

ROCKING RESPONSE OF MASONRY BLOCK STRUCTURES USING MATHEMATICAL PROGRAMMING

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Abstract. *In this paper a variational formulation for dynamic analysis is adopted to investigate rocking behaviour of masonry block structures under lateral loads. The model is composed of rigid bodies interacting at potential contact points located at the vertexes of the block interfaces. A no-tension and associative frictional behaviour with infinite compressive strength is assumed at contact interfaces. The contact dynamic problem is governed by equilibrium equations, which relate external, inertial and contact forces, and by kinematic equations, which ensure compatibility between contact displacement rates and block degrees of freedom. Mathematical programming is used to solve the optimization problem arising from the formulation of the variational problem associated to dynamics of the block assemblages. To evaluate the accuracy and computational efficiency of the implemented formulation, a validation study is presented for rigid blocks subjected to rocking behaviour under different acceleration pulse types and for an in-plane wall panel problem from the literature. A good agreement in terms of failure mechanism and response time histories was observed. The computational efficiency and the stability of the implemented procedure were found to be encouraging, thus suggesting that the proposed model may be used to model dynamic behaviour of masonry block assemblages with a large number of rigid bodies.*

1 INTRODUCTION

In case of seismic events, the response of masonry structures, such as historical monuments made of stone blocks or facades in masonry buildings subjected to local failure mechanism, is typically characterized by rocking behaviour [1-3].

Different modelling approaches are available in the literature to investigate the dynamic response of masonry block structures subjected to rocking. Among those, non-smooth contact dynamics (NSCD) represents an alternative modelling approach to discontinuous finite element modelling as well as to the discrete element method (DEM) [4-8].

The NSCD method has been applied to masonry block structures since the beginning of its development and it is now receiving a growing attention in the research community on masonry structures [9-11]. This is due to different reasons, also including the availability of fast and accurate algorithm for the numerical solution of the formulations which have been proposed in the literature for the mathematical programming problem arising from the conditions governing the contact dynamics [12-15].

In this paper a simple formulation for dynamic analysis of masonry block structures is adopted to investigate rocking behaviour of masonry block structures. The aim of the study was to evaluate its accuracy and computational efficiency when applied to in-plane loaded wall panels. The adopted modelling approach for contact interfaces represents an extension to dynamics of the point-based formulation used in the rigid block model which has been developed for limit analysis of masonry block structures in [16-18].

The static and kinematic variables as well as the relationships governing the behaviour of the rigid block model and the limit analysis formulation are presented in Sections 2 and 3. In Section 4 we present a validation study on single rigid block subjected to free rocking motion and to different acceleration pulse types. Finally, an application to a numerical case study of an in-plane wall panel considered in the literature is illustrated to show the ability of the implemented formulation to capture rocking behaviour of multi-block assemblages.

2 THE RIGID BLOCK DYNAMIC MODEL

We consider a multi-body assemblage made of rectangular rigid blocks i interacting at potential contact points k located at the vertexes of the interface j (Fig. 1).

A no-tension and associative frictional behaviour with infinite compressive strength is assumed at contact interfaces.

The dynamic model is formulated following the approach proposed in [19, 20] for granular materials, though now detailed for two-dimensional assemblages of rectangular blocks.

The contact variables are the internal forces acting at each contact point k , which are located at a vertex of interface j of block i (Fig. 2a). These variables are collected in vector \mathbf{c} and include the shear force component t_k and the normal force n_k along the local coordinate axes.

The kinematic variables associated in a virtual work sense to the contact forces are the relative displacement rates at the contact points, namely the tangential and normal displacement rates Δu_{ik} and Δu_{nk} (Fig. 2b), which are collected in the vector $\Delta \mathbf{u}$.

