ECCOMAS Congress 2016
VII European Congress on Computational Methods in Applied Sciences and Engineering
M. Papadrakakis, V. Papadopoulos, G. Stefanou, V. Plevris (eds.)
Crete Island, Greece, 5–10 June 2016

THE INFLUENCE OF UNCERTAINTY OF DESIGN PARAMETERS ON DYNAMIC CHARACTERISTICS OF STRUCTURE WITH DAMPERS

Magdalena Łasecka-Plura[1], Roman Lewandowski[2]

[1] Institute of Structural Engineering, Poznan University of Technology
Piotrowo 5, 60-965 Poznan, Poland
magdalena.lasecka-plura@put.poznan.pl

[2] Institute of Structural Engineering, Poznan University of Technology
Piotrowo 5, 60-965 Poznan, Poland
roman.lewandowski@put.poznan.pl

Keywords: Structures with dampers, fractional models of dampers, uncertainty of design parameters.

Abstract. *In this paper, the influence of the uncertainty of design parameters on the dynamic characteristics of structures with viscoelastic dampers mounted on them is considered. The fractional derivatives are used to describe the models of dampers. The uncertainty of their design parameters is introduced as an interval value. The lower and upper bounds of objective dynamic characteristics are obtained using the first- and second-order Taylor series expansion. Typical calculations are presented and compared with the results obtained using the vertex method.*

1 INTRODUCTION

In many practical situations, design parameters are uncertain. The uncertainty of parameters is caused by the inaccuracy of measurements, assembly errors, defect of material, temperature dependence, etc. The problem of uncertain parameters is widely described in numerous research papers. For issues with an unknown variability of the parameters, the interval analysis is often used. In this method, it suffices to know that the parameter values change within certain limits. Moore [1] did a pioneering work of interval analysis. In the last two decades, many methods which enable this theory to be used in engineering problems have been developed. In paper [2], the vertex method is presented. The method is based on the theorem of “inclusion monotonic”. It assumes that the lower and upper bounds of the objective function should be calculated as the end-point combination of the design interval parameters. The method is also used in [3]. However, when the number of interval design parameters is large, the computational cost of the combination of all the parameters is very high. In this approach, the number of the required combinations equals 2^r, where r denotes the number of the interval parameters.

Alternative methods are required for non-monotonic problems. In papers [4,5], optimization-based interval analysis methods are presented. The lower and upper bounds of the objective function are obtained as the minimization and maximization of calculation using the interval parameters as constraints for the optimization problem.

For dynamic issues with uncertain design parameters, the lower and upper bounds of the objective function are often obtained using the Taylor series expansion. The method was extended in paper [6] and its application to optimization was shown. In [3, 7], the authors improved the method using the second-order Taylor series expansion. In [7], Fujita and Takewaki introduced an application of the method to the analysis of structures with passive dampers. A similar analysis of the structures with uncertain parameters was presented in [8] using the robustness analysis.

In this paper, the Taylor series expansion of the first- and second-order is used for determination of the lower and upper bounds of dynamic characteristics, such as natural frequencies, non-dimensional damping factors, and eigenvectors. The structure is modeled as a shear frame with dampers described by fractional derivatives. In the numerical example, the uncertainties of dampers’ parameters are taken into consideration. The obtained results are compared with the vertex method of which the results are close to the exact solution. Such analysis is carried out for structures with dampers described by fractional derivatives for the first time.

2 DESCRIPTION OF STRUCTURES WITH DAMPERS

2.1 Models of dampers

Many rheological models of dampers have been proposed in the literature. The most frequently used models are the Kelvin and the Maxwell models. The first one consists of a spring and a dashpot connected in parallel whereas the second one is built of a serially connected spring and dashpot (Fig. 1).

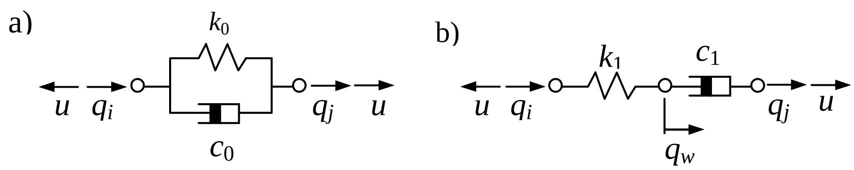

Figure 1. Classical models of dampers a) Kelvin model, b) Maxwell model.

In many instances, it is difficult to describe the rheological properties of viscoelastic dampers by basic classical models and it is necessary to use the generalized models which consist of numerous Kelvin or Maxwell elements. The number of parameters which are required for the description of the properties of dampers is rather high and sometimes it is more efficient to use the fractional models [9]. A damping element (called “a springpot element”) is described by two constants, c and α, where α denotes the order of the fractional derivative. The fractional Kelvin and Maxwell models are shown in Fig. 2.

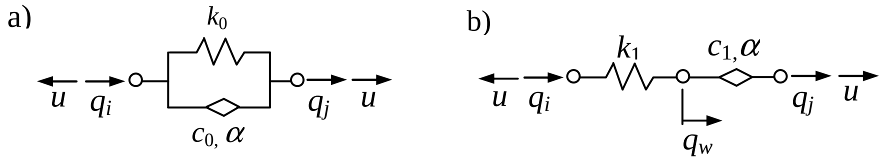

Figure 2. Fractional models of dampers a) Kelvin model, b) Maxwell model.

The equation of motion for the fractional Kelvin model can be written in the following way:

$$u(t) = k_0 \Delta q(t) + c_0 D_t^\alpha \Delta q(t) \tag{1}$$

and for the fractional Maxwell model, it could be described as follows:

$$u(t) + \frac{c_1}{k_1} D_t^\alpha u(t) = c_1 D_t^\alpha \Delta q(t) \tag{2}$$

where k_0 and k_1 denote stiffnesses, c_0 and c_1 denote damping factors, $\Delta q(t) = q_j(t) - q_i(t)$ is relative displacement of nodes of damper and the symbol D_t^α denotes the Riemann-Liouville fractional derivative of the order α with respect to time [10, 11].

After taking the Laplace transform, Equations (1) and (2) can be written in the forms:

$$\bar{u}(s) = k_0 \Delta \bar{q}(s) + c_0 s^\alpha \Delta \bar{q}(s) \tag{3}$$

$$\bar{u}(s) + \frac{c_1}{k_1} s^\alpha \bar{u}(s) = c_1 s^\alpha \Delta \bar{q}(s) \tag{4}$$

where: $\Delta \bar{q}(s) = L[\Delta q(t)]$, $\bar{u}(s) = L[u(t)]$, $s^\alpha \bar{u}(s) = L[D_t^\alpha u(t)]$, and s – Laplace variable. Finally, it is possible to write the general relationship:

$$\bar{u}(s) = G(s) \Delta \bar{q}(s) \tag{5}$$

where:

$$G(s) = k_0 + c_0 s^\alpha \tag{6}$$

for the fractional Kelvin model (see Fig. 2a) and

$$G(s) = k_1 \frac{c_1 s^\alpha}{k_1 + c_1 s^\alpha} \tag{7}$$

for the fractional Maxwell model (see Fig. 2b).

