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NUMERICAL PROPERTIES OF A DISCONTINUOUS GALERKIN FOMULATION FOR ELECTRO-THERMAL COUPLED PROBLEMS

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Abstract. *Discontinuous Galerkin (DG) methods are attractive tools to integrate several PDEs in engineering sciences, due to their high order accuracy and their high scalability in parallel simulations. The main interest of this work is to derive a constant and stable Discontinuous Galerkin method for two-way electro-thermal coupling analyses.*
A fully coupled nonlinear weak formulation for electro-thermal problems is developed based on continuum mechanics equations which are discretized using the Discontinuous Galerkin method. Toward this end, the weak form is written in terms of energetically conjugated fields gradients and fluxes.
In order to validate the effectiveness of the formulation and illustrate the algorithmic properties, a numerical test for composite materials is performed.

1 INTRODUCTION

There has been increased a recent interest in electro-thermal coupling due to its application in energy conversion [3, 4, 7, 8].
The objective of this paper is to extend the DG method to study steady state and response of electro-thermal material model taking into account the Peltier and Seebeck effects.
Some of the features of DG methods include the use of structured grid and high order approximation, hp adaptivity as there is no need for continuity requirement across element interfaces, and the flexibility in terms of mesh design. However, for practitioners, it is important to have methods available which yield reliable results for a wide variety of problems. By using stabilization techniques and inter element flux definitions, the shortcomings of non-stabilized DG methods can be overcome.
Recently, DG has been used to solve coupled problems. For instance a primal DG method with interior penalty (IP) terms has been proposed in [10] to solve coupled reactive transport in porous media. A DG method has also been used in [9] to solve the thermo-elastic coupling problems due to temperature and pressure dependent thermal contact resistance.
We have recently extended the DG method to solve electro-thermal coupled problems in terms of energetically conjugated fields gradients and fluxes [1], which to the authors knowledge, has not been introduced yet. In [1], the numerical properties of the nonlinear elliptic problem, following the method proposed in [2, 5], have been derived. In particular, the convergence rates of the error in both the energy and L^2-norms have been shown to be optimal with respect to the mesh size in terms of the polynomial degree approximation k (respectively in order $k-1$ and k).
The new formulation proposed in this paper is able to capture the electro-thermal effects, which describe the direct conversion of electric potential difference into temperature difference and vice versa. In addition it is able to effectively capture the electro-thermal behavior of the composite materials. This paper is structured as follows. In Section 2, the electro-thermal governing equations are discussed. In Section 3, the DG weak form of electro-thermal coupling is presented. Numerical results in the response of composite materials are shown in Section 4. Section 5 is dedicated to the concluding remarks.

2 Governing equations

Let us consider a volume Ω and let the boundary of the domain $\partial\Omega$ be the union of two disjoint sets: the Dirichlet boundary, $\partial_{\mathrm{D}}\Omega$ and Neumann boundary, $\partial_{\mathrm{N}}\Omega$.
The first balance equation is the electrical charge conservation equation

$$\nabla \cdot \mathbf{j}_{\mathrm{e}} = 0 \qquad \forall\, \mathbf{x} \in \Omega, \tag{1}$$

where $\mathbf{j}_{\mathrm{e}}$ denotes the flow of electrical current density vector, which is defined as

$$\mathbf{j}_{\mathrm{e}} \quad = \mathbf{l} \cdot (-\nabla \mathrm{V}) + \alpha \mathbf{l} \cdot (-\nabla \mathrm{T}). \tag{2}$$

In this equation α is the Seebeck coefficient, and $\mathbf{l}$ is the electric conductivity .
The second balance equation is the conservation of the energy flux

$$\nabla \cdot \mathbf{j}_{\mathrm{y}} \quad = -\partial_{\mathrm{t}} \mathrm{y} \qquad \forall\, \mathbf{x} \in \Omega. \tag{3}$$