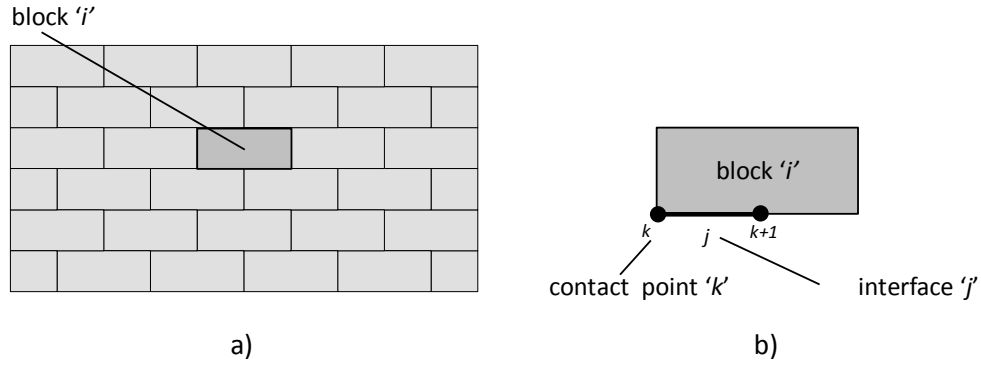


Figure 1: a) Rigid block assemblage; b) Rigid block i , interface j and contact point k .

External loads applied to the centroid of rigid block i are collected in vector of external forces \mathbf{f}_{ext} (Fig. 2a).

The position at the centroid of block i is collected in the vector \mathbf{x}_i :

$$\mathbf{x}_i(t) = [x_i(t) \quad z_i(t) \quad \alpha_i(t)]^T \quad (1)$$

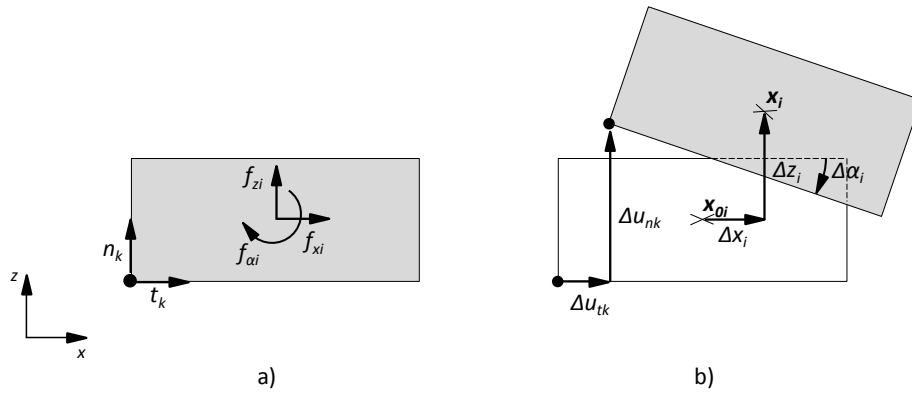


Figure 2: a) Contact forces and b) kinematic variables at block centroid i , and contact point k .

The equations of motions are discretized with respect to time using the θ -method and assuming:

$$\dot{\mathbf{x}}(t) = \frac{1}{\theta} \left[\frac{\Delta \mathbf{x}}{\Delta t} - (1 - \theta) \dot{\mathbf{x}}_0 \right] \quad (2)$$

where $\Delta \mathbf{x} = \mathbf{x} - \mathbf{x}_0$ is the displacement vector, \mathbf{x}_0 and $\dot{\mathbf{x}}_0$ are the known position and velocity at time t_0 and $\theta \geq 0.5$.

On the basis of the incremental expression (2), the equations of motion of the rigid block assemblage interacting at potential contact points can be posed as follows:

$$\bar{\mathbf{M}} \Delta \mathbf{x} + \mathbf{A}_0 \mathbf{c} = \bar{\mathbf{f}}_0 \quad (3)$$

where \mathbf{A}_0 is the equilibrium matrix corresponding to contact forces;

$$\bar{\mathbf{M}} = \frac{1}{\theta \Delta t^2} \mathbf{M};$$

\mathbf{M} is the mass matrix collecting the mass m_i and the mass moment of inertia J_i of each block;
 $\bar{\mathbf{f}}_0 = \mathbf{f}_{ext} + \bar{\mathbf{M}}\dot{\mathbf{x}}_0\Delta t$.

