. Facing this type of problems, *numerical zoom* and *finite element patches* have been used in various fields [10, 3, 15, 4]. We propose a linearization scheme that allows to use the patch method to solve a class of nonlinear problems. These methods are sufficiently flexible to be easily integrated in current plasma simulation codes such as SPIS, and allow the resolution of local singularities to simulate their impact on the global problem.

We showcase this numerical method in a simulation of a plasma around a negatively charged array of solar generator interconnects. These spacecraft components are typically highly interacting with the surrounding plasma, and are too small to be resolved with classical numerical schemes.

2 THE NONLINEAR PATCH METHOD

2.1 Overview

The patch method [19, 7] is a method to solve elliptic multiscale problems using multiple grids. It is a flexible domain decomposition method similar to the Chimera method used in computational fluid dynamics [17].

The algorithm used in this method is quite straightforward. The problem is successively solved on a coarse mesh and on a finer, local mesh called *patch* to compute corrections added the complete solution. An extensive analysis of this method has been carried out in [19], including the derivation of convergence properties. This algorithm can be seen as a multiplicative Schwarz domain decomposition method, without conformity between the meshes. Convergence has been proven, and links between the convergence rate and the meshes geometry have been established [19] in the linear case. A fast converging variant of the patch method using harmonic functions has been presented in [9].

We here introduce an extension of the patch method to solve a class of nonlinear problems. We propose a linearization strategy that provides good convergence while limiting the number of linearizations. The proposed method is similar to a patch method variant of Newton-MG [11], but does not require the full convergence of the multigrid solver loop between linearizations.

2.2 Definitions

Consider a nonlinear problem on the domain $\Omega \subset \mathrm{R}^n$, which weak formulation can be written

$$\langle a(\mathbf{u}) \,|\, \varphi \rangle = l(\varphi),\ \forall \varphi \in H_0^1(\Omega) \tag{1}$$

Here, $a \in \mathcal{C}^1(H_0^1(\Omega))$ is a nonlinear operator, and l is a continuous form on $H_0^1(\Omega)$. For all $\mathbf{u} \in H_0^1(\Omega)$, the Jacobian $J_\mathbf{u}$ of a near $\mathbf{u}$ is supposed continuous and coercive.

Suppose the solution $\mathbf{u}$ of the problem exists and varies rapidly in a small sub-domain $\Lambda \subset \Omega$, but varies slowly in $\Omega \backslash \Lambda$. We define a *coarse mesh* of Ω, with associated function space $V_H \subset H_0^1(\Omega)$, and a *fine mesh* of Λ called *patch*, associated to the function space $V_h \subset H_0^1(\Lambda)$. We set $V = V_H + V_h$ and approximate the solution $\mathbf{u}$ of the continuous problem with the solution $\mathbf{u}_{Hh}$ of

$$\langle a(\mathbf{u}_{Hh}) \,|\, \varphi \rangle = l(\varphi),\ \forall \varphi \in V \tag{2}$$

One should note that, even when a is linear, this problem is not trivial because it is generally not possible to construct a finite element basis of the function space V. The patch method builds the solution $\mathbf{u}_{Hh}$ by solving iteratively on V_H and V_h, adding a source term to the problem, corresponding to the coupling between the meshes.

The other difficulty arrising in this problem is the nonlinearity of the operator. Several linearization schemes have been studied and provide convergence of the method toward the solution of the nonlinear problem:

- Unconditionnal linearization: provided the initial error is not too big, it can be shown that linearizing the problem between each iteration will allow the solution to converge. Using this scheme, a high number of linearization are required, but numerical results show good stability.
- Linearization at convergence: similarly to the Newton-MG method, one can choose to linearize the problem only when the previous linearized problem has been solved to a given accuracy. This scheme minimizes the number of linearizations, but will require a high number of overall iterations. Numerically, we have found that the stability of this scheme is relatively poor.
- Conditionnal linearization: we propose a convergence criterion based on the relative linearization and solving errors. As we will show in this paper, this scheme requires a limited number of linearizations while ensuring good overall convergence and stability.