The right hand side of this equilibrium equation is the time derivative of the internal energy density y

$$\mathrm{y} \quad = \mathrm{y}_0 + \mathrm{c}_{\mathrm{v}}\, \mathrm{T}, \tag{4}$$

which consists of the constant y_0 independent of the temperature and of the electric potential, and of the volumetric heat capacity c_v multiplied by the absolute temperature T.
Moreover the energy flux $\mathbf{j}_y$, which is a combination of the inter exchanges between the thermal and electric energies, is defined as

$$\mathbf{j}_y \ = \mathbf{q} + \ V\mathbf{j}_e, \tag{5}$$

where, $\mathbf{q}$ is the heat flux, defined as

$$\begin{aligned} \mathbf{q} \ &= \mathbf{k}\cdot(-\nabla T) + \beta_\alpha \mathbf{j}_e = \mathbf{k}\cdot(-\nabla T) + \alpha T\mathbf{j}_e && (6)\\ &= (\mathbf{k} + \alpha^2\, T\mathbf{l})\cdot(-\nabla T) + \alpha T\mathbf{l}\cdot(-\nabla V). && (7)\end{aligned}$$

In this equation $\mathbf{k}$ denotes the thermal conductivity and $\beta_\alpha = \alpha T$ is Peltier coefficient.
First we rewrite the equations (2, 5, 6) under the form

$$\mathbf{j} = \begin{pmatrix} \mathbf{j}_e \\ \mathbf{j}_y \end{pmatrix} = \begin{pmatrix} \mathbf{l} & \alpha\mathbf{l} \\ V\mathbf{l} + \alpha T\mathbf{l} & \mathbf{k} + \alpha V\mathbf{l} + \alpha^2 T\mathbf{l} \end{pmatrix} \begin{pmatrix} -\nabla V \\ -\nabla T \end{pmatrix}. \tag{8}$$

The conservation laws of this problem can then be formulated as finding $V,\, T \in H^2(\Omega) \times H^{2^+}(\Omega)$ such that

$$\begin{aligned} \nabla\cdot\mathbf{j}_e = 0 \qquad & \forall\, \mathbf{x}\in\Omega, && (9)\\ \nabla\cdot\mathbf{j}_y = \nabla\cdot\mathbf{q} + \mathbf{j}_e\cdot\nabla V = -\partial_t y \qquad & \forall\, \mathbf{x}\in\Omega, && (10)\\ T = \bar{T} > 0, \quad V = \bar{V} \qquad & \forall\, \mathbf{x}\in\partial_D\Omega, && (11)\\ \mathbf{q}\cdot\mathbf{n} = \bar{q}, \quad \mathbf{j}_e\cdot\mathbf{n} = \bar{j}_e, \quad \mathbf{j}_y\cdot\mathbf{n} = \bar{j}_y \qquad & \forall\, \mathbf{x}\in\partial_N\Omega, && (12)\end{aligned}$$

where $\bar{T}$ and $\bar{V}$ are the prescribed temperature and electric potential respectively, and $\mathbf{n}$ is the outward unit normal to the boundary $\partial\Omega$.
It should be noted that $H^{2^+}(\Omega)$ is the manifold to which T belongs, which is always positive.
The set of equations (9, 10) can be rewritten under a matrix form using Eq. (8)

$$\mathrm{div}\,(\mathbf{j}) = \begin{pmatrix} 0 \\ -\partial_t y \end{pmatrix} = \mathbf{i}. \tag{13}$$

Let us define the vector of the unknown fields $\mathbf{M}(2\times 1) = \begin{pmatrix} f_V \\ f_T \end{pmatrix}$, with $f_V = -\frac{V}{T}$ and $f_T = \frac{1}{T}$, [1, 3]. Indeed it can be shown that the fluxes $\mathbf{j}_e$, $\mathbf{j}_y$ and the fields gradients $\nabla(-\frac{V}{T})$, $\nabla(\frac{1}{T})$ are conjugated pairs. Therefore the gradients of the fields vector $\nabla\mathbf{M}(6\times 1)$ in terms of $(\nabla f_V, \nabla f_T)$ is defined by

$$(\ \nabla\mathbf{M}\ ) = \begin{pmatrix} \nabla f_V \\ \nabla f_T \end{pmatrix} = \begin{pmatrix} \nabla(-\frac{V}{T}) \\ \nabla(\frac{1}{T}) \end{pmatrix} = \begin{pmatrix} \frac{-1}{T} & \frac{V}{T^2} \\ 0 & \frac{-1}{T^2} \end{pmatrix} \begin{pmatrix} \nabla V \\ \nabla T \end{pmatrix}. \tag{14}$$