The non-penetration condition at potential contact point k is formulated in a linearized explicit form, that is assuming that the geometry is the same for two subsequent time steps.

In matrix form, the condition can be expressed imposing that the normal component of the relative displacement at a contact point k has not to be greater than the initial gap g_{0k} , as follows:

$$\mathbf{N}_0^T \Delta \mathbf{u} \leq \mathbf{g}_0 \quad (4)$$

where \mathbf{N}_0^T is the matrix collecting the initial normal associated with surfaces j and \mathbf{g}_0 is the vector of initial gaps.

For contact interactions, a complementarity condition is also included which ensures that contact forces are positive only if the gap is closed otherwise are zero:

$$\begin{aligned} n &\geq 0 \\ \text{diag}(n) (\mathbf{N}_0^T \Delta \mathbf{u} - \mathbf{g}_0) &= 0 \end{aligned} \quad (5)$$

The behaviour at contact interfaces undergoing sliding failure is governed by failure conditions which are expressed according to the Coulomb friction law.

In vector notation, the limit conditions for sliding failure can be written as:

$$\pm \mathbf{t} \leq \mu \mathbf{n} \quad (6)$$

where μ is the friction coefficient.

3 FORMULATION AND IMPLEMENTATION OF THE VARIATIONAL PROBLEM OF RIGID BLOCK DYNAMICS

Under the assumption of associative flow rule for displacement rates, the equilibrium equations (3), kinematic conditions (4-5) and sliding friction conditions (6) are equivalent to the following discrete mixed force–displacement problem [19, 20]:

$$\begin{aligned} \min_{\Delta \mathbf{x}} \max_c \quad & \frac{1}{2} \Delta \mathbf{x}^T \bar{\mathbf{M}} \Delta \mathbf{x} - \Delta \mathbf{x}^T \bar{\mathbf{f}}_0 + \Delta \mathbf{x}^T \mathbf{A}_0 \mathbf{c} - \mathbf{g}_0 \\ \text{subject to} \quad & \pm \mathbf{t} - \mu \mathbf{n} \leq 0 \quad n \geq 0 \end{aligned} \quad (7)$$

The problem (7) can be uncoupled into two dual quadratic programming problems, corresponding to the kinematic and force-based formulation of the contact dynamic problem.

To calculate and update positions of the blocks and contact gaps, an iterative procedure was implemented to solve the mathematical program (7).

The procedure was implemented in a computer code, *DynoBlock_2D*, which provides as outputs the time histories of contact forces and kinematic variables as well as the plots of the failure mechanisms at different time steps.

The quadratic programming problems associated to (7) were solved using the primal–dual interior-point solver in MOSEK [23].

The analyses were carried out using a PC containing a 3.3GHz Intel Xeon E3-1245 processor with 8 GB of RAM. The value of algorithm parameter θ for time discretization was set equal to 0.7 and time increment was set equal to 0.002 s.

4 DYNAMICS OF THE SINGLE RIGID BLOCK

To validate the proposed formulation, the case study of a single rigid block subjected to free rocking motion and to ground acceleration pulses was considered.

The rigid block dimensions are 0.20×1.00 m and the unit weight is 18.0 kN/m^3 (Fig. 3). The friction coefficient μ adopted at the contact interface is equal to 0.7.