Let us introduce the following notation:

$$\forall (\varphi, \mathbf{u}_\ell, \delta\mathbf{u}) \in V^3, \langle R_{\mathbf{u}_\ell}(\delta\mathbf{u}) \,|\, \varphi \rangle = l(\varphi) - \langle a(\mathbf{u}_\ell) + J_{\mathbf{u}_\ell}\delta\mathbf{u} \,|\, \varphi \rangle \tag{3}$$

In equation 3, $R_{\mathbf{u}_\ell}(\delta\mathbf{u})$ can be seen as a residual in the linearized problem at $\mathbf{u}_\ell$, with approximate solution $\delta\mathbf{u}$. It is readily seen that equation 2 is equivalent to $R_{\mathbf{u}_{Hh}}(\mathbf{0}) = \mathbf{0}$.

2.3 Proposed algorithm

The proposed iteratif algorithm can be written as the following:

1. Find $\mathbf{u}_h \in V_h$ s.t. $\forall \varphi_h \in V_h$,

$$\langle J_{\mathbf{u}_\ell}\mathbf{u}_h \,|\, \varphi_h \rangle = \left\langle R_{\mathbf{u}_\ell}(\mathbf{u}_n \quad - \mathbf{u}_\ell) \,\middle|\, \varphi_h \right\rangle$$

2. Let $\mathbf{u}_{n+\frac{1}{2}} = \mathbf{u}_n \quad + \omega \mathbf{u}_h$

3. Find $\mathbf{u}_H \in V_H$ s.t. $\forall \varphi_H \in V_H$,

$$\langle J_{\mathbf{u}_\ell} \mathbf{u}_H \,|\, \varphi_H \rangle = \left\langle R_{\mathbf{u}_\ell}(\mathbf{u}_{n+\frac{1}{2}} - \mathbf{u}_\ell) \,\middle|\, \varphi_H \right\rangle$$

4. Let $\mathbf{u}_{n+1} = \mathbf{u}_{n+\frac{1}{2}} + \omega \mathbf{u}_H$

5. If $\|R_{\mathbf{u}_\ell}(\mathbf{u}_{n+1} - \mathbf{u}_\ell)\| < \epsilon \|R_{\mathbf{u}_\ell}(\mathbf{0})\|$, let $\mathbf{u}_\ell = \mathbf{u}_{n+1}$

Note that the steps 1 and 3 are resolutions of the linearized problem, respectively on the V_h and V_H subspaces. The relaxation parameter ω has been studied in [19], and an optimal value is computed from geometrical properties of the meshes used.

It is readily apparent that the biggest numerical challenge involved in this method is the computation of the mixed terms $a(u_H, \varphi_h)$ and $a(u_h, \varphi_H)$. In a FEM code, these terms can be approximated with a quadrature rule on the intersection of a coarse element and a patch element. The computation of such intersection is trivial when using 2D structured meshes, but become much more difficult when using 3D unstructured meshes. See [6] for an efficient algorithm solving this problem.

2.4 Convergence properties

Let us prove the convergence of the proposed method. Let $\mathbf{u}_\ell$ be an initial linearization and $\mathbf{u}'_\ell$ be the next computed linearization point. We note $\mathbf{u}^*_\ell$ the solution of the linearized problem around $\mathbf{u}_\ell$. It has been proven in [19] that the following holds:

Lemma 1. *Let $\mathbf{u}^*_\ell$ be the solution of the linearized problem. There is a constant $C \in]0, 1[$ independent of the initial approximation $\mathbf{u}_\ell$, such that*

$$\|\mathbf{u}'_\ell - \mathbf{u}^*_\ell\| \leq C \|\mathbf{u}_\ell - \mathbf{u}^*_\ell\|$$

The Taylor expansion of a near $\mathbf{u}$ can be written:

$$a(\mathbf{u} + \delta\mathbf{u}) = a(\mathbf{u}) + J_{\mathbf{u}_\ell} \delta\mathbf{u} + h_{\mathbf{u}_\ell}(\delta\mathbf{u}) \tag{4}$$

where $J_{\mathbf{u}}$ is the Jacobian of a in $\mathbf{u}$ and $h_{\mathbf{u}}$ is a quadratic residual verifying:

$$\lim_{\delta\mathbf{u} \to 0} \frac{\|h_{\mathbf{u}}(\delta\mathbf{u})\|}{\|\delta\mathbf{u}\|} = 0 \tag{5}$$