Hence, the fluxes defined by Eq. (8) can be expressed in terms of f_V, f_T, yielding

$$\mathbf{j} \ = \begin{pmatrix} \mathbf{l}T & V T\mathbf{l} + \alpha T^2\mathbf{l} \\ V T\mathbf{l} + \alpha T^2\mathbf{l} & T^2\mathbf{k} + 2\alpha T^2 V\mathbf{l} + \alpha^2 T^3\mathbf{l} + T V^2\mathbf{l} \end{pmatrix} \begin{pmatrix} \nabla f_V \\ \nabla f_T \end{pmatrix} = \ \mathbf{Z} \ \ \nabla\mathbf{M}. \tag{15}$$

The fluxes vector $\mathbf{j}(6\times 1)$ is the product of the gradients fields vector $\nabla\mathbf{M}$, which derive from the state variables (f_V, f_T), by a coefficients matrix $\mathbf{Z}(6\times 6)$ which is temperature and electric

potential dependent.
The symmetric coefficients matrix $\mathbf{Z}(\mathrm{V},\mathrm{T})$ in Eq. (15), which is positive definite if $\mathrm{T} > 0$, can be rewritten in terms of $(\mathrm{f_V},\mathrm{f_T}) = (-\frac{\mathrm{V}}{\mathrm{T}},\frac{1}{\mathrm{T}})$, as $\mathrm{T} = \frac{1}{\mathrm{f_T}}, \mathrm{V} = -\frac{\mathrm{f_V}}{\mathrm{f_T}}$:

$$\mathbf{Z}(\mathrm{f_V},\mathrm{f_T}) = \begin{pmatrix} \frac{1}{\mathrm{f_T}}\mathbf{l} & -\frac{\mathrm{f_V}}{\mathrm{f_T^2}}\mathbf{l} + \alpha\frac{1}{\mathrm{f_T^2}}\mathbf{l} \\ -\frac{\mathrm{f_V}}{\mathrm{f_T^2}}\mathbf{l} + \alpha\frac{1}{\mathrm{f_T^2}}\mathbf{l} & \frac{\mathbf{k}}{\mathrm{f_T^2}} - 2\alpha\frac{\mathrm{f_V}}{\mathrm{f_T^3}}\mathbf{l} + \alpha^2\frac{1}{\mathrm{f_T^3}}\mathbf{l} + \frac{\mathrm{f_V^2}}{\mathrm{f_T^3}}\mathbf{l} \end{pmatrix}. \tag{16}$$

Therefore, the strong form (8, 13) can be expressed as

$$\begin{aligned} \mathrm{div}(\mathbf{j}) &= \mathbf{i} && \forall\, \mathbf{x} \in \Omega, \\ \mathbf{M} &= \bar{\mathbf{M}} && \forall\, \mathbf{x} \in \partial_{\mathrm{D}}\Omega, \\ \bar{\mathbf{n}}\,\mathbf{j} &= \bar{\mathbf{j}} && \forall\, \mathbf{x} \in \partial_{\mathrm{N}}\Omega, \end{aligned} \tag{17}$$

where $\bar{\mathbf{n}} = \begin{pmatrix} \mathbf{n} & 0 \\ 0 & \mathbf{n} \end{pmatrix}, \bar{\mathbf{M}} = \begin{pmatrix} \bar{\mathrm{f}}_{\mathrm{V}} \\ \bar{\mathrm{f}}_{\mathrm{T}} \end{pmatrix} \in \mathrm{L}^2(\partial_{\mathrm{D}}\Omega) \times \mathrm{L}^{2^+}(\partial_{\mathrm{D}}\Omega)$, meaning that $\bar{\mathrm{f}}_{\mathrm{T}}$ belongs to the space which is always positive, and $\bar{\mathbf{j}} = \begin{pmatrix} \bar{\mathrm{j}}_{\mathrm{e}} \\ \bar{\mathrm{j}}_{\mathrm{y}} \end{pmatrix}$.

