

MODELLING STIFFENED LIGHTWEIGHT STRUCTURES WITH ISOGEOMETRIC ANALYSIS VIA MORTAR METHODS

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Abstract. *A method to model stiffened panels with NURBS-based isogeometric analysis is presented. Stiffeners and panel are modeled as separate patches and coupled by Mortar methods. The stiffeners and panel meshes can be non-conforming. Therefore modeling of the geometry is simple even with the NURB's limitation to structured meshes. The feasibility of the method is shown for static and modal analyses. The method is validated against results obtained from standard finite element approaches with conforming meshes for a panel with one stiffener and a panel with multiple stiffeners. For all cases good accordance to the FEM results is observed with a difference of less than 1%.*

1 INTRODUCTION

Isogeometric analysis (IGA) is a method intended to bridge the gap between finite element analysis and computer aided design (CAD). It uses the methods that are used in CAD to describe geometries for numerical analysis with the finite element method (FEM). The original formulation from Hughes et. al. [1] uses Non-Uniform Rational B-Splines (NURBS) basis functions as they are most commonly used in CAD. Lately also T-Splines, a generalization of NURBS, are used as they enable local h-refinement for IGA. The Idea behind IGA is that standard FEM approximate geometries with Lagrange polynomials where an exact geometry description is already available by CAD.

Compared to standard FEM the application of NURBS basis functions has several advantages. Due to the mathematical properties of NURBS higher polynomial orders p can be applied what may lead to higher accuracy per degree of freedom and higher convergence rates [2]. Furthermore higher inter-element continuities can be achieved. While Lagrange polynomials are limited to C^0 inter-element continuity NURBS can be constructed with up to C^{p-1} continuity. This results in smooth surface representation and enables the usage of governing equations, that need higher order derivatives. Among other fields of application this property is appealing for analysis of thin-walled structures as many shell theories include second order derivatives [3].

This potential cannot be fully used as NURBS are limited to structured meshes. Many lightweight structures are stiffened structures and therefore consists of a base panel with several stiffeners connected to it which cannot be modeled as one structured mesh. One possibility to model this kind of structures is to model stiffener and base panel as separate domains and connect them afterwards. In the simplest case the common border of the two parts are conforming and therefore can simply be connected by sharing control points. This may be a possible solution for classical FEM but if the limitation to structured meshes is taken into account this may lead to massive mesh distortion in the base plate or the need of splitting the base-plate into many parts. Therefore coupling methods are needed that enables the coupling of the stiffener's foot onto the base plate in an arbitrary position. Mortar methods are frequently used for coupling of non-conforming edges in IGA [4]. They are also used in contact formulations [5]. Therefore it is not only possible to couple two boundary edges of NURBS patches but also to couple displacements on a shell surface e.g. example in problems involving contact of shells. Depending on the structural behavior that should be achieved only transversal or transversal and rotational displacements can be coupled.

The aim of the paper is to use coupling methods for connecting a stiffener on a base panel without having to adapt the baseplate mesh to the stiffener position or splitting it into sub-regions. This is desirable because each coupling of subregions with Lagrange multipliers introduce extra degrees of freedom and breaks the high inter-element continuity locally. At least the here proposed methods can only enforce a C^0 continuity of the displacement field over the coupling edge. This is not problematic in this context of Mindlin-Reissner which are only requiring C^0 continuous transversal and rotational displacements.

This gives not only great flexibility in modeling structures with many stiffeners but also enables the movement of stiffeners without having to recompute big parts of the system. Therefore the proposed method is especially interesting for optimization tasks. The feasibility of the method is demonstrated by comparing IGA results for stiffened panels with results obtained by classic FEM-models with conforming meshes.

2 Fundamentals

The examples presented in this paper are based on standard Mindlin-Reissner shell kinematic as also used in FEM. These type of shell theory represents the geometry as its mid-surface and a thickness parameter t , describing the thickness of the shell normal to the mid-surface. It accounts for the mid-surface curvature. Mindlin-Reissner shells often have 6 displacement variables. Three of them describing the mid-surface displacement in physical space u_x, u_y, u_z and three describing the rotation of a material fiber that was normal to the mid-surface in the undeformed state $\theta_x, \theta_y, \theta_z$ as described in [3]. As the three rotation parameters correspond to the Euler angels, which have the well-known non-uniqueness problem a drilling penalty method is used. In difference to shell elements used in classical FEM the mid-surface normal, also denoted as the shell director, is not interpolated from nodal values but directly computed what is possible due to the higher inter-element continuity. Even there exist more evolved shell models for IGA as locking-free 5-Parameter Mindlin-Reissner shells in this paper 6-parameter shell theory is used because the presented examples contain geometries with C^0 continuous locations which are complicated to handle with 5-Parameter shells. It should be noted that there are also more elaborated 5-parameter theories that can handle such regions (e.g. [7])

2.1 B-Splines and NURBS

This work uses NURBS based geometry description which is based on B-Splines. Following the isoparametric approach these are also used for interpolation of displacements. As the examples shown in this paper are based on shell theory the geometries consist of surfaces and curves in three dimensional space. B-splines describe geometries in a parametric way. A curve is therefore defined over one parametric coordinate ξ as the product of geometric control points \mathbf{P}_i and basis functions $N_{i,p}$ of order p as

$$\mathbf{C}(\xi) = \sum_{i=1}^n N_{i,p}(\xi) \mathbf{P}_i \quad . \quad (1)$$

These n basis functions form a basis of the space of all piecewise polynomials of degree p on an interval defined by a knot vector

$$\Xi = [\xi_1, \xi_2, \dots, \xi_{n+p+1}], \quad \xi_i \leq \xi_{i+1} \quad . \quad (2)$$

The positions ξ_i are called knots where each knot may appear several times consecutively. The knots denote positions in parametric space where two polynomials are connected in a C^{p-m} continuous way where m is called multiplicity and denotes how often a knot appears in the knot vector. Therefore continuity of the B-Spline curve and of the B-Spline basis can be controlled by the knot vector. The non-zero knot-spans are something close to what is referred to as elements in classic FEM. Therefore they are often called elements even though there exist some differences in principle. The basis functions can be computed recursively from the knot vector with the Cox-de Boor formula [1].

A B-Spline surface is simply constructed by the tensorial product of two B-Spline curves

$$\mathbf{S}(\xi, \eta) = \sum_i \sum_j N_{i,p}(\xi) M_{j,q}(\eta) \mathbf{P}_{ij} \quad . \quad (3)$$

Both curves may have different orders p and q . The control points \mathbf{P}_{ij} can be understood as a structured control mesh in the two dimensional case. The Surface spans over a rectangular parametric region with the parameters $[\xi, \eta]$. Therefore B-Splines and NURBS are limited to structured control meshes and regions that can be parametrized by a rectangular parameter space.

The limitation to structured control meshes can be overcome using T-Splines what enables the possibility of local refinement [8] whereas the limitation to regions that can be parametrized by a rectangular parameter space cannot.

The NURBS basis functions are computed from the B-Spline basis functions as

$$R_{i,p}(\xi) = \frac{N_{i,p}(\xi)w_i}{\sum_{k=1}^n N_{k,p}(\xi)w_k} \quad . \quad (4)$$

For each geometric control point an additional parameter w_i is introduced called the weight of the control point.

NURBS are a generalization of B-Splines. They share many properties as the inter-element continuity of basis functions, partition of unity and non-negativity. In contrast to B-splines they allow the exact representation of conic sections (e.g. circles) and therefore the geometric continuity can be higher than the continuity of the basis functions. The geometric region that is described by one set of NURBS basis functions and