Moreover, since a is of class $\mathcal{C}^1$, the Jacobian $J_{(\cdot)}$ is continue, so

$$\lim_{\delta\mathbf{u} \to 0} \|J_{\mathbf{u}} - J_{\mathbf{u}+\delta\mathbf{u}}\| = 0 \tag{6}$$

Using the Taylor expansion around $\mathbf{u}$ and $\mathbf{v}$ gives:

$$\begin{aligned} a(\mathbf{u}) - a(\mathbf{v}) &= J_{\mathbf{v}}(\mathbf{u} - \mathbf{v}) + h_{\mathbf{v}}(\mathbf{u} - \mathbf{v}) \\ &= J_{\mathbf{u}}(\mathbf{u} - \mathbf{v}) - h_{\mathbf{u}}(\mathbf{v} - \mathbf{u}) \end{aligned}$$

Hence we have the following equation:

$$\forall (\mathbf{u}, \mathbf{v}) \in V^2, h_{\mathbf{u}}(\mathbf{v} - \mathbf{u}) = (J_{\mathbf{u}} - J_{\mathbf{v}})(\mathbf{u} - \mathbf{v}) - h_{\mathbf{v}}(\mathbf{u} - \mathbf{v}) \tag{7}$$

Lemma 2. *Let* $(\mathbf{u}_n)_{n\in\mathbb{N}} \in V^{\mathbb{N}}$ *a convergent sequence and note* $\mathbf{u}_\infty$ *its limit. Then,*

$$\lim_{n\to+\infty} \frac{\|h_{\mathbf{u}_n}(\mathbf{u}_\infty - \mathbf{u}_n)\|}{\|\mathbf{u}_n - \mathbf{u}_\infty\|} = 0$$

Proof. From equation 7, we have

$$\forall(\mathbf{u},\mathbf{v}) \in V^2,\ \|h_{\mathbf{u}}(\mathbf{v}-\mathbf{u})\| \leq \|J_{\mathbf{v}} - J_{\mathbf{u}}\| \cdot \|\mathbf{u}-\mathbf{v}\| + \|h_{\mathbf{v}}(\mathbf{u}-\mathbf{v})\|$$

Hence, $\forall n \in \mathbb{N}$,

$$\frac{\|h_{\mathbf{u}_n}(\mathbf{u}_\infty - \mathbf{u}_n)\|}{\|\mathbf{u}_n - \mathbf{u}_\infty\|} \leq \|J_{\mathbf{u}_\infty} - J_{\mathbf{u}_n}\| + \frac{\|h_{\mathbf{u}_\infty}(\mathbf{u}_n - \mathbf{u}_\infty)\|}{\|\mathbf{u}_n - \mathbf{u}_\infty\|}$$

Using equations 5 and 6, this upper bound converges to zero when $\mathbf{u}_n \to \mathbf{u}_\infty$. □

Lemma 3. *Let* $\alpha \in]C, 1[$*, where* C *is the bound in lemma 1. Then*

$$\|\mathbf{u}_\ell^* - \mathbf{u}_{Hh}\| < \frac{\alpha - C}{1+\alpha}\|\mathbf{u}_\ell - \mathbf{u}_\ell^*\| \Rightarrow \frac{\|\mathbf{u}_\ell' - \mathbf{u}_{Hh}\|}{\|\mathbf{u}_\ell - \mathbf{u}_{Hh}\|} < \alpha$$

Proof. Suppose $\|\mathbf{u}_\ell^* - \mathbf{u}_{Hh}\| < \frac{\alpha-C}{1+\alpha}\|\mathbf{u}_\ell - \mathbf{u}_\ell^*\|$, then

$$\begin{aligned}
\|\mathbf{u}_\ell' - \mathbf{u}_{Hh}\| &= \|\mathbf{u}_\ell' - \mathbf{u}_\ell^* + \mathbf{u}_\ell^* - \mathbf{u}_{Hh}\| \\
&\leq \|\mathbf{u}_\ell' - \mathbf{u}_\ell^*\| + \|\mathbf{u}_\ell^* - \mathbf{u}_{Hh}\| \\
&< C\|\mathbf{u}_\ell - \mathbf{u}_\ell^*\| + \frac{\alpha - C}{1+\alpha}\|\mathbf{u}_\ell - \mathbf{u}_\ell^*\| \\
&= \alpha\left(1 - \frac{\alpha - C}{1+\alpha}\right)\|\mathbf{u}_\ell - \mathbf{u}_\ell^*\| \\
&< \alpha\left(\|\mathbf{u}_\ell - \mathbf{u}_\ell^*\| - \|\mathbf{u}_\ell^* - \mathbf{u}_{Hh}\|\right) \\
&\leq \alpha\|\mathbf{u}_\ell - \mathbf{u}_{Hh}\|
\end{aligned}$$