3 Weak Discontinuous Galerkin (DG) form for electro-thermal coupled problems

Let us introduce the finite element space $\Omega_{\mathrm{h}} = \cup_{\mathrm{e}}\Omega^{\mathrm{e}}$, associated with the triangulation of the domain Ω. If subscript I denotes the boundary between two elements, $\partial\Omega_{\mathrm{e}} = \partial_{\mathrm{N}}\Omega^{\mathrm{e}} \cup \partial_{\mathrm{D}}\Omega^{\mathrm{e}} \cup \partial_{\mathrm{I}}\Omega^{\mathrm{e}}$, and $\partial_{\mathrm{I}}\Omega_{\mathrm{h}} = \cup_{\mathrm{e}}\partial_{\mathrm{I}}\Omega^{\mathrm{e}} \setminus \partial\Omega_{\mathrm{h}}$, where $\partial_{\mathrm{I}}\Omega_{\mathrm{h}}$ is the intersecting boundary of the finite elements. The finite discontinuous polynomial approximation $\mathbf{M}_{\mathrm{h}} = \begin{pmatrix} \mathrm{f}_{\mathrm{V_h}} \\ \mathrm{f}_{\mathrm{T_h}} \end{pmatrix} \in \mathrm{X}^{\mathrm{k}}$ of the solution is thus defined in the space

$$\mathrm{X}^{\mathrm{k}} = \left\{ \mathbf{M}_{\mathrm{h}} \in \mathrm{L}^2(\Omega_{\mathrm{h}}) \times \mathrm{L}^{2^+}(\Omega_{\mathrm{h}}) \mid_{\mathbf{M}_{\mathrm{h}}|_{\Omega^{\mathrm{e}}} \in \mathbb{P}^{\mathrm{k}}(\Omega^{\mathrm{e}}) \times \mathbb{P}^{\mathrm{k}^+}(\Omega^{\mathrm{e}})\ \forall \Omega^{\mathrm{e}} \in \Omega_{\mathrm{h}}} \right\}, \tag{18}$$

where $\mathbb{P}^{\mathrm{k}}(\Omega^{\mathrm{e}})$ is the space of polynomial functions of order up to k and $\mathbb{P}^{\mathrm{k}^+}$ means that the polynomial approximation remains positive. Moreover we can define the kinematically admissible counterpart:

$$\mathrm{X}_{\mathrm{c}}^{\mathrm{k}} = \left\{ \delta\mathbf{M}_{\mathrm{h}} \in \mathrm{X}^{\mathrm{k}} \mid_{\delta\mathbf{M}_{\mathrm{h}} = \mathrm{I_h}\bar{\mathbf{M}}|_{\partial_{\mathrm{D}}\Omega^{\mathrm{e}}}} \right\}, \tag{19}$$

where $\mathrm{I_h}\bar{\mathbf{M}}$ is the interpolation of $\bar{\mathbf{M}}$ on $\partial_{\mathrm{D}}\Omega_{\mathrm{h}}$.
Let us derive the weak form of the governing equations by multiplying the first equation by a kinematically admissible test function $\delta\mathbf{M} \in \mathrm{X}_{\mathrm{c}}^{\mathrm{k}}$. Then performing a volume integral and using the divergence theorem on each element Ω^{e}, lead to state the problem as finding $\mathbf{M}_{\mathrm{h}} = \begin{pmatrix} \mathrm{f}_{\mathrm{V_h}} \\ \mathrm{f}_{\mathrm{T_h}} \end{pmatrix} \in \mathrm{X}^{\mathrm{k}}$ such that