2.2 Equation of motion for structures with dampers

The equation of motion for structures with viscoelastic dampers can be written in the following form [12]:

$$\mathbf{M}_s\ddot{\mathbf{q}}(t)+\mathbf{C}_s\dot{\mathbf{q}}(t)+\mathbf{K}_s\mathbf{q}(t)=\mathbf{p}(t)+\mathbf{f}(t) \tag{8}$$

where: $\mathbf{M}_s$, $\mathbf{C}_s$ and $\mathbf{K}_s$ denote the mass, the damping and the stiffness matrix of structure, respectively. The structure is modeled as a shear frame with mass lumped at the storey level. $\mathbf{q}(t)=[q_1 \quad \ldots \quad q_n]^T$ is the vector of displacements of the structure, $\mathbf{p}(t)=[p_1 \quad \ldots \quad p_n]^T$ is the vector of excitation forces, $\mathbf{f}(t)=[f_1 \quad \ldots \quad f_n]^T$ is the vector of the interaction forces between the frame and the dampers (see Fig. 3) and n is the number of the degrees of freedom of the considered system.

Vector $\mathbf{f}(t)$ is a sum of the vectors $\mathbf{f}_k(t)$. Each of them is formed if damper k only is located on the frame, i.e.:

$$\mathbf{f}(t)=\sum_{k=1}^{m}\mathbf{f}_k(t)\,. \tag{9}$$

where m is the number of dampers.

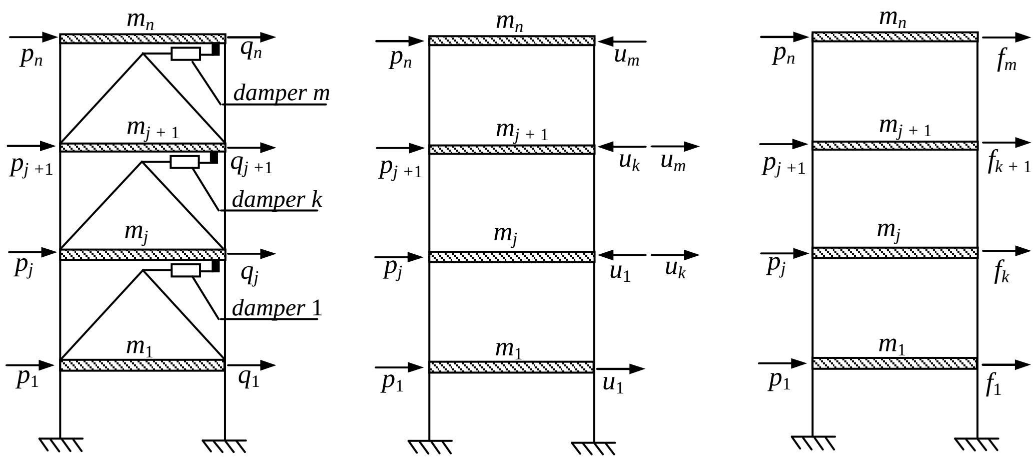

Figure 3. Diagram of frame with dampers.

For the damper located between the floors j and $j+1$ (see Fig. 3), it is possible to write:

$$\mathbf{f}_k(t)=\mathbf{e}_k u_k(t), \qquad \mathbf{e}_k=\begin{bmatrix}0 & \ldots & e_j=1 & e_{j+1}=-1 & \ldots & 0\end{bmatrix}^T. \tag{10}$$

After taking the Laplace transform, the equation of motion can be written as:

$$\left(s^2\mathbf{M}_s+s\mathbf{C}_s+\mathbf{K}_s\right)\overline{\mathbf{q}}(s)=\overline{\mathbf{p}}(s)+\overline{\mathbf{f}}(s) \tag{11}$$

where: $\overline{\mathbf{q}}(s)=L[\mathbf{q}(t)]$, $\overline{\mathbf{p}}(s)=L[\mathbf{p}(t)]$ and $\overline{\mathbf{f}}(s)=L[\mathbf{f}(t)]$.
The vector $\overline{\mathbf{f}}(s)$ is in the following form:

$$\overline{\mathbf{f}}(s)=\sum_{k=1}^{m}\overline{\mathbf{f}}_k(s)=\sum_{k=1}^{m}G_k(s)\mathbf{L}_k\overline{\mathbf{q}}(s) \tag{12}$$

and $\mathbf{L}_k=\mathbf{e}_k\mathbf{e}_k^{\mathrm{T}}$ is the matrix of the location of dampers.

After substituting Equation (12) to (11), the equation of the motion of structure with viscoelastic dampers is:

$$\mathbf{D}(s)\overline{\mathbf{q}}(s) = \overline{\mathbf{p}}(s) \tag{13}$$

where:

$$\mathbf{D}(s) = s^2\mathbf{M}_s + s\mathbf{C}_s + \mathbf{K}_s + \sum_{k=1}^{m}\mathbf{G}_k \,, \qquad \mathbf{G}_k(s) = G_k(s)\mathbf{L}_k \,. \tag{14}$$

If the vector of external forces equals zero ($\overline{\mathbf{p}}(s) = \mathbf{0}$), then Equation (13) leads to the following nonlinear eigenproblem:

$$\mathbf{D}(s)\,\overline{\mathbf{q}}(s) = \mathbf{0}\,. \tag{15}$$

Methods to solve a nonlinear eigenproblem for a structure with damping forces described using fractional derivatives were proposed in [13-15]. In this paper, the nonlinear eigenproblem was solved by the continuation method, described in detail in paper [15].

After determination of eigenvalues (s_i) and eigenvectors ($\mathbf{q}_i$), it is possible to calculate natural frequencies (ω_i) and non-dimensional damping factors (γ_i) according to the formulae:

$$\begin{aligned} \omega_i^2 &= \mu_i^2 + \eta_i^2 \\ \gamma_i &= -\mu_i / \omega_i \end{aligned} \tag{16}$$

where $\mu_i = \mathrm{Re}(s_i)$, $\eta_i = \mathrm{Im}(s_i)$.

3 UNCERTAIN PARAMETERS

3.1 Interval analysis - definition

Let us assume some design parameters have changed, though only within a specified range. These variations can be written as interval parameters in the form:

$$\mathbf{p}^I = \left[\,\mathbf{p}^c - \Delta\mathbf{p},\ \mathbf{p}^c + \Delta\mathbf{p}\right] \tag{17}$$

or as follows:

$$\mathbf{p}^I = \left[\underline{\mathbf{p}}, \overline{\mathbf{p}}\right] \tag{18}$$

where $\mathbf{p}^I = \left[p_1^I, p_2^I, ..., p_r^I\right]$ denotes the values of interval parameters, r denotes the number of interval design parameters, $\mathbf{p}^c$ denotes middle values of interval parameters:

$$\mathbf{p}^c = \frac{\overline{p} + \underline{p}}{2}\,, \tag{19}$$

$\Delta\mathbf{p}$ denotes half of the varied range of interval parameters (radius value):

$$\Delta\mathbf{p} = \frac{\overline{p} - \underline{p}}{2}\,, \tag{20}$$

$\underline{\mathbf{p}}$ and $\overline{\mathbf{p}}$ denote the lower and upper bounds of the interval parameters, respectively.

