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EVOLUTIONARY TOPOLOGY OPIMIZATION USING PARAMETERIZED B-SPLINE SURFACE

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Abstract. *In solid mechanics, the topology optimization process involves two separate fields: displacement fields as well as density fields (material distribution). The level set topology optimization method is based on implicit functions whose iso-lines describe the respective geometry. The traditional level set method is performed by using the Hamilton-Jacobi transport equation, which implies the use of gradient optimization methods that are prone to getting stuck in local extremes. Furthermore, resulting optimal shapes are strongly dependent on the initial guess.*

In order to provide a robust topology optimization method insensitive to local optima, parameterized level set functions and evolutionary optimization methods (genetic algorithms) are used. Using B-spline parameterized level set functions enables the number of variables to be reduced and the use of genetic algorithms possible.

The method is presented on a simple 2D cantilever plate. The objective function is minimum of compliance in one case and minimum of volume in the second case. Due the geometric change, the finite element method requires remeshing. To avoid remeshing, the "Ersatz material" technique is applied. It changes the material properties by setting Young's module of elasticity to zero for elements of the FE mesh where the material should be removed.

The proposed method of topology optimization using genetic algorithms is feasible, but a very time-consuming process. For more complex problems, the proposed procedure with a smaller number of generations can be used as a generator of the initial solutions for subsequent faster procedures.

1 INTRODUCTION

The task of topology optimization is to distribute a material in order to optimize the respective response. In structural mechanics the response is usually displacements. Topology optimization as the inverse process of synthesis is a very complex and creative process, which involves many aspects such as experience, ideas, rules, methods, analysis, interaction and other conceptual elements.In literature several methods are developed for topology optimization :

- Bubble method, where in order to optimize e.g. plate with holes, the optimizer is changing the size of the gaps in the observed domain [1], [2], [3], [4], [5]
- Evolutionary structural optimization method (ESO) [6] and [7], where the material is removed (nibbling on) using a material property (modulus of elasticity) where the finite element method gives a smaller stress response. The main problem using this method for topology optimization is that after each optimization step, the geometry is irreversible.
- Solid Isotropic Material with Penalization method (SIMP) [2], [3], [8], [9], [10] using the compliance minimization problem to determine the best material distribution that provides a better stiffness/weight ratio. The variable that describes the distribution of material is the material stiffness, which can have the value of solid material or zero (no material).
- Level set methods [6], [11], [12], [13], [14], [15], [16], [17], [18], [19] based on implicit functions whose iso-lines describe the geometry. Thus, the level set function is able to describe the topological changes of the geometry during optimization process.

A topology description function method [20] was developed from the Level set method. Some researchers studied non gradient methods for topology optimization [21], but in that case the problem is time consuming. A possible solution lies in the parallelization of process [22].

Topology optimization problems that are discussed in the literature can include several areas: parameterization of geometry of the domain, interpolation methods, solving equilibrium and optimization algorithm to optimize shape and topology.
Topology optimization is nearest to the abstract conceptual phase in design. In the phase of shape optimization, effective parameterization of shape becomes crucial for efficient optimization of geometry. Parameterization can be done using some of the functions: NURBS, B-spline, Bezier surface, Radial basis function (RBF) [14], [16], [23], [24].

2 MATHEMATICAL FORMULATION

Topology optimization is a complex process that involves a series of numerical procedures. In the next flowchart, the numerical procedure for topology optimization in structural mechanics is presented.