$$-\sum_{\mathrm{e}} \int_{\Omega^{\mathrm{e}}} \nabla\delta\mathbf{M}_{\mathrm{h}}^{\mathrm{T}}\mathbf{j}(\mathbf{M}_{\mathrm{h}}, \nabla\mathbf{M}_{\mathrm{h}})\, \mathrm{d}\Omega + \sum_{\mathrm{e}} \int_{\partial\Omega^{\mathrm{e}}} \mathbf{n}^{\mathrm{T}}\delta\mathbf{M}_{\mathrm{h}}^{\mathrm{T}}\mathbf{j}(\mathbf{M}_{\mathrm{h}}, \nabla\mathbf{M}_{\mathrm{h}})\, \mathrm{dS} = 0 \quad \forall\delta\mathbf{M}_{\mathrm{h}} \in \mathrm{X}_{\mathrm{c}}^{\mathrm{k}}. \tag{20}$$

Let us define suitable face operators which are required for the definition of the proceeding DG. For two adjacent elements $+$, $-$, who share a common face $\partial_{\mathrm{I}}\Omega^{\mathrm{s}}$, the averages of $\mathbf{M}$, $\mathbf{j}_{\mathrm{e}}$ are given respectively by

$$\langle\mathbf{M}\rangle = \frac{1}{2}(\mathbf{M}^+ + \mathbf{M}^-), \qquad \langle\mathbf{j}_{\mathrm{e}}\rangle = \frac{1}{2}(\mathbf{j}_{\mathrm{e}}^+ + \mathbf{j}_{\mathrm{e}}^-),$$

and the jumps are defined respectively as

$$[\![\mathbf{M}]\!] = \frac{1}{2}(\mathbf{M}^{+} - \mathbf{M}^{-}), \qquad [\![\mathbf{j}_{\mathrm{e}}]\!] = \frac{1}{2}(\mathbf{j}_{\mathrm{e}}^{+} - \mathbf{j}_{\mathrm{e}}^{-}).$$

For simplicity, we introduce the vector $\mathbf{M}_{\mathbf{n}} = \begin{pmatrix} \mathbf{n}^{-} & 0 \\ 0 & \mathbf{n}^{-} \end{pmatrix} \mathbf{M}$, where $\mathbf{n}^{-}$ is the unit normal vector pointing from element $-$ toward element $+$ and $\bar{\mathbf{M}}_{\mathrm{h}_{\mathbf{n}}} = \bar{\mathbf{n}}\mathrm{I}_{\mathrm{h}}\bar{\mathbf{M}}$.
Then the surface integral of Eq. (20) is rewritten using the boundary condition stated in Eq. (17) and the notation introduced above, leading to

$$\int_{\partial_{\mathrm{N}}\Omega_{\mathrm{h}}} \delta\mathbf{M}_{\mathrm{h}}^{\mathrm{T}}\bar{\mathbf{j}}\,\mathrm{dS} + \int_{\partial_{\mathrm{D}}\Omega_{\mathrm{h}}} \bar{\mathbf{M}}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}\mathbf{j}(\mathbf{M}_{\mathrm{h}}, \nabla\mathbf{M}_{\mathrm{h}})\,\mathrm{dS} - \int_{\Omega_{\mathrm{h}}} \mathbf{i}\delta\mathbf{M}_{\mathrm{h}}^{\mathrm{T}}\mathrm{d}\Omega = \int_{\Omega_{\mathrm{h}}} \nabla\delta\mathbf{M}_{\mathrm{h}}^{\mathrm{T}}\mathbf{j}(\mathbf{M}_{\mathrm{h}}, \nabla\mathbf{M}_{\mathrm{h}})\mathrm{d}\Omega + \int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}} [\![\delta\mathbf{M}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}\mathbf{j}(\mathbf{M}_{\mathrm{h}}, \nabla\mathbf{M}_{\mathrm{h}})]\!]\,\mathrm{dS} \quad \forall\delta\mathbf{M}_{\mathrm{h}} \in \mathrm{X}_{\mathrm{c}}^{\mathrm{k}}. \tag{21}$$