Let us consider an objective function $F(\mathbf{p})$ which depends on the design parameters. If the parameters' values change within a specified range, the considered function can be written in the form $F^I(\mathbf{p}^I)$. The interval analysis leads us to find the lower and upper bounds of the objective function in a feasible domain of the interval parameters $(-\Delta\mathbf{p}, \Delta\mathbf{p})$. The interval problem can be described as:

$$F^I(\mathbf{p}^I) = \left[\min_{\mathbf{p}\in\mathbf{p}^I} F(\mathbf{p}),\ \max_{\mathbf{p}\in\mathbf{p}^I} F(\mathbf{p}) \right] \tag{21}$$

3.2 Interval analysis based on approximation of Taylor series expansion

An approximation of the objective function $\tilde{F}(\mathbf{p})$ using the first-order Taylor series expansion around the middle values of parameters $\mathbf{p}^c$ is defined in the following form:

$$\tilde{F}(\mathbf{p}) = F(\mathbf{p}^C) + \sum_{i=1}^{r} \frac{\partial F(\mathbf{p}^C)}{\partial p_i} \Delta p_i \tag{22}$$

where $\Delta p_i = p_i - p_i^C$ and $\partial F(\mathbf{p}^C)/\partial p_i$ denotes a first-order differentiation of the function $F(\mathbf{p}^C)$ at middle values of design parameters with respect to the changing parameter p_i.

From Equation (22), it is possible to write the increment of the objective function calculated using the first-order Taylor series expansion

$$\Delta F(\mathbf{p}) = \sum_{i=1}^{r} \frac{\partial F(\mathbf{p}^C)}{\partial p_i} \Delta p_i \tag{23}$$

as a sum of increments of the objective function for each one-dimensional perturbation. Hence, the increment of the objective function for the variation of parameter p_1 can be evaluated from:

$$\Delta F_1(p_1^I, p_2^C, ..., p_n^C) = \frac{\partial F(\mathbf{p}^C)}{\partial p_1}(p_1 - p_1^C). \tag{24}$$

According to an interval analysis, the following formula can be written:

$$\Delta F_1^I(p_1^I, p_2^C, ..., p_n^C) = \begin{bmatrix} \min\left[\Delta F_1(\underline{p}_1, p_2^C, ..., p_n^C), \Delta F_1(\overline{p}_1, p_2^C, ..., p_n^C)\right] \\ \max\left[\Delta F_1(\underline{p}_1, p_2^C, ..., p_n^C), \Delta F_1(\overline{p}_1, p_2^C, ..., p_n^C)\right] \end{bmatrix} \tag{25}$$

It is possible to write a similar relationship for each interval design parameter. Finally, substituting $\Delta F_i(p_i^I)$ to Equation (22) leads us to the equation:

$$F^I(\mathbf{p}^I) = \left[F(\mathbf{p}^C) + \sum_{i=1}^{r} \Delta \underline{F}_i(p_i^I),\ \ F(\mathbf{p}^C) + \sum_{i=1}^{r} \Delta \overline{F}_i(p_i^I) \right] \tag{26}$$

According to formula (26), the lower and upper bounds of the objective interval function, approximated using the first-order Taylor series expansion, can be written as:

$$\begin{aligned} \underline{F}(\mathbf{p}^I) &= F(\mathbf{p}^C) - \sum_{i=1}^{r} \left| \frac{\partial F(\mathbf{p}^C)}{\partial p_i} \Delta p_i \right| \\ \overline{F}(\mathbf{p}^I) &= F(\mathbf{p}^C) + \sum_{i=1}^{r} \left| \frac{\partial F(\mathbf{p}^C)}{\partial p_i} \Delta p_i \right| \end{aligned} \tag{27}$$

Some researchers [3,7] proposed an approximation of the objective function $\tilde{\tilde{F}}(\mathbf{p})$ using the second-order Taylor series expansion around the middle values of parameters $\mathbf{p}^c$:

$$\tilde{\tilde{F}}(\mathbf{p}) = F\left(\mathbf{p}^C\right) + \sum_{i=1}^{r} \frac{\partial F\left(\mathbf{p}^C\right)}{\partial p_i} \Delta p_i + \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} \frac{\partial^2 F\left(\mathbf{p}^C\right)}{\partial p_i \partial p_j} \Delta p_i \Delta p_j \tag{28}$$

where $\partial^2 F\left(\mathbf{p}^C\right) / \left(\partial p_i \partial p_j\right)$ build the Hessian matrix of the function $F\left(\mathbf{p}^C\right)$ with respect to the changing parameters p_i and p_j. The matrix consists of a second-order differentiation of the function $F\left(\mathbf{p}^C\right)$ at middle values of the design parameters. The increment of the objective function can be written in the following form:

$$\Delta F(\mathbf{p}) = \sum_{i=1}^{r} \frac{\partial F\left(\mathbf{p}^C\right)}{\partial p_i} \Delta p_i + \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} \frac{\partial^2 F\left(\mathbf{p}^C\right)}{\partial p_i \partial p_j} \Delta p_i \Delta p_j \,. \tag{29}$$

Based on inclusion monotonic, in order to find the lower and upper bounds of the objective function $\tilde{\tilde{F}}(\mathbf{p})$, it is necessary to calculate all end-point combinations of the interval parameters p_i and p_j. The number of combinations is the same as the number of combinations using the vertex method and it equals 2^r. The computational cost in this approach is very high, especially for high numbers of the interval parameters. Therefore, a simplification of the above method was proposed in [3]. In that paper, the non-diagonal elements of the Hessian matrix are neglected.

An approximation of the objective function $\tilde{\tilde{\tilde{F}}}(\mathbf{p})$ using the second-order Taylor series expansion with only the diagonal elements of the Hessian matrix is as follows:

$$\tilde{\tilde{F}}(\mathbf{p}) = F\left(\mathbf{p}^C\right) + \sum_{i=1}^{r} \left(\frac{\partial F\left(\mathbf{p}^C\right)}{\partial p_i} \Delta p_i + \frac{1}{2} \frac{\partial^2 F\left(\mathbf{p}^C\right)}{\partial p_i^2} \Delta p_i^2 \right). \tag{30}$$

The increment of the objective function can be written in the following form:

$$\Delta F(\mathbf{p}) = \sum_{i=1}^{r} \left(\frac{\partial F\left(\mathbf{p}^C\right)}{\partial p_i} \Delta p_i + \frac{1}{2} \frac{\partial^2 F\left(\mathbf{p}^C\right)}{\partial p_i^2} \Delta p_i^2 \right). \tag{31}$$

For variability of parameter p_1, Equation (31) is as follows:

$$\Delta F_1(\mathbf{p}) = \frac{\partial F\left(\mathbf{p}^C\right)}{\partial p_1} \left(p_1 - p^C\right) + \frac{1}{2} \frac{\partial^2 F\left(\mathbf{p}^C\right)}{\partial p_1^2} \left(p_1 - p^C\right)^2 . \tag{32}$$

It should be noted that, when using relationship (25) for calculating the lower and upper bounds of the objective function, the number of combinations is reduced to 2*r*.