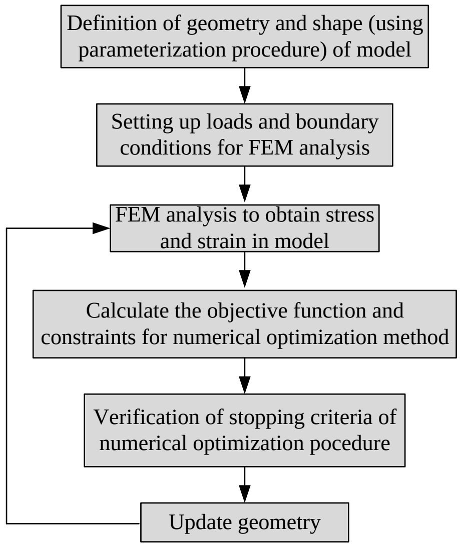

Figure 1: Flowchart of topology optimization in structural mechanics

2.1 Level set function (Shape definition)

Level set is an implicit function that describes the geometry. The Level set function lies in n+1 dimension with respect to geometry that describes, and its iso-line defines the geometry.
Where n is dimension of the observed domain. This feature of the level set function provides the description of the topological changes of the geometry during the optimization process (creating new holes in the domain), without requiring a new parameterization of the geometry. This feature is very important because otherwise the optimization process will have a variable number of unknowns (Figure 2).

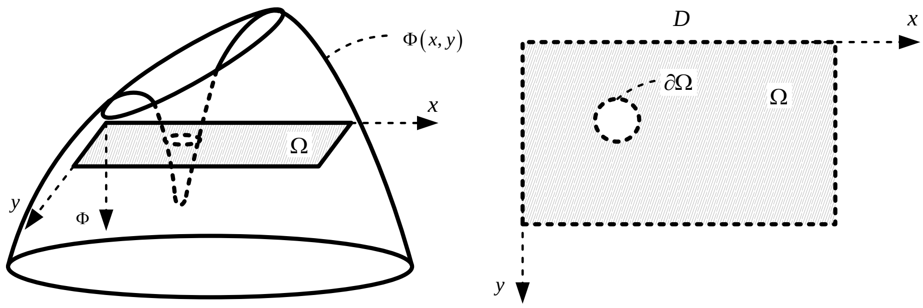

Figure 2: The general implicit function Φ which penetrates the domain D

A rectangular domain D, shown in Figure 2, can be partitioned in the following way:
- Material (inside) which is defined as $\Phi(\mathbf{x},t)<0 \ \ \forall x \in \Omega \backslash \partial\Omega$,
- Void (outside) which is defined as $\Phi(\mathbf{x},t)>0 \ \ \forall x \in D \backslash \Omega$, and
- Boundary $\Phi(\mathbf{x},t)=0 \ \ \forall x \in \partial\Omega \cap D$.

where Ω is the area filled with the material, ∂Ω is the border of the area Ω, and D is the fixed domain within which the shape and topology of the observed area Ω are defined.
The information whether the observed point lies inside or outside the area can be provided using the Heaviside step function H [12]- [20]:

$$H(\phi(\boldsymbol{x})) = \begin{cases} 1 & if \ \ \phi(\boldsymbol{x}) \leq 0, \\ 0 & if \ \ \phi(\boldsymbol{x}) > 0, \end{cases} \tag{1}$$

At a time t, the contour function is:

$$\Phi(\boldsymbol{x}, t) = 0 \tag{2}$$

2.2 Parametric shape B spline surface

Polynomial expressions, such as the NURBS and B spline surfaces, can be used to describe more complex geometries. In this paper, level set functions are described using B-spline surfaces. In that case, a representive point of the B spline surface or approximation point $\boldsymbol{P}(x,y)$, , is defined in the following way [24]:

$$\boldsymbol{P}(x,y)=\sum_{i_n=0}^{n}\sum_{i_m=0}^{m}\boldsymbol{N}_{i_n,j_n}(x)\cdot\boldsymbol{N}_{i_m,j_m}(y)\cdot\boldsymbol{p}_{i_n i_m} \tag{3}$$

wherw $0\leq i_n\leq$ n and $0\leq i_m\leq$ m are rectangular patch. The polynomial components of $\boldsymbol{P}(x,y)$ are defined piecewise on the sets $[x_i, x_{i+1}]\times[y_i, y_{i+1}]$. The vector $x\in[x_0, x_n]$ and $y\in[y_0, y_m]$ is called a knot vector, and it is refined in x and y direction, and $\boldsymbol{p}_{i_n i_m}$ are control points of surface.