Applying the mathematical identity $[\![\mathrm{ab}]\!] = [\![\mathrm{a}]\!]\langle\mathrm{b}\rangle + [\![\mathrm{b}]\!]\langle\mathrm{a}\rangle$, and neglecting the second term because only consistency in $\delta\mathbf{M}_{\mathrm{h}}^{\mathrm{T}}$ needs to be enforced, the interface flux related to Eq. (21) becomes$[\![\delta\mathbf{M}_{\mathrm{h}_{\mathbf{n}}}]\!]\langle\mathbf{j}(\mathbf{M}_{\mathrm{h}}, \nabla\mathbf{M}_{\mathrm{h}})\rangle$.
Due to the discontinuous nature of the approximation functions in the DG finite element, interelement discontinuity is allowed, so the continuity of unknown variables in the DG formulation is enforced weakly by adding symmetrization and stabilization terms at the interior element boundary $\partial_{\mathrm{I}}\Omega_{\mathrm{h}}$. Using Eq. (15), the stabilized form can be stated as finding $\mathbf{M}_{\mathrm{h}} \in \mathrm{X}^{\mathrm{k}}$ such that

$$\begin{aligned} &\int_{\partial_{\mathrm{N}}\Omega_{\mathrm{h}}} \delta\mathbf{M}_{\mathrm{h}}^{\mathrm{T}}\bar{\mathbf{j}}\mathrm{d}\Omega + \int_{\partial_{\mathrm{D}}\Omega_{\mathrm{h}}} \bar{\mathbf{M}}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}\mathbf{j}(\mathbf{M}_{\mathrm{h}}, \nabla\mathbf{M}_{\mathrm{h}})\mathrm{ds} = \int_{\Omega_{\mathrm{h}}} \nabla\delta\mathbf{M}^{\mathrm{T}}\mathbf{j}(\mathbf{M}_{\mathrm{h}}, \nabla\mathbf{M}_{\mathrm{h}})\mathrm{d}\Omega \\ &+ \int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}} [\![\delta\mathbf{M}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}]\!]\langle\mathbf{j}(\mathbf{M}_{\mathrm{h}}, \nabla\mathbf{M}_{\mathrm{h}})\rangle\,\mathrm{dS} + \int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}} [\![\mathbf{M}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}]\!]\langle\mathbf{Z}(\mathbf{M}_{\mathrm{h}})\nabla\delta\mathbf{M}_{\mathrm{h}}\rangle\,\mathrm{dS} \\ &+ \int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}} [\![\delta\mathbf{M}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}]\!]\left\langle\frac{\mathcal{B}}{\mathrm{h_s}}\mathbf{Z}(\mathbf{M}_{\mathrm{h}})\right\rangle[\![\mathbf{M}_{\mathrm{h}_{\mathbf{n}}}]\!]\,\mathrm{dS} \quad \forall\delta\mathbf{M}_{\mathrm{h}} \in \mathrm{X}_{\mathrm{c}}^{\mathrm{k}}. \end{aligned} \tag{22}$$

In this equation $\mathcal{B}$ is the stability parameter, which has to be sufficiently high to guarantee stability, and $\mathrm{h_s}$ is the mesh size.
[1].
Then, the IP discontinuous Galerkin method corresponding to problem Eq. (17) is defined as: find $\mathbf{M}_{\mathrm{h}} \in \mathrm{X}^{\mathrm{k}}$ such that

$$\mathrm{a}_3(\mathbf{M}_{\mathrm{h}}, \delta\mathbf{M}_{\mathrm{h}}) = \mathrm{b}_3(\delta\mathbf{M}_{\mathrm{h}}) - \int_{\Omega_{\mathrm{h}}} \mathbf{i}\delta\mathbf{M}_{\mathrm{h}}^{\mathrm{T}}\mathrm{d}\Omega + \mathrm{c}_3(\mathbf{M}_{\mathrm{h}}; \bar{\mathbf{M}}_{\mathrm{h}}) \quad \forall\delta\mathbf{M}_{\mathrm{h}} \in \mathrm{X}_{\mathrm{c}}^{\mathrm{k}}, \tag{23}$$