3.3 Dynamic characteristics of structures with uncertain design parameters

In this section, formulae describing the lower and upper bounds of the chosen dynamic characteristics are determined. The natural frequency (ω), non-dimensional damping factor (γ), and eigenvectors (**q**) are taken into consideration. According the formula (26), it is possible to write:

$$\omega^I(\mathbf{p}^I)=\left[\omega(\mathbf{p}^C)+\sum_{i=1}^{r}\Delta\underline{\omega}_i(p_i^I),\ \ \omega(\mathbf{p}^C)+\sum_{i=1}^{r}\Delta\overline{\omega}_i(p_i^I)\right] \tag{33}$$

for the natural frequency,

$$\gamma^I(\mathbf{p}^I)=\left[\gamma(\mathbf{p}^C)+\sum_{i=1}^{r}\Delta\underline{\gamma}_i(p_i^I),\ \ \gamma(\mathbf{p}^C)+\sum_{i=1}^{r}\Delta\overline{\gamma}_i(p_i^I)\right] \tag{34}$$

for the non-dimensional damping factor and

$$\mathbf{q}^I(\mathbf{p}^I)=\left[\mathbf{q}(\mathbf{p}^C)+\sum_{i=1}^{r}\Delta\underline{\mathbf{q}}_i(p_i^I),\ \ \mathbf{q}(\mathbf{p}^C)+\sum_{i=1}^{r}\Delta\overline{\mathbf{q}}_i(p_i^I)\right] \tag{35}$$

for the eigenvector.

When the objective function is approximated using the first-order Taylor series expansion, the increments can be written as:

$$\Delta\omega(\mathbf{p})=\sum_{i=1}^{r}\frac{\partial\omega(\mathbf{p}^C)}{\partial p_i}\Delta p_i\,, \tag{36}$$

$$\Delta\gamma(\mathbf{p})=\sum_{i=1}^{r}\frac{\partial\gamma(\mathbf{p}^C)}{\partial p_i}\Delta p_i\,, \tag{37}$$

$$\Delta\mathbf{q}(\mathbf{p})=\sum_{i=1}^{r}\frac{\partial\mathbf{q}(\mathbf{p}^C)}{\partial p_i}\Delta p_i\,. \tag{38}$$

When the objective function is approximated using the second-order Taylor series expansion, the increments are as follows:

$$\Delta\omega(\mathbf{p})=\sum_{i=1}^{r}\frac{\partial\omega(\mathbf{p}^C)}{\partial p_i}\Delta p_i+\frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}\frac{\partial^2\omega(\mathbf{p}^C)}{\partial p_i\partial p_j}\Delta p_i\Delta p_j\,, \tag{39}$$

$$\Delta\gamma(\mathbf{p})=\sum_{i=1}^{r}\frac{\partial\gamma(\mathbf{p}^C)}{\partial p_i}\Delta p_i+\frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}\frac{\partial^2\gamma(\mathbf{p}^C)}{\partial p_i\partial p_j}\Delta p_i\Delta p_j\,, \tag{40}$$

$$\Delta\mathbf{q}(\mathbf{p})=\sum_{i=1}^{r}\frac{\partial\mathbf{q}(\mathbf{p}^C)}{\partial p_i}\Delta p_i+\frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}\frac{\partial^2\mathbf{q}(\mathbf{p}^C)}{\partial p_i\partial p_j}\Delta p_i\Delta p_j\,. \tag{41}$$

And when the objective function is approximated using the second-order Taylor series expansion with only the diagonal elements of the Hessian matrix, the increments are described as:

$$\Delta\omega(\mathbf{p})=\sum_{i=1}^{r}\left(\frac{\partial\omega(\mathbf{p}^C)}{\partial p_i}\Delta p_i+\frac{1}{2}\frac{\partial^2\omega(\mathbf{p}^C)}{\partial p_i^2}\Delta p_i^2\right), \tag{42}$$

$$\Delta\gamma(\mathbf{p})=\sum_{i=1}^{r}\left(\frac{\partial\gamma(\mathbf{p}^C)}{\partial p_i}\Delta p_i+\frac{1}{2}\frac{\partial^2\gamma(\mathbf{p}^C)}{\partial p_i^2}\Delta p_i^2\right), \tag{43}$$

$$\Delta\mathbf{q}(\mathbf{p})=\sum_{i=1}^{r}\left(\frac{\partial\mathbf{q}(\mathbf{p}^C)}{\partial p_i}\Delta p_i+\frac{1}{2}\frac{\partial^2\mathbf{q}(\mathbf{p}^C)}{\partial p_i^2}\Delta p_i^2\right). \tag{44}$$

The calculation of the lower and upper bounds of the dynamic characteristics requires the sensitivities of the first- and the second-order of the objective function $\left(\partial F\left(\mathbf{p}^C\right)/\partial p_i\, ,\partial^2 F\left(\mathbf{p}^C\right)/\left(\partial p_i \partial p_j\right)\right)$ with respect to the design parameters. These sensitivities are obtained by the direct differentiation method, which is described in detail in [16]. The formulae for sensitivities with respect to the parameters of the Kelvin damper model are shown in Appendix A.

4 EXAMPLE

In the example, an eight-storey frame with three bays is considered (see Fig. 4). The frame is designed according to the EC8 Part 1. Construction data, with the exception of the unit mass of the floor, were adopted on the basis of [17]. The height of the columns is 3m and the span of the beams is 5m. Young's modulus (E) for concrete is 31GPa. Dimensions of columns and their replacement stiffnesses are shown in Table 1.

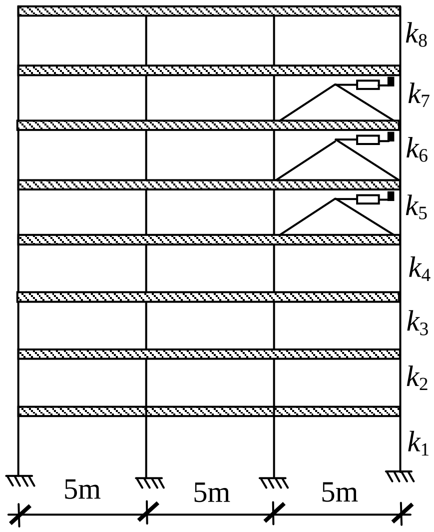

Figure 4. Diagram of the considered frame.

Storey level	Lateral column [cm]	Central column [cm]	Replacement stiffness [kN/m]
1,2	50x50	60x60	441119
3,4	45x45	53x53	275351
5,6	40x40	45x45	152948
7,8	35x35	40x40	93244

Table 1. Dimensions and replacement stiffness columns.