The shape function of zero order in x direction is:

$$\boldsymbol{N}_{i,0}(x)=\begin{cases}1, & x_i\leq x<x_{i+1}\\ 0, & otherwise\end{cases}, \tag{4}$$

and for other orders ($0\leq i\leq n+j$),

$$\boldsymbol{N}_{i,j}(x)=\frac{x-x_i}{x_{i+j-1}-x_i}\boldsymbol{N}_{i,j-1}(x)+\frac{x_{i+j}-x}{x_{i+j}-x_{i+1}}\boldsymbol{N}_{i+1,j-1}(x) \tag{5}$$

The spline order is $1\leq j_x\leq n$ and $1\leq j_y\leq m$, and for single node is presented uniformly as:

$$x_i=\begin{cases}0, & 0\leq i\leq j\\ \dfrac{i-j}{n+1-j}, & j+1\leq i\leq n\\ 1 & n+1\leq i\leq n+j+1\end{cases} \tag{6}$$

Analog are able to introduce the shape functions for second direction (y).

B splines have the property of local control since basic functions have non zero values only locally for a few adjacent control points $\boldsymbol{p}$.

2.3 Objective function and constraints

In structural mechanics, the objective function for topology optimization problems is usually represented as minimization of compliance with constraints related to displacements and available volume of material [2]:

$$\min_{\substack{\boldsymbol{v}\in U\\ E\in E_{ad}}} l(\boldsymbol{v},\Phi) \tag{7}$$

$$a_B(\boldsymbol{u},\boldsymbol{v},\Phi)=l(\boldsymbol{v},\Phi) \quad \Rightarrow$$

$$\int_D \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{E}(\boldsymbol{X})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})H(\Phi)d\Omega=\int_\Omega \boldsymbol{f}_v\cdot\boldsymbol{v}\cdot H(\Phi)d\Omega+\int_\Gamma \boldsymbol{f}_s\cdot\boldsymbol{v}\cdot\delta(\Phi)|\nabla\Phi|d\Gamma \tag{8}$$

The Previous expression represents a weak variational form. The index B indicates that bilinear form a_B depends on the design variables. U is respective virtual (kinematic admissible) displacement field, $\boldsymbol{f}_v$ are the body forces, $\boldsymbol{f}_s$ is boundary traction on the traction part Γ of the boundary, E_{ad} is admissible stiffness, $\boldsymbol{E}$ is stiffness matrix, $\boldsymbol{u}$ is equilibrium displacement field, $\boldsymbol{v}$ is virtual displacement field (kinematically permissible), $a(\boldsymbol{u},\boldsymbol{v})$ is virtual work of internal forces, $l(\boldsymbol{v},\Phi)$ is virtual work of external load, $\boldsymbol{X}$ are design variables of shape and topology (represented distribution of material), and $\delta(\Phi)$ is partial derivation of the Heaviside function $H(\Phi)$ (Dirac function).

The linear form of external loads is equal to the sum of the external load $\boldsymbol{f}_v$ per unit volume Ω and external load $\boldsymbol{f}s$ per unit contour Γ.
From the above follows that for steady state bilinear forms optimal allocation of material is seeking to provide a maximum stiffness [3]:

$$\max_{E\in E_{ad}}\left(\min_{u\epsilon U,E}\{a_B(\boldsymbol{u},\boldsymbol{v})-l(\boldsymbol{v})\}\right) \tag{9}$$

In this case, the displacement field and allowable distribution of material are constraints. The function to be minimized is compliance, which provides the optimal distribution of material that will satisfy equilibrium (nested formulation).
The topology optimization problem can be defined as the minimum of volume with respect to the stress constraints. If the domain is divided into finite elements (FE), the minimum volume formulation is:

$$\min_{\mathrm{E}} \mathrm{V}$$
$$\boldsymbol{K}(E_e)\boldsymbol{u}=\boldsymbol{f}_v \quad E_e\in E_{ad} \tag{10}$$

with respect to the stress constraint:

$$\sigma<\sigma_{allowed} \tag{11}$$

where $\boldsymbol{K}$ is stiffness matrix which depends on the material properties, $\boldsymbol{E}_e$ represents the material properties of elements.