$$\begin{aligned} \mathrm{a}_3(\mathbf{M}_{\mathrm{h}}, \delta\mathbf{M}_{\mathrm{h}}) \quad &= \int_{\Omega_{\mathrm{h}}} \nabla\delta\mathbf{M}_{\mathrm{h}}^{\mathrm{T}}\mathbf{j}(\mathbf{M}_{\mathrm{h}}, \nabla\mathbf{M}_{\mathrm{h}})\mathrm{d}\Omega + \int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}} [\![\delta\mathbf{M}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}]\!]\langle\mathbf{j}(\mathbf{M}_{\mathrm{h}}, \nabla\mathbf{M}_{\mathrm{h}})\rangle\,\mathrm{dS} \\ &+ \int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}} [\![\mathbf{M}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}]\!]\langle\mathbf{Z}(\mathbf{M}_{\mathrm{h}})\nabla\delta\mathbf{M}_{\mathrm{h}}\rangle\,\mathrm{dS} + \int_{\partial_{\mathrm{I}}\Omega_{\mathrm{h}}} [\![\delta\mathbf{M}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}]\!]\left\langle\frac{\mathcal{B}}{\mathrm{h_s}}\mathbf{Z}(\mathbf{M}_{\mathrm{h}})\right\rangle[\![\mathbf{M}_{\mathrm{h}_{\mathbf{n}}}]\!]\,\mathrm{dS} \\ &\forall\delta\mathbf{M}_{\mathrm{h}} \in \mathrm{X}_{\mathrm{c}}^{\mathrm{k}}, \end{aligned} \tag{24}$$

$$\mathrm{c}_3(\mathbf{M}_{\mathrm{h}}; \bar{\mathbf{M}}_{\mathrm{h}}) = \int_{\partial_{\mathrm{D}}\Omega_{\mathrm{h}}} \bar{\mathbf{M}}_{\mathrm{h}_{\mathbf{n}}}^{\mathrm{T}}\mathbf{j}(\mathbf{M}_{\mathrm{h}}, \nabla\mathbf{M}_{\mathrm{h}})\mathrm{ds}, \text{ and} \tag{25}$$

$$\mathrm{b}_3(\delta\mathbf{M}_{\mathrm{h}}) = \int_{\partial_{\mathrm{N}}\Omega_{\mathrm{h}}} \delta\mathbf{M}_{\mathrm{h}}^{\mathrm{T}}\bar{\mathbf{j}}\mathrm{d}\Omega. \tag{26}$$

The demonstration of numerical properties such as the optimal error estimate, stability of the formulation and uniqueness of the solution for β high enough is reported in [1].

4 Numerical Results

The studied problem involves the production of heat by supplying electric power. The boundary conditions shown in Fig. 1 are applied on a composite material consisting of a combination of two materials: matrix (e.g., polymer) which is a non-conductive material and conductive fillers (e.g., carbon fiber).