The mass per unit length of every floor equals 60000 kg/m. The dampers are attached on the fifth, sixth and seventh floors of the structure. Such locations were chosen according to paper [17]. A fractional Kelvin model of damper is taken into consideration (see Fig. 2a). The dampers‘ parameters were adopted in such a way that the first non-dimensional damping factor was $\gamma_1 \geq 0.04$. The following parameters are adopted for the dampers:

$k_0 = 20000\,\text{kN/m}$, $c_0 = 27000\,\text{kNs}^{\alpha}/\text{m}$ and $\alpha = 0.6$. The values of first natural frequencies ω_1 and the non-dimensional damping factor γ_1 for the structure without dampers and for the structure with the fractional Kelvin model are shown in Table 2. Damping in the structure is neglected.
The uncertainties of dampers' parameters are considered. They are shown in Table 3. The damper parameters are marked k_{0i} and c_{0i}, where i denotes the damper number. The order of the fractional derivative remains constant.

Structure without dampers	Structure with fractional dampers	
natural frequency ω_1 [rad/s]	natural frequency ω_1 [rad/s]	non-dimensional damping factor γ_1
3.10396	3.34162	0.04045

Table 2. First natural frequencies and non-dimensional damping factor without and with fractional dampers.

damper parameter	uncertainty [%]
c_{01}	10
c_{02}	15
c_{03}	15
k_{01}	15
k_{02}	20
k_{03}	15

Table 3. Uncertainties of damper parameters.

The parameters are assumed to vary independently. The interval parameters for dampers are listed in Tables 4-5.

bounded value	middle value	radius value
$c_{01}^I = [24300, 29700]$	$c_{01}^C = 27000$	$\Delta c_{01} = 2700$
$c_{02}^I = [22950, 31050]$	$c_{02}^C = 27000$	$\Delta c_{02} = 4050$
$c_{03}^I = [22950, 31050]$	$c_{03}^C = 27000$	$\Delta c_{03} = 4050$

Table 4. Interval parameters of damping factor [kNs$^{\alpha}$/m].

bounded value	middle value	radius value
$k_{01}^I = [17000, 2300]$	$k_{01}^C = 20000$	$\Delta k_{01} = 2000$
$k_{02}^I = [16000, 2400]$	$k_{02}^C = 20000$	$\Delta k_{02} = 1000$
$k_{03}^I = [17000, 2300]$	$k_{03}^C = 20000$	$\Delta k_{03} = 2000$

Table 5. Interval parameters of stiffness [kN/m].

The parameters' middle values are taken to find the solutions given in Table 2. In Table 6, the results of interval analysis are presented. The results obtained by the vertex method, first-order Taylor series expansion, second-order Taylor series expansion and second-order Taylor series expansion with only the diagonal element of the Hessian matrix are compared. The critical combinations of the interval parameters are shown for each case. The solution of the vertex method is assumed as a comparative solution and the error is calculated in relation to that.

	lower $\underline{\omega}_1$			upper $\overline{\omega}_1$		
	value of $\underline{\omega}_1$ [rad/s]	error [%]	Combination of parameters	value of $\overline{\omega}_1$ [rad/s]	error [%]	Combination of parameters
Vertex method	3.31306	-	$\underline{c}_{01}$, $\underline{c}_{02}$, $\underline{c}_{03}$ $\underline{k}_{01}$, $\underline{k}_{02}$, $\underline{k}_{03}$	3.36841	-	$\overline{c}_{01}$, $\overline{c}_{02}$, $\overline{c}_{03}$ $\overline{k}_{01}$, $\overline{k}_{02}$, $\overline{k}_{03}$
First-order Taylor series expansion	3.31832	0.16	$\underline{c}_{01}$, $\underline{c}_{02}$, $\underline{c}_{03}$ $\underline{k}_{01}$, $\underline{k}_{02}$, $\underline{k}_{03}$	3.36493	0.10	$\overline{c}_{01}$, $\overline{c}_{02}$, $\overline{c}_{03}$ $\overline{k}_{01}$, $\overline{k}_{02}$, $\overline{k}_{03}$
Second-order Taylor series expansion	3.31780	0.14	$\underline{c}_{01}$, $\underline{c}_{02}$, $\underline{c}_{03}$ $\underline{k}_{01}$, $\underline{k}_{02}$, $\underline{k}_{03}$	3.36441	0.12	$\overline{c}_{01}$, $\overline{c}_{02}$, $\overline{c}_{03}$ $\overline{k}_{01}$, $\overline{k}_{02}$, $\overline{k}_{03}$
Simplified second-order Taylor series expansion	3.31801	0.15	$\underline{c}_{01}$, $\underline{c}_{02}$, $\underline{c}_{03}$ $\underline{k}_{01}$, $\underline{k}_{02}$, $\underline{k}_{03}$	3.36462	0.11	$\overline{c}_{01}$, $\overline{c}_{02}$, $\overline{c}_{03}$ $\overline{k}_{01}$, $\overline{k}_{02}$, $\overline{k}_{03}$

Table 6. Lower and upper bounds of ω_1.

	lower $\underline{\gamma}_1$			upper $\overline{\gamma}_1$		
	value of $\underline{\gamma}_1$	error [%]	Combination of parameters	value of $\overline{\gamma}_1$	error [%]	Combination of parameters
Vertex method	0.03587	-	$\underline{c}_{01}$, $\underline{c}_{02}$, $\underline{c}_{03}$ $\overline{k}_{01}$, $\overline{k}_{02}$, $\overline{k}_{03}$	0.04480	-	$\overline{c}_{01}$, $\overline{c}_{02}$, $\overline{c}_{03}$ $\underline{k}_{01}$, $\underline{k}_{02}$, $\underline{k}_{03}$
First-order Taylor series expansion	0.03656	1.92	$\underline{c}_{01}$, $\underline{c}_{02}$, $\underline{c}_{03}$ $\overline{k}_{01}$, $\overline{k}_{02}$, $\overline{k}_{03}$	0.04435	1.00	$\overline{c}_{01}$, $\overline{c}_{02}$, $\overline{c}_{03}$ $\underline{k}_{01}$, $\underline{k}_{02}$, $\underline{k}_{03}$
Second-order Taylor series expansion	0.03639	1.45	$\underline{c}_{01}$, $\underline{c}_{02}$, $\underline{c}_{03}$ $\overline{k}_{01}$, $\overline{k}_{02}$, $\overline{k}_{03}$	0.04417	1.41	$\overline{c}_{01}$, $\overline{c}_{02}$, $\overline{c}_{03}$ $\underline{k}_{01}$, $\underline{k}_{02}$, $\underline{k}_{03}$
Simplified second-order Taylor series expansion	0.03633	1.28	$\underline{c}_{01}$, $\underline{c}_{02}$, $\underline{c}_{03}$ $\overline{k}_{01}$, $\overline{k}_{02}$, $\overline{k}_{03}$	0.04412	1.52	$\overline{c}_{01}$, $\overline{c}_{02}$, $\overline{c}_{03}$ $\underline{k}_{01}$, $\underline{k}_{02}$, $\underline{k}_{03}$

Table 7. Lower and upper bounds of γ_1.

Based on the obtained results, it should be noted that the solutions using the first-order Taylor series expansion and the second-order Taylor series expansion are close to the solution calculated with the use of the vertex method. In each of the four cases, the critical combination of the interval parameters is the same.