2.4 Level set method

The objective function, using a Lagrange multiplier method, now looks like [14]:

$$\underbrace{min}_{\Phi(\mathrm{x})}: \qquad L(\boldsymbol{u},\Phi,\lambda)=\frac{1}{2}\int_D\left(\varepsilon(\boldsymbol{u})\right)^T\boldsymbol{E}\varepsilon(\boldsymbol{v})H(\Phi)d\,\Omega+\lambda\left[\int_D H(\Phi)d\,\Omega-\zeta V_0\right] \tag{12}$$

If the level set function (2) is dynamically changed in time, then a continuous velocity field $v_n=dx/dt$ is introduced and time-dependent change of the level set surface is obtained (Hamilton-Jacobi partial differential equation H-J) [15], [16]:

$$\frac{\partial\Phi(\boldsymbol{x},t)}{\partial t}+\nabla\Phi(\boldsymbol{x},t)\cdot\frac{d\boldsymbol{x}}{dt}=0, \tag{13}$$

where t is the virtual time step.
The H-J partial differential equation can be solved using the "upwinding" procedure (finite differences method in the spatial domain and time stepping scheme), or using a parameterization function to describe level set surface $\Phi(\mathbf{x},t)$.
The upwinding procedure in a domain that is optimized cannot create a new hole, which is its major disadvantage.
The H-J equation can be parameterized using some of the parametric equations (e.g. B-spline) (3), so that separation of space and time variables is achieved. That procedure provides an uncoupled system of ordinary differential equations [14], [15], from (3) and (13):

$$\boldsymbol{\phi}^T(\boldsymbol{x})\frac{d\boldsymbol{P}(t)}{dt}+v_n\left|\left(\nabla\boldsymbol{\phi}(\boldsymbol{x})\right)^T\boldsymbol{P}(t)\right|=0 \tag{14}$$

where $\boldsymbol{P}(t)$ are B-spline approximation function.
The velocity (v_n) can be obtained by considering the minimum of the potential energy (imposition of "external" velocity field).
Using a permutation of boundaries of domain on the objective function (12), the velocity of all points of level set function [25] is obtained:

$$v_n=\boldsymbol{\varepsilon}^T\boldsymbol{E}\boldsymbol{\varepsilon}-\lambda \tag{15}$$

From previous equations can be seen that the level set method is based on a gradient optimization method. Therefore, the main disadvantage of the level set method as well as all gradient methods of topology optimization is dependence on the initial solution.
A topology optimization method using B-spline parameterization of the level set surface is proposed. The optimization variables are control points of the B-spline. Moving the B-spline nodes causes a change of Level set function, as well as the observed domain (iso-line of level set function). Such a formulation allows a usage of genetic algorithms as optimization method. Genetic algorithms are global optimization methods, but numerically highly expensive. B-spline parameterization methods are reducing the required number of variables, which allows application of genetic algorithm optimization in reasonable time.
In the presented method, optimization variables are changing the geometry by using the level set function (2). The binary code of one individual, *z* coordinate of parameter $\boldsymbol{p}$ from equation (3) with resolution of 8 bits, is presented in Figure 3:

$\mathbf{p}_{z1}$								$\mathbf{p}_{z2}$									$\mathbf{p}_{zN}$							
0	0	1	0	1	1	0	1	0	0	0	0	1	0	1	1	...	0	1	1	0	0	0	1	0

Figure 3 Optimization variables in binary code

Using the operators of selection, crossover and mutation, the genetic algorithm changes the population (bits in binary code) of individuals in the next generations. Some of binary coded phenotypes are shown in Figure 4.