□

Lemma 4.

$$\forall \mathbf{u}_\ell \in V,\ J_{\mathbf{u}_\ell}(\mathbf{u}_\ell^* - \mathbf{u}_{Hh}) = h_{\mathbf{u}_\ell}(\mathbf{u}_{Hh} - \mathbf{u}_\ell)$$

Proof. $\forall \mathbf{v} \in V$,

$$\begin{aligned}
\langle J_{\mathbf{u}_\ell}(\mathbf{u}_\ell^* - \mathbf{u}_{Hh}) \mid \mathbf{v}\rangle &= \langle J_{\mathbf{u}_\ell}(\mathbf{u}_\ell - \mathbf{u}_{Hh}) \mid \mathbf{v}\rangle + \langle J_{\mathbf{u}_\ell}(\mathbf{u}_\ell^* - \mathbf{u}_\ell) \mid \mathbf{v}\rangle \\
&= \langle J_{\mathbf{u}_\ell}(\mathbf{u}_\ell - \mathbf{u}_{Hh}) \mid \mathbf{v}\rangle + l(v) - \langle a(\mathbf{u}_\ell) \mid \mathbf{v}\rangle \quad \mathbf{u}_\ell^* \text{ solution of linearized problem} \\
&= \langle J_{\mathbf{u}_\ell}(\mathbf{u}_\ell - \mathbf{u}_{Hh}) + a(\mathbf{u}_{Hh}) - a(\mathbf{u}_\ell) \mid \mathbf{v}\rangle \quad \mathbf{u}_{Hh} \text{ solution of initial problem} \\
&= \langle h_{\mathbf{u}_\ell}(\mathbf{u}_{Hh} - \mathbf{u}_\ell) \mid \mathbf{v}\rangle
\end{aligned}$$

□

Theorem 5. *Let* $\alpha \in]C, 1[$*, where* C *is the bound in lemma 1. Then*

$$\frac{\|h_{\mathbf{u}_\ell}(\mathbf{u}_{Hh} - \mathbf{u}_\ell)\|}{\mathcal{C}_{J_{\mathbf{u}_\ell}}\|\mathbf{u}_{Hh} - \mathbf{u}_\ell\|} < \frac{\alpha - C}{1 + 2\alpha - C} \Rightarrow \|\mathbf{u}_\ell' - \mathbf{u}_{Hh}\| < \alpha\|\mathbf{u}_\ell - \mathbf{u}_{Hh}\|$$

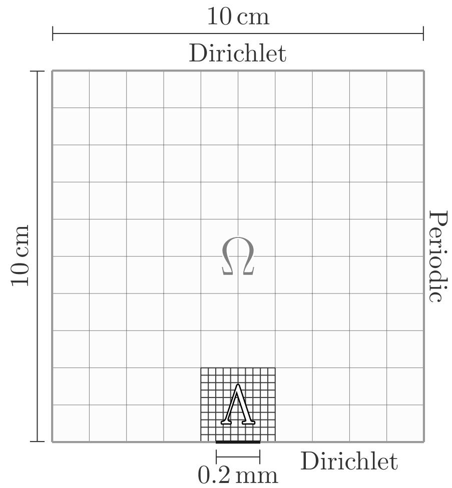

Figure 1: Computational domain and meshes

Proof. If $\mathbf{u}_\ell^* = \mathbf{u}_{Hh}$, lemma 3 gives the result. Suppose that $\mathbf{u}_\ell^* \neq \mathbf{u}_{Hh}$, then