The lower and upper bounds of the first eigenvector are also calculated. The middle values of the eigenvector's elements are evaluated for the middle values of dampers' parameters. The lower and upper values of the eigenvector obtained by the vertex method $[\underline{\mathbf{q}}_1, \overline{\mathbf{q}}_1]$ are calculated for the relevant natural frequencies $[\underline{\omega}_1, \overline{\omega}_1]$ obtained by this method (solid lines). The obtained results are compared with the values calculated using the first-order Taylor series expansion (dashed line). The comparison is shown in Figures 6-7.

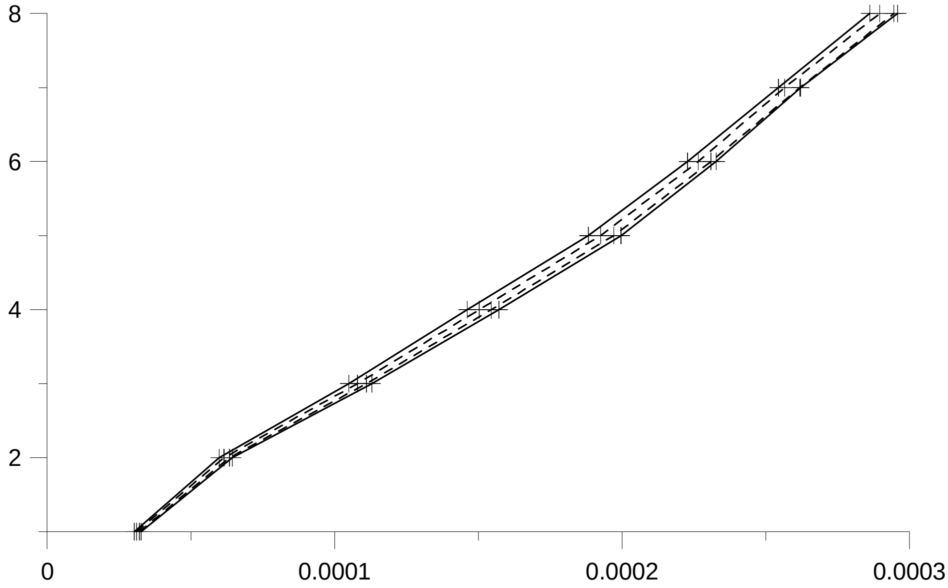

Figure 6. Lower and upper bounds of the real part of eigenvector $\mathbf{q}_1$.

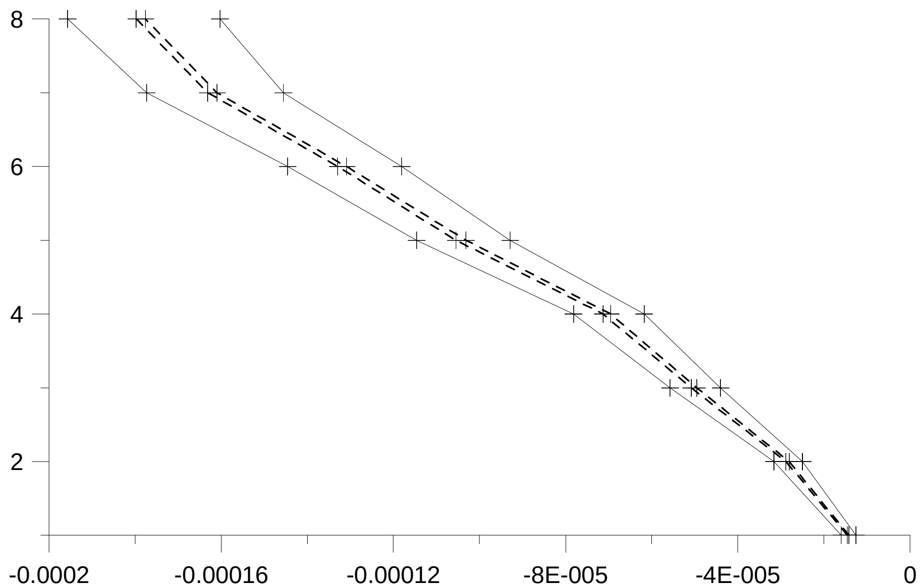

Figure 7. Lower and upper bounds of the imaginary part of eigenvector $\mathbf{q}_1$.

Evaluations of the lower and upper bounds of the eigenvector, carried out using the vertex method and the first-order Taylor series expansion differ by about 2-3% in the case of the real parts and 9-13% in the case of the imaginary parts.

5 CONCLUSIONS

The influence of uncertainties of the design parameters on the dynamic characteristics of structures with viscoelastic dampers are investigated. The presented formulae enable determination of the lower and upper bounds of the objective function. The different methods of evaluation of the lower and upper bounds are shown. The bounds are determined using the vertex method, first order Taylor series expansion, second-order Taylor series expansion and simplified second-order Taylor series expansion. The vertex method was chosen as a comparative solution but it should be noted that the exact solution in the form of a critical combination of the interval parameters can be obtained for values of parameters between the limit values. The methods based on the Taylor series expansion provide results which are close to those obtained by the vertex method. Errors of the calculations performed are similar in each of the three presented approximating functions but the second-order Taylor series expansion requires high computational costs, especially if all the elements of the Hessian matrix are taken into account.

REFERENCES

[1] R.E. Moore, *Interval analysis*, Englewood Cliffs, New York Prentice Hall, 1966.

[2] W. Dong, H. Shah, Vertex method for computing functions of fuzzy variables. *Fuzzy Sets and Systems*, **24** (1), 65-78, 1987.

[3] S.H. Chen, L. Ma, G.W. Meng, R. Guo, An efficient method for evaluating the natural frequency of structures with uncertain-but-bounded parameters, *Computers and Structures*, **87**, 582-590, 2009.

[4] S. Rao, L. Chen, Numerical solution of fuzzy linear equations in engineering analysis, *International Journal for Numerical Methods in Engineering*, **43**, 391-408, 1998.

[5] U. Akpan, T. Koko, I. Orisamolu, B. Gallant, Fuzzy finite element analysis of smart structures, *Smart Materials and Structures*, **10**, 273-284, 2001.

[6] S.H. Chen, J. Wu, Interval optimization of dynamic response for structures with interval parameters, *Computers and Structures*, **82**, 1-11, 2004.

[7] K. Fujita, I. Takewaki, An efficient methodology for robustness evaluation by advanced interval analysis using updated second-order Taylor series expansion, *Engineering Structures*, **33**, 3299-3310, 2011.

[8] K. Fujita, I. Takewaki, Robust passive damper design for building structures under uncertain structural parameter environments, *Earthquakes and Structures*, **3** (6), 805-820, 2012.

[9] N. Makris, M.C. Constantinou, fractional derivative Maxwell model for viscous damper, *ASCE Journal of Structural Engineering*, **117**, 2708-2724, 1991.

[10] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.

[11] J.S. Leszczyński, *An introduction to fractional mechanics*, Czestochowa University of Technology, 2011.

[12] R. Lewandowski, Z. Pawlak, Dynamic analysis of frames with viscoelastic dampers modeled by rheological models with fractional derivatives, *Journal of Sound and Vibration*, 330, 923-936, 2011.