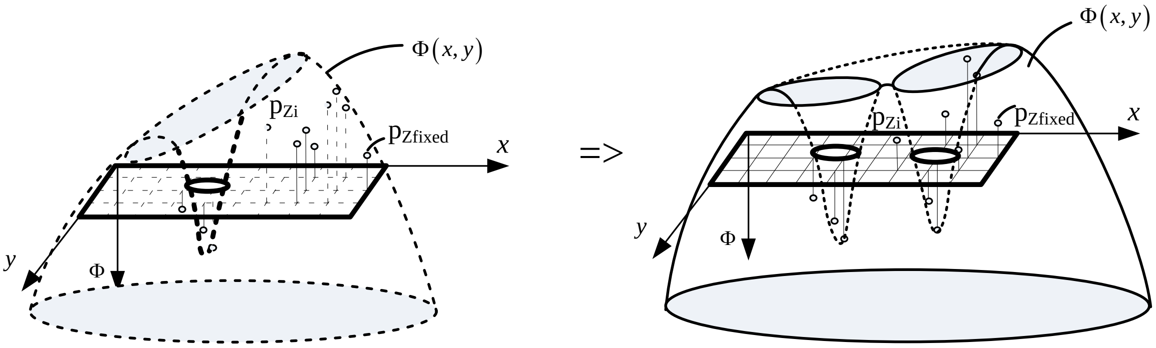

Figure 4 Geometry change with B-spline parametrized Level set implicit function

Such a description of the geometry also allows topological changes of the model.
The objective function (12) in each time step is calculated utilizing FEM analysis and simple structurally active elements counter.
Also, the finite element method requires considerable time for executing calculations, especially in situations when the change of geometry occurs in each step of the optimization procedure, which results in a high time-cost re-meshing method. To avoid re-meshing, each element of stiffness matrix is represented with appropriate value (1):

- null – if level set function larger than zero, or
- elasticity modulus of material (E) – if level set function equal or smaller than zero.

This procedure is called „Ersatz material“, and it is not numerically expensive.

3 TEST CASE AND RESULTS

As a test case for the proposed topology optimization procedure, a 2D cantilever plate is defined as shown below:

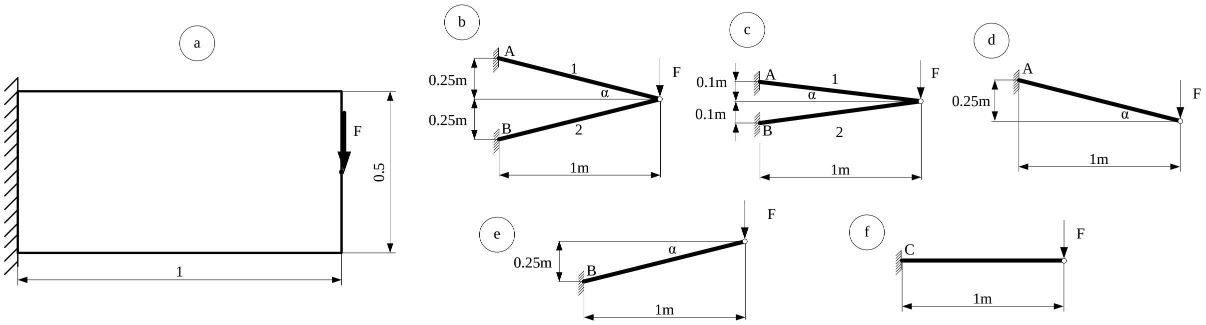

Figure 5: The geometry, loads and boundary conditions of cantilever plate for topology optimization (a) and some of discrete solutions (b, c, d, e)

Some of the expected potential discrete solutions are shown in Figure 5 b, c, d, e, f which can be found as optimal local extremes.