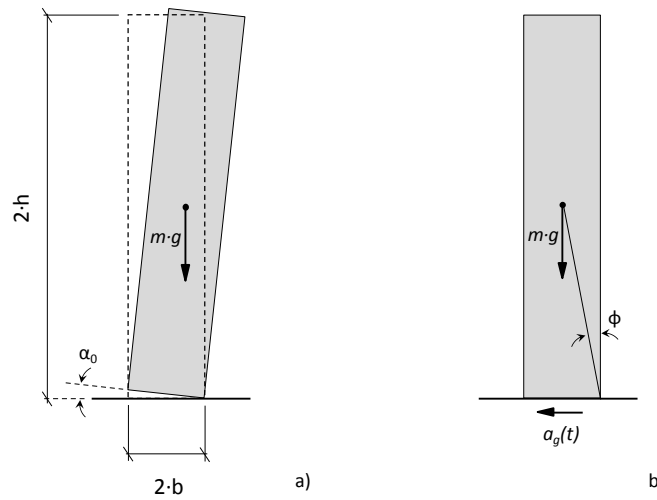


Figure 3: Rigid block subjected to free rocking motion and to ground acceleration pulses.

To analyze the free rocking motion response of the block subjected to gravity acceleration g an initial angle $\alpha_0 = 0.01$ rad was considered.

The rotation time history of rigid block obtained from the proposed formulation is shown in Fig. 4. The comparison with closed form solution, which is referred to a value of the coefficient of restitution $\sqrt{r} = 0.925$ relating the reduction of the angular velocity before and after the impact, shows a good agreement [21].

The response obtained for the rigid block resting on the base and subjected to a constant horizontal ground acceleration a_g was investigated as well for validation. The objective of this set of simulations was to test the accuracy of the formulation in predicting the minimum value of the horizontal acceleration for the motion to be initiated and the magnitudes of rectangular pulse excitations with duration of 1.0s needed for overturning.

The numerical simulations showed that the minimum value of the horizontal acceleration a_g to begin tilting of the block was $0.20g$. As expected, this value is in accordance with the condition $a_g / g \geq \phi$, corresponding to the simple condition that the overturning moment induced by the lateral force on the contact has to be greater than the corresponding resistant moment related to the weight.

The rotation time histories for different values of ground accelerations in the case of rectangular pulse excitations are shown in Figure 5.

Case study	Model size ($b \times c$)	a_g (g)					
		Acceleration value for rocking initiation		Overturning acceleration amplitude for constant pulse with duration 1.0 s		Overturning acceleration amplitude for a sinusoidal pulse with duration 1.0 s	
		Differential equations	Proposed	Differential equations	Proposed	Differential equations	Proposed
Single block	1×2	0.200	0.201	0.204	0.204	0.237	0.237

Table 1: Single rigid block: acceleration values for overturning and CPU times.

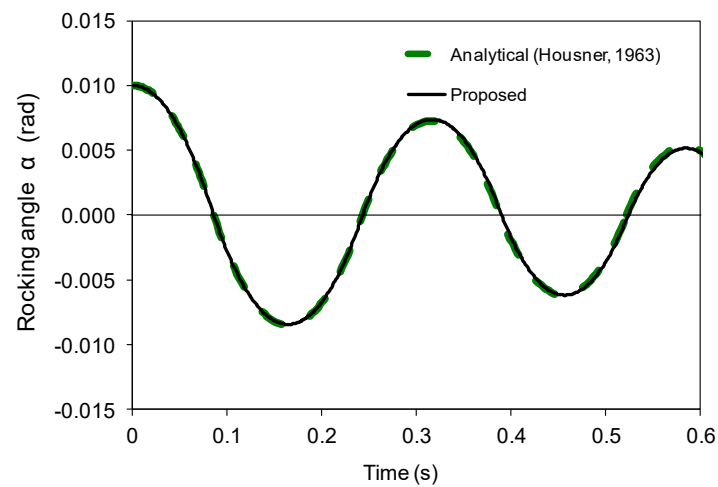


Figure 4: Rotation time history of the rigid block subjected to free rocking motion: comparison of proposed and analytical formulations.