$$\begin{aligned}
\frac{\|\mathbf{u}_\ell - \mathbf{u}_\ell^*\|}{\|\mathbf{u}_{Hh} - \mathbf{u}_\ell^*\|} &\geq \frac{\|\mathbf{u}_{Hh} - \mathbf{u}_\ell\| - \|\mathbf{u}_{Hh} - \mathbf{u}_\ell^*\|}{\|\mathbf{u}_{Hh} - \mathbf{u}_\ell^*\|} \\
&= \frac{\|\mathbf{u}_{Hh} - \mathbf{u}_\ell\|}{\|\mathbf{u}_{Hh} - \mathbf{u}_\ell^*\|} - 1 \\
&\geq \frac{\mathcal{C}_{J_{\mathbf{u}_\ell}} \|\mathbf{u}_{Hh} - \mathbf{u}_\ell\|}{\|J_{\mathbf{u}_\ell}(\mathbf{u}_\ell^* - \mathbf{u}_{Hh})\|} - 1 \\
&= \frac{\mathcal{C}_{J_{\mathbf{u}_\ell}} \|\mathbf{u}_{Hh} - \mathbf{u}_\ell\|}{\|h_{\mathbf{u}_\ell}(\mathbf{u}_{Hh} - \mathbf{u}_\ell)\|} - 1 \quad \text{from lemma 4}
\end{aligned}$$

Thus,

$$\frac{\|h_{\mathbf{u}_\ell}(\mathbf{u}_{Hh} - \mathbf{u}_\ell)\|}{\mathcal{C}_{J_{\mathbf{u}_\ell}} \|\mathbf{u}_{Hh} - \mathbf{u}_\ell\|} < \frac{\alpha - C}{1 + 2\alpha - C} \;\Rightarrow\; \|\mathbf{u}_\ell^* - \mathbf{u}_{Hh}\| < \frac{\alpha - C}{1 + \alpha} \|\mathbf{u}_\ell - \mathbf{u}_\ell^*\|$$

and lemma 3 gives the result. □

Using lemma 2, and since $\mathbf{u} \mapsto J_\mathbf{u}$ is continue, there exists a neighborhood B of $\mathbf{u}_{Hh}$ verifying the left side expression in theorem 5, ensuring that, for every $\mathbf{u}_\ell \in B$, the proposed algorithm converges towards the solution $\mathbf{u}_{Hh}$.

3 NUMERICAL EXAMPLE

3.1 A nonlinear multiscale plasma problem

The simulation of multiscale plasmas can be done with a traditional FEM Poisson solver, using a nonuniform mesh with caracteristic sizes of elements spanning several orders of magnitude. This is the method currently used in the Spacecraft Plasma Interaction Software (SPIS) to solve local sub-centimeter scales in complete satellite simulations [16]. When reaching to sub-millimeter scales, the discrepancies on the size of the elements lead to an ill-conditioned problem, which can be difficult to solve accurately. It is of interest to investigate new numerical methods, such as the nonlinear patch method, which can avoid this issue by using multiple meshes to solve each scale of the simulation. We will present here an application of the non-linear patch method to solve the Poisson-Boltzmann equation describing the electrostatic potential in a plasma at thermodynamic equilibrium[12]. This nonlinear equation is often used in spacecraft-plasma simulations with low or negative potentials, modelling both the electrostatic field and the electronic density distribution.

Here we present the simulation of a plasma around a negatively-biased array of solar cell interconnects, which are small conductors exposed to the spacecraft environment. The Poisson-Boltzmann equation can be written as:

$$-\Delta\varphi - \frac{q_e c_e^0}{\epsilon_0} \exp\left(-\frac{q_e \varphi}{k_B T}\right) = \frac{\rho_i}{\epsilon_0} \tag{8}$$

where ϵ_0 is the medium permitivity, φ is the electrostatic potential, q_e is the electric charge of an electron, c_e^0 the electronic density at 0V, ρ_i the ionic charge density, and $k_B T$ the electron temperature.

The computational domain and the used structured meshes are presented in figure 1. The spacecraft is modelled by a Dirichlet condition applied on the south boundary, were small local discontinuities represent the interconnects. On the north boundary, an homogenous Dirichlet condition models the space plasma. A patch is added locally, next to the interconnect.

The ion charge density will be computed using a Particle in Cell (PiC) solver, starting with an uniform density $c_i^0 = c_e^0$.