For structural topology optimization of the cantilevered beam defined in Figure 5, a B-spline parameterized level set surface was used. Optimization variables are the *z* coordinates of B-spline control points $\boldsymbol{p}_{ij}$ in Eq. (3). Changing of the B-spline control points has the effect to change the geometry of model. As the geometry of the model is discretized by finite elements, each element of discrete domain has its own material properties, respecting the Heaviside function (1) of the FE geometric center point. If the coordinate of the center point of the observed FE takes a negative value of the approximated level set function, the value of Young's modulus of elasticity E= 70[GPa] is assigned to the element. Otherwise, if the coordinate of the center point of the observed FE takes a positive value of the approximated level set function, then zero value is assigned to the element material property (E= 0, i.e. that element is non-load-bearing part of the structure).

In this paper, a genetic algorithm for structural topology optimization is used. Genetic algorithms (GA), as global search methods for nonlinear optimization, are not sensitive to local extremes of the objective function.

All numerical procedures were performed using in-house MATLAB code, inspired by [8], [9]. The number of finite elements of the discretized 2D plate (Figure 5) is 3200. If the objective function is defined as the minimum of the volume with stress constraint, and 576 B-spline control points i.e. 576 optimization variables are chosen to describe a level set function the proposed procedure provides:

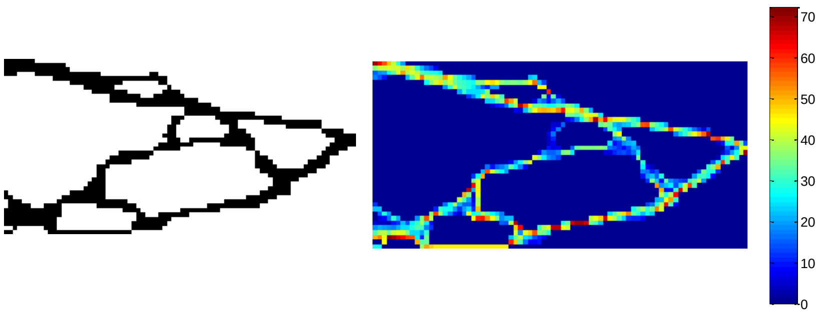

Figure 6: Topology optimization of 2D cantilever plate with 576 variables of optimization for minimum volume problem

Figure 6 shows the solution of topology optimization 2D cantilever plate using a genetic algorithm after 200 generations. The number of the population is 2304, and allowable stress is 70 GPa. The time required for the represented numerical procedure was 35 hours on Intel 3.10 GHz 4 core CPU. From Figure 6 it can be seen that convergence is not achieved due to large number of optimization variables. In order to achieve convergence, the number of generations should be increased, but in that case the numerical procedure might be too long. However, an expected solution is obtained. The reduction of volume is 20.3% (648 elements of the initial 3200 elements).
The numerical procedure can be accelerated by using parallelization of FE processes on a cluster [22], or by reducing the number of variables. If 144 B-spline control points, i.e. optimization variables, are chosen to describe a level set function, the time required for represented numerical procedure was reduced to 14 hours (Figure 7):

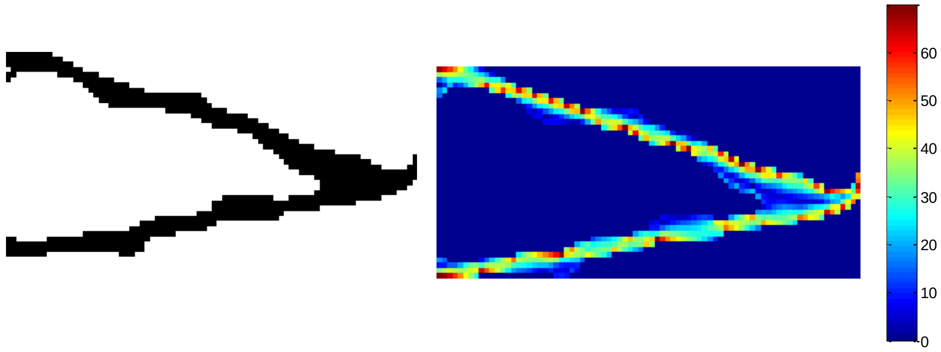

Figure 7 Topology optimization of 2D cantilever plate with 144 variables of optimization