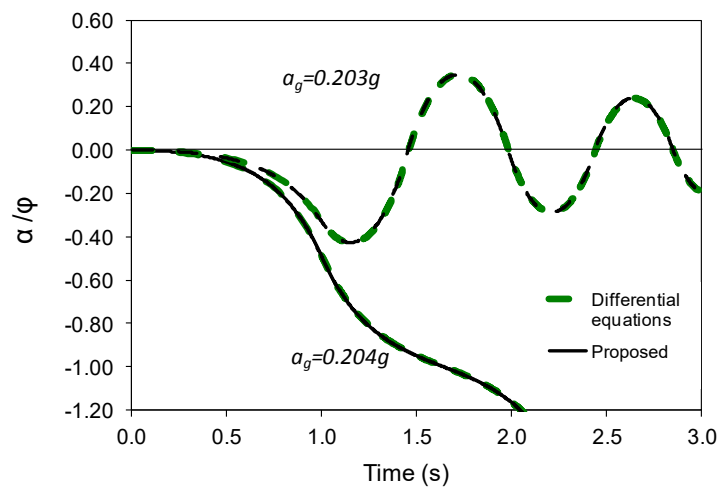


Figure 5: Rotation time histories of the rigid block subjected to constant (rectangular) acceleration pulses with duration of 1.0 s.

A similar study was carried out for sinusoidal pulse excitation. Also in this case the results are in good agreement with the response obtained from numerical integration of the differential equation of motion (see Table 1).

5 APPLICATION TO IN-PLANE LOADED MASONRY WALL

In this section we consider for validation a block panel analysed by Ferris and Tin-Loi [22] using mathematical programming with equilibrium constraints (MPEC) for computational limit analysis and subsequently investigated in [16] using a second order cone programming formulation (SOCP).

The main goal of this study was to compare results from the present QP dynamic formulation with those obtained from previously developed limit equilibrium analysis (LA) procedures and to illustrate the ability of the proposed formulation in capturing the rocking response in case of rectangular pulse excitation.

The wall panel under investigation is illustrated in Figure 6 (example no.3 in [22]). The size of a full block is $0.4 \times 0.175\text{m}$, the friction coefficient is 0.65 and the unit weight is 1.0kN/m^3 . To simulate the load cases of the limit analysis formulation, in the contact dynamic QP formulation the blocks are subject to a constant value of horizontal acceleration and to the gravity acceleration. The analysis is repeated increasing the horizontal acceleration by step of $0.001g$ until rocking motion is initiated. Once the minimum value of acceleration is obtained, a constant acceleration pulse with duration of 1.1 seconds is applied to the panel to analyze the evolution of the failure mechanism over time.

The sizes of the numerical model, expressed as the number of blocks b and contact points c , are reported in Table 2, together with the numerical results in terms of accelerations and ultimate load factors obtained from limit equilibrium analysis, as well as CPU times.

The computed acceleration magnitude to initiate rocking motion is in good agreement with the load factors obtained using previously presented limit equilibrium formulations (Table 2). Indeed, the computed acceleration values are comprised in the range of associative and non-associative solutions from limit analysis problems.

Case study	Model size ($b \times c$)	Limit equilibrium analysis using SOCP [16]				Dynamic analysis using QP (proposed)	
		Associative friction solutions		Non-associative friction solutions			
		λ	CPU Time (s)	λ	CPU Time (s)	a_g (g)	CPU Time [*] (s)
Example no. 3 [22]	46×240	0.404	0.1	0.356	0.9	0.387	97.5

*Referred to 3.3GHz CPU, per unit time duration of dynamic analysis (second) in the case of QP formulation.

Table 2. In-plane loaded masonry wall: comparison of collapse load factors from LA formulations and acceleration obtained from CD-QP problems.