3.2 Numerical results

Using $(\varphi)_i$ and $(\psi)_i$ as basis of the subspaces V_h and V_H, we note

$$\|R_{\mathbf{u}_\ell}(\mathbf{u})\|^2 = \sum_i \langle R_{\mathbf{u}_\ell}(\mathbf{u}) \mid \varphi_i \rangle^2 + \sum_i \langle R_{\mathbf{u}_\ell}(\mathbf{u}) \mid \psi_i \rangle^2 \tag{9}$$

Hence, at iteration n using linearization at $\mathbf{u}_\ell$, $\|R_{\mathbf{u}_\ell}(\mathbf{0})\|$ is the norm of the linearization residual, and $\|R_{\mathbf{u}_\ell}(\mathbf{u}_n)\|$ is the norm of the residual in the linearized system.

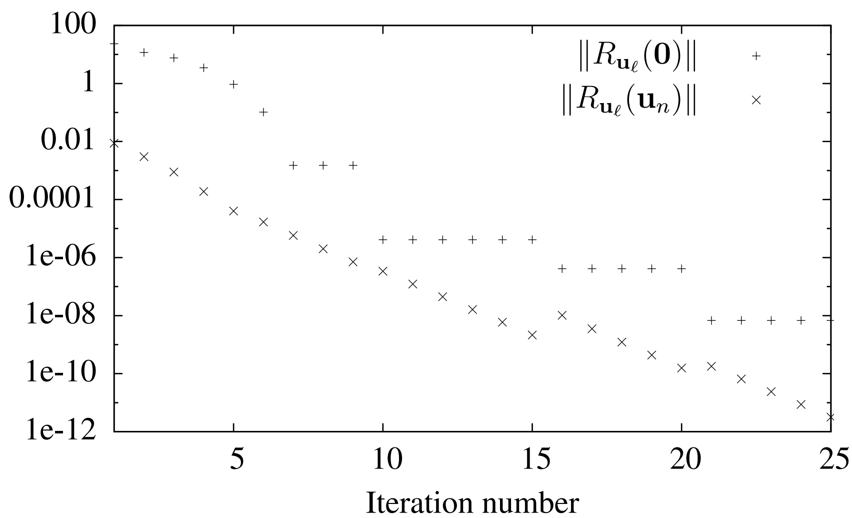

Figure 2: Convergence of the residuals

Figure 2 shows the convergence of these residuals using our method for a coarse mesh of 11×11 cells, and a refined patch of 120x120 cells covering a 3x3 coarse cells area near the interconnect. The relinearization criterion has been set to $\epsilon = 10^{-2}$.

On this figure, $\|R_{\mathbf{u}_\ell}(\mathbf{0})\|$ only changes on iterations involving linearization. Note that during the first 8 iterations in this example, the problem is systematicaly linearized, and the method is equivalent to a coarse Newton method. This is due to the fact that one patch iteration is enough to make the solving error small relative to the linearization error. During these initial iterations, the overall convergence is quadratic, as a typical Newton method convergence.

As soon as the linearization error reaches the same level as the solving error, our scheme allows for more patch iterations between linearizations. This can be seen in figure 2, where $\|R_{\mathbf{u}_\ell}(\mathbf{0})\|$ is levelling for several iterations, starting at iteration numbers 8, 10, 16 and 21.

Notice that linearizing may increase the patch residual $\|R_{\mathbf{u}_\ell}(\mathbf{u}_n)\|$, for instance at iteration number 16, but that the linearization residual $\|R_{\mathbf{u}_\ell}(\mathbf{0})\|$ is monotonically decreasing. Temporary increase of patch residual is due to a previous over-resolution of the system, and can be avoided by increasing the ϵ convergence parameter. Doing so will increase the frequency of linearization, and allows to weight the cost of linearizing against the cost of a patch iteration. A good value for the ϵ parameter would be the smallest one that ensures strict monotony for the patch residual.

Overall, the convergence is assymptotically linear, as with the linear patch method.

4 CONCLUSION

A linearization scheme has been proposed to solve nonlinear multiscale problems using the patch finite elements method, and a theoretical convergence result has been given. The resulting nonlinear patch method allows nonlinear multiscale, such as the simulation of small satellite elements in the space environment, to be solved efficiently with the required precision. This method is being implemented in the numerical engine of SPIS, and will provide a mean for industrial and scientific partners to simulate multiscale effects in the interaction between spacecrafts and their surrounding plasma.

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