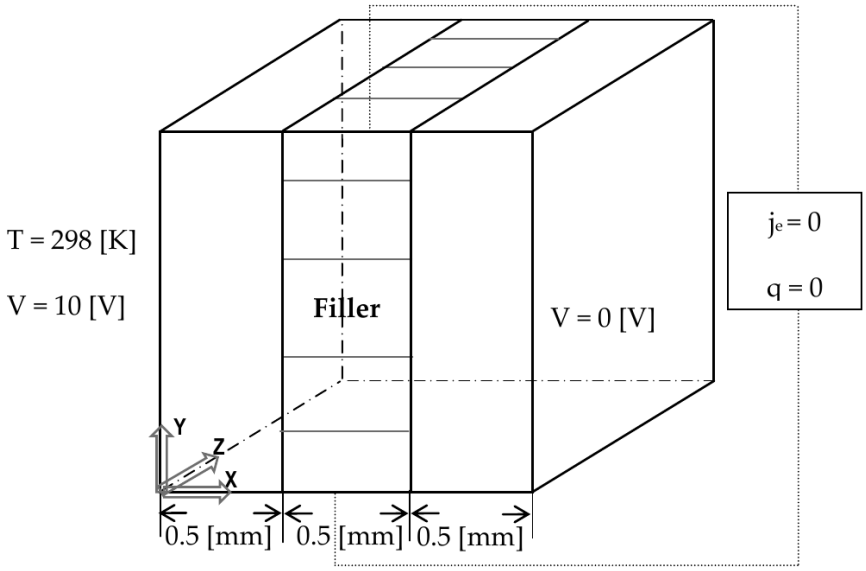


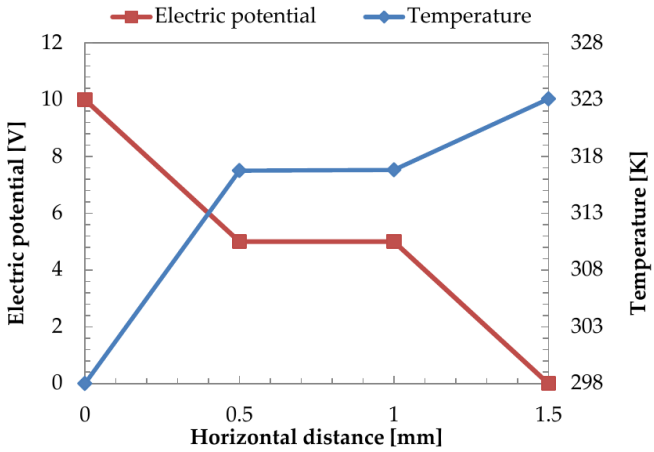
Figure 1: Electrothermal composite domain and the boundary conditions.

A finite element mesh consisting of 114 quadratic elements is used, and a stabilization parameter $\beta = 100$ is considered, to solve the DG discretization. The different material parameters used in this test are listed in Table 1, where all the material properties are assumed to be independent of temperature including the Seebeck coefficient.

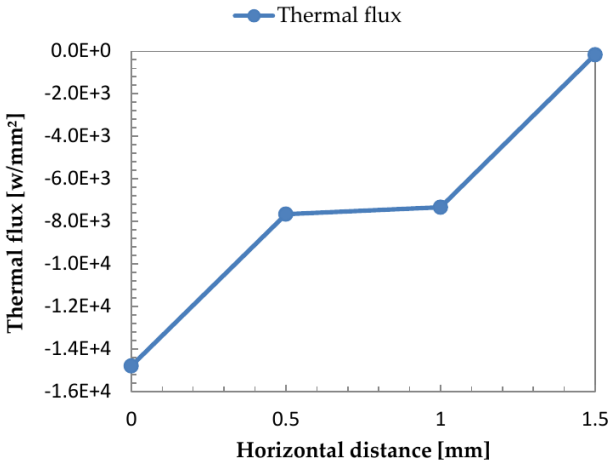
Material	$\mathbf{l}[\mathrm{S/m}]$	$\mathbf{k}[\mathrm{W/(K \cdot m)}]$	$\alpha[\mathrm{V/K}]$
Carbon fiber	diag(100000)	diag(40)	3×10^{-6}
Polymer	diag(0.1)	diag(0.2)	3×10^{-7}

Table 1: Composite material phases parameters.

Fig. 2 presents the distributions of the temperature and the of electric potential. When an electric potential of 10 [V] is applied on one side, the temperature on the other side increases from 298 [K] to 323 [K], We have also noticed that the temperature and electric potential, see Fig. 2(a), as well as the thermal flux, see Fig. 2(b), are almost constant in the fiber, as its electrical conductivity is high, and transfer gradually in the polymer matrix which is a non conductive material. Also a constant electric current density has been obtained with a value of $1.5 \times 10^3\,[\mathrm{A/m^2}]$.



(a)



(b)

Figure 2: (a) Distribution of the electrical potential and temperature in the electrothermal domain for composite materials, (b) The distribution of the thermal flux in the electrothermal domain for composite materials.

5 CONCLUSIONS

In this work

- A DG method was developed for electro-thermal coupling using energetically conjugated fluxes and fields.
- The result formulation was then applied to predict behavior of electro-thermal composite, and was able to compute the temperature and electric potential distributions as well as their fluxes, as a function of spatial coordinates.
- In the future, this formulation will be used for Electo-thermo-mechanical coupling in composite materials.

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