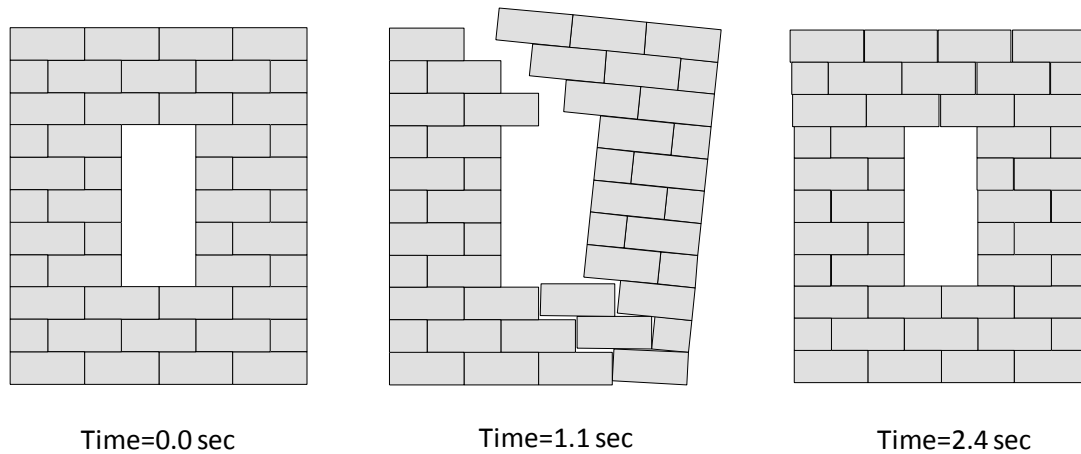


Figure 6: Wall sample n.3 [22]: failure mechanism time history obtained from the contact dynamic formulation under a rectangular pulse excitation of $0.387g$ with duration of $1.1s$.

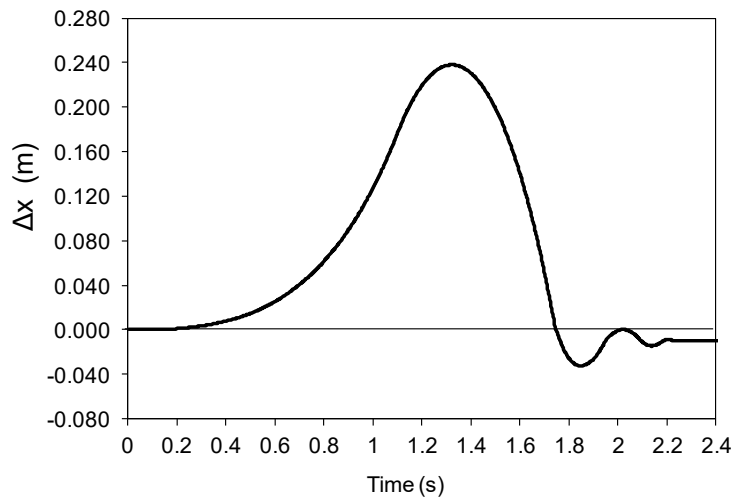


Figure 7: Time history of the top drift displacement.

The time history of the failure mechanism and the drift displacement at the top for the rectangular pulse excitation with duration of $1.1 s$ are shown in Figures 6 and 7.

It is interesting to point out the residual drift after pulse excitation and the stability of the predicted response, which shows no chattering effect and is only slightly affected by the few mechanical and algorithm parameters inherent the numerical model, namely the friction coefficient, the θ factor and the size of time step.

6 CONCLUSIONS

- A 2D formulation for contact dynamic analysis of block masonry structures with no-tension and associative frictional joints and infinite compressive strength was presented. The formulation is based on the contact point model for interfaces and uses QP programming to solve the discretized equation of motion under kinematic and static constraints.

- The model was validated against numerical case studies from the literature involving single and multi-block assemblages and a good agreement in terms of failure mechanism and response time histories was observed.
- The computational efficiency and the convergence stability of the implemented procedure were found to be encouraging, also considering the small number of mechanical and algorithm parameters associated to the adopted formulation.

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