Number of population is 1296 over 200 generations, which was sufficient for the procedure to converge to the global optimum. The reduction of volume is 20% (640 elements). Better optimization results were achieved using a reduced numbers of variables (144) due to a lower search field. A better, final solution can be provided using numerical filters.
The next example provides solution for a minimum compliance problem with volume constraint ($\zeta = 0.3$, Eq. 11) –Figure 8:

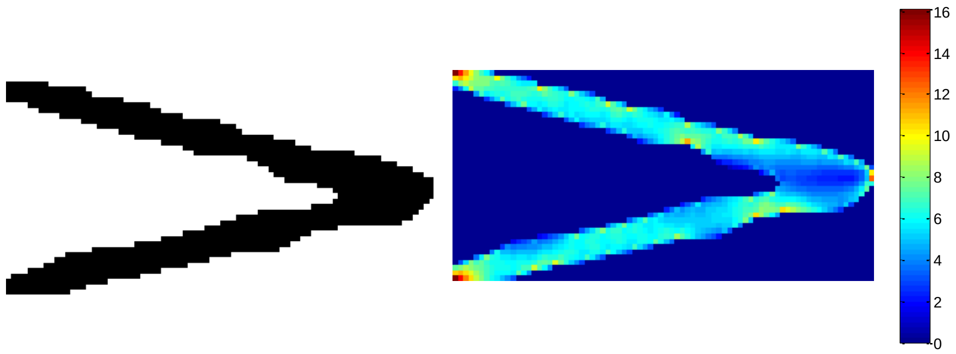

Figure 8: Topology optimization of 2D cantilever plate with 144 optimization variables for minimum compliance problem

Figure 8 shows that the parametrized level set function and the genetic algorithm are able to provide a good solution of topology optimization. The number of the population is 2304 with 200 generations. The time required for represented numerical procedure was 14 hours on Intel 3.10 GHz 4 core CPU.

The solutions presented in Figure 7 and Figure **8** are very similar to the structural type shown in Figure 5b. Simple structural analysis, considering four types of the truss or beam structures shown in Figure 5b-5f, can confirm that this structural type is optimal one with respect to the previously defined objectives and constraints. This conclusion can be used for justification of the quality of the solutions obtained by the method presented in this paper. Moreover, proposed method may serve as a generator of initial solutions for the traditional gradient topology optimization methods.

4 CONCLUSIONS

Topology optimization as a process of synthesis of shape and topology in structural mechanics is very complex and demanding numerical procedure. In this work a method for topology optimization of a 2D structure (cantilever plate) using a paremetrized level set surface for geometric description was proposed. The optimization variables in the proposed procedure are *z*-coordinates of B-spline control points $\boldsymbol{p}_{ij}$ defined in Eq. (3). The parameters of the B-spline surface that describe the level set surface produce a change of 2D plate geometry. The procedure avoids the Hamilton-Jacobi transport equation and allows the application of non-gradient and evolutionary optimization methods. Parametrization of level set surface using B-spline results in reduced number of variables of optimization process, as well as good shape representation of model. Moreover, reducing the number of the optimization variables decreases the required numerical analysis time. A reduced number of variables allows application of numerically demanding genetic algorithm method. The global optimization method (GA) gives a great advantage over conventional topology optimization methods mostly based on gradient methods. Also, the proposed method allows the application of the minimum volume as objective function, which is, due to the discrete character of the objective function and the complexity of the problem, impossible for gradient optimization methods without additional numerical procedures. Instead of remeshing FE during the geometry changes, the "Ersatz material" techniques are used as computation ally cheaper procedures.

The results show the applicability of the proposed topology optimization method by using a parameterized level set function and genetic algorithm. The main application of the proposed method of topology optimization is to obtain good initial solutions for the traditional level set method, which has a large dependence on the selected initial solution. The acceleration of the proposed procedure is possible by using a cluster of computers, or additional numerical algorithms for parameterization and optimization.

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