[13] A. Fenander, Modal synthesis when modeling damping by use of fractional derivatives, *AIAA Journal*, **34**, 1051-1058, 1996.

[14] F. Cortes, M.J. Elajabarrieta, Finite element formulations for transient dynamic analysis in structural systems with viscoelastic treatments containing fractional derivative models, *International Journal of Numerical Methods in Engineering*, **69**, 2173-2195, 2007.

[15] Z. Pawlak, R. Lewandowski, The continuation method for the eigenvalue problem of structures with viscoelastic dampers, *Computers and Structures*, **125**, 53-61, 2013.

[16] R. Lewandowski, M. Łasecka-Plura, Design sensitivity analysis of structures with viscoelastic dampers, *Computers and Structures*, **164**, 95-107, 2016.

[17] Y. Ribakov, G. Agranovich, A method for efficient placement of active dampers in seismically excited structures, *Structural Control and Health Monitoring*, **17**, 513-531, 2010.

ACKNOWLEDGMENTS

The study is supported by the National Science Centre, Poland, as part of Project No. DEC/2013/09/B/ST8/01733, carried out in the years 2014-2016. This support is gratefully acknowledged.

APPENDIX A

In order to calculate the sensitivities of dynamic characteristics with respect to the design parameter, the following sets of equations can be solved [16]:

$$\begin{bmatrix} \mathbf{D}(s) & \dfrac{\partial \mathbf{D}(s)}{\partial s}\overline{\mathbf{q}}(s) \\ \overline{\mathbf{q}}^T(s)\dfrac{\partial \mathbf{D}(s)}{\partial s} & \dfrac{1}{2}\overline{\mathbf{q}}^T(s)\dfrac{\partial^2 \mathbf{D}(s)}{\partial s^2}\overline{\mathbf{q}}(s) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \overline{\mathbf{q}}(s)}{\partial p_i} \\ \dfrac{\partial s}{\partial p_i} \end{Bmatrix} = \begin{Bmatrix} \mathbf{h}_1(s) \\ h_2(s) \end{Bmatrix} \tag{A.1}$$

for the first-order sensitivities and

$$\begin{bmatrix} \mathbf{D}(s) & \dfrac{\partial \mathbf{D}(s)}{\partial s}\overline{\mathbf{q}}(s) \\ \overline{\mathbf{q}}^T(s)\dfrac{\partial \mathbf{D}(s)}{\partial s} & \dfrac{1}{2}\overline{\mathbf{q}}^T(s)\dfrac{\partial^2 \mathbf{D}(s)}{\partial s^2}\overline{\mathbf{q}}(s) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 \overline{\mathbf{q}}(s)}{\partial p_i \partial p_j} \\ \dfrac{\partial^2 s}{\partial p_i \partial p_j} \end{Bmatrix} = \begin{Bmatrix} \mathbf{h}_3(s) \\ h_4(s) \end{Bmatrix} \tag{A.2}$$

for the second-order sensitivities.
The right-hand sides of the above equations are as follows:

$$\mathbf{h}_1(s) = -\mathbf{R}_1(s)\overline{\mathbf{q}}(s), \tag{A.3}$$

$$h_2(s) = -\frac{1}{2}\overline{\mathbf{q}}^T(s)\mathbf{R}_2(s)\overline{\mathbf{q}}(s), \tag{A.4}$$

$$\mathbf{h}_3(s) = -\left[\mathbf{R}_{31}(s)\overline{\mathbf{q}}(s) + \mathbf{R}_{32}(s)\frac{\partial \overline{\mathbf{q}}(s)}{\partial p_i} + \mathbf{R}_{33}(s)\frac{\partial \overline{\mathbf{q}}(s)}{\partial p_j}\right], \tag{A.5}$$

$$h_4(s) = -\left[\overline{\mathbf{q}}(s)^T \mathbf{R}_{41}(s)\overline{\mathbf{q}}(s) + \overline{\mathbf{q}}(s)^T \mathbf{R}_{42}(s)\frac{\partial \overline{\mathbf{q}}(s)}{\partial p_i} + \overline{\mathbf{q}}(s)^T \mathbf{R}_{43}(s)\frac{\partial \overline{\mathbf{q}}(s)}{\partial p_j} + \frac{\partial \overline{\mathbf{q}}(s)^T}{\partial p_i}\frac{\partial \mathbf{D}(s)}{\partial s}\frac{\partial \overline{\mathbf{q}}(s)}{\partial p_j} \right] \quad \text{(A.6)}$$

where:

$$\mathbf{R}_1(s) = \frac{\partial \mathbf{D}(s)}{\partial p_i}, \qquad \mathbf{R}_2(s) = \frac{\partial^2 \mathbf{D}(s)}{\partial p_i \partial s}, \quad \text{(A.7)}$$

$$\mathbf{R}_{31}(s) = \frac{\partial^2 \mathbf{D}(s)}{\partial p_i \partial p_j} + \frac{\partial^2 \mathbf{D}(s)}{\partial p_i \partial s}\frac{\partial s}{\partial p_j} + \frac{\partial^2 \mathbf{D}(s)}{\partial p_j \partial s}\frac{\partial s}{\partial p_i} + \frac{\partial^2 \mathbf{D}(s)}{\partial s^2}\frac{\partial s}{\partial p_i}\frac{\partial s}{\partial p_j}, \quad \text{(A.8)}$$

$$\mathbf{R}_{32}(s) = \frac{\partial \mathbf{D}(s)}{\partial p_j} + \frac{\partial \mathbf{D}(s)}{\partial s}\frac{\partial s}{\partial p_j}, \qquad \mathbf{R}_{33}(s) = \frac{\partial \mathbf{D}(s)}{\partial p_i} + \frac{\partial \mathbf{D}(s)}{\partial s}\frac{\partial s}{\partial p_i} \quad \text{(A.9)}$$

$$\mathbf{R}_{41}(s) = \frac{1}{2}\left[\frac{\partial^3 \mathbf{D}(s)}{\partial s^3}\frac{\partial s}{\partial p_i}\frac{\partial s}{\partial p_j} + \frac{\partial^3 \mathbf{D}(s)}{\partial s^2 \partial p_j}\frac{\partial s}{\partial p_i} + \frac{\partial^3 \mathbf{D}(s)}{\partial s^2 \partial p_i}\frac{\partial s}{\partial p_j} + \frac{\partial^3 \mathbf{D}(s)}{\partial s \partial p_i \partial p_j} \right], \quad \text{(A.10)}$$

$$\mathbf{R}_{42}(s) = \frac{\partial^2 \mathbf{D}(s)}{\partial s \partial p_j} + \frac{\partial^2 \mathbf{D}(s)}{\partial s^2}\frac{\partial s}{\partial p_j}, \qquad \mathbf{R}_{43}(s) = \frac{\partial^2 \mathbf{D}(s)}{\partial s \partial p_i} + \frac{\partial^2 \mathbf{D}(s)}{\partial s^2}\frac{\partial s}{\partial p_i}. \quad \text{(A.11)}$$

In the case of the Kelvin model of damper, the matrices $\mathbf{R}_1(s)$ and $\mathbf{R}_2(s)$ which are required for solving the first-order sensitivities, are as follows:

$$\mathbf{R}_1(s) = s^{\alpha_i}\mathbf{L}_i, \qquad \mathbf{R}_2(s) = \alpha_i s^{\alpha_i - 1}\mathbf{L}_i \quad \text{(A.12)}$$

when the design parameter p_i is c_{0i} and

$$\mathbf{R}_1(s) = \mathbf{L}_i, \qquad \mathbf{R}_2(s) = \mathbf{0} \quad \text{(A.13)}$$

when the design parameter p_i is k_{0i}.

When the second-order sensitivities is of interest, the matrices $\mathbf{R}_{31}(s)$, $\mathbf{R}_{32}(s)$, $\mathbf{R}_{33}(s)$, $\mathbf{R}_{41}(s)$, $\mathbf{R}_{42}(s)$ and $\mathbf{R}_{43}(s)$ can be written as:

$$\mathbf{R}_{31}(s) = \frac{\partial^2 \mathbf{D}}{\partial s^2}\frac{\partial s}{\partial c_{0i}}\frac{\partial s}{\partial c_{0j}} + \alpha_j s^{\alpha_j - 1}\mathbf{L}_j\frac{\partial s}{\partial c_{0i}} + \alpha_i s^{\alpha_i - 1}\mathbf{L}_i\frac{\partial s}{\partial c_{0j}}, \quad \text{(A.14)}$$

$$\mathbf{R}_{32}(s) = s^{\alpha_j}\mathbf{L}_j + \frac{\partial \mathbf{D}(s)}{\partial s}\frac{\partial s}{\partial c_{0j}}, \qquad \mathbf{R}_{33}(s) = s^{\alpha_i}\mathbf{L}_i + \frac{\partial \mathbf{D}(s)}{\partial s}\frac{\partial s}{\partial c_{0i}}, \quad \text{(A.15)}$$

$$\mathbf{R}_{41}(s) = \frac{1}{2}\left[\alpha_j(\alpha_j - 1)s^{\alpha_j - 2}\mathbf{L}_j\frac{\partial s}{\partial c_{0i}} + \alpha_i(\alpha_i - 1)s^{\alpha_i - 2}\mathbf{L}_i\frac{\partial s}{\partial c_{0j}} + \frac{\partial^3 \mathbf{D}}{\partial s^3}\frac{\partial s}{\partial c_{0i}}\frac{\partial s}{\partial c_{0j}} \right], \quad \text{(A.16)}$$

$$\mathbf{R}_{42}(s) = \alpha_j s^{\alpha_j - 1}\mathbf{L}_j + \frac{\partial^2 \mathbf{D}(s)}{\partial s^2}\frac{\partial s}{\partial c_{0j}}, \qquad \mathbf{R}_{43}(s) = \alpha_i s^{\alpha_i - 1}\mathbf{L}_i + \frac{\partial^2 \mathbf{D}(s)}{\partial s^2}\frac{\partial s}{\partial c_{0i}} \quad \text{(A.17)}$$

when the design parameters p_i and p_j are c_{0i} and c_{0j}, respectively, and

$$\mathbf{R}_{31}(s) = \frac{\partial^2 \mathbf{D}}{\partial s^2}\frac{\partial s}{\partial c_{0i}}\frac{\partial s}{\partial k_{0j}} + \alpha_i s^{\alpha_i - 1}\mathbf{L}_i\frac{\partial s}{\partial k_{0j}}, \quad \text{(A.18)}$$

$$\mathbf{R}_{32}(s)=\mathbf{L}_j+\frac{\partial\mathbf{D}(s)}{\partial s}\frac{\partial s}{\partial k_{0j}}, \quad \mathbf{R}_{33}(s)=s^{\alpha_i}\mathbf{L}_i+\frac{\partial\mathbf{D}(s)}{\partial s}\frac{\partial s}{\partial c_{0i}}, \tag{A.19}$$

$$\mathbf{R}_{41}(s)=\frac{1}{2}\left[\alpha_i(\alpha_i-1)s^{\alpha_i-2}\mathbf{L}_i\frac{\partial s}{\partial k_{0j}}+\frac{\partial^3\mathbf{D}}{\partial s^3}\frac{\partial s}{\partial c_{0i}}\frac{\partial s}{\partial k_{0j}}\right], \tag{A.20}$$

$$\mathbf{R}_{42}(s)=\frac{\partial^2\mathbf{D}(s)}{\partial s^2}\frac{\partial s}{\partial k_{0j}}, \quad \mathbf{R}_{43}(s)=\alpha_i s^{\alpha_i-1}\mathbf{L}_i+\frac{\partial^2\mathbf{D}(s)}{\partial s^2}\frac{\partial s}{\partial c_{0i}} \tag{A.21}$$

when the design parameters p_i and p_j are c_{0i} and k_{0j}, respectively, and

$$\mathbf{R}_{31}(s)=\frac{\partial^2\mathbf{D}}{\partial s^2}\frac{\partial s}{\partial k_{0i}}\frac{\partial s}{\partial k_{0j}}, \quad \mathbf{R}_{32}(s)=\mathbf{L}_j+\frac{\partial\mathbf{D}(s)}{\partial s}\frac{\partial s}{\partial k_{0j}}, \quad \mathbf{R}_{33}(s)=\mathbf{L}_i+\frac{\partial\mathbf{D}(s)}{\partial s}\frac{\partial s}{\partial k_{0i}} \tag{A.22}$$

$$\mathbf{R}_{41}(s)=\frac{1}{2}\left[\frac{\partial^3\mathbf{D}}{\partial s^3}\frac{\partial s}{\partial k_{0i}}\frac{\partial s}{\partial k_{0j}}\right], \quad \mathbf{R}_{42}(s)=\frac{\partial^2\mathbf{D}(s)}{\partial s^2}\frac{\partial s}{\partial k_{0j}}, \quad \mathbf{R}_{43}(s)=\frac{\partial^2\mathbf{D}(s)}{\partial s^2}\frac{\partial s}{\partial k_{0i}} \tag{A.23}$$

when the design parameters p_i and p_j are k_{0i} and k_{0j}, respectively.
The derivatives of matrix $\mathbf{D}(s)$ with respect to s in the case of the Kelvin model of damper are as follows:

$$\frac{\partial\mathbf{D}(s)}{\partial s}=2s\mathbf{M}+\mathbf{C}+\sum_{k=1}^{m}c_{0k}\alpha_k s^{\alpha_k-1}\mathbf{L}_k, \tag{A.24}$$

$$\frac{\partial^2\mathbf{D}(s)}{\partial s^2}=2\mathbf{M}+\sum_{k=1}^{m}c_{0k}\alpha_k(\alpha_k-1)s^{\alpha_k-2}\mathbf{L}_k, \tag{A.25}$$

$$\frac{\partial^3\mathbf{D}(s)}{\partial s^3}=\sum_{k=1}^{m}c_{0k}\alpha_k(\alpha_k-1)(\alpha_k-2)s^{\alpha_k-3}\mathbf{L}_k. \tag{A.26}$$

The derived formulae enable determination of all the elements of the Hessian matrix of dynamic characteristics with respect to the chosen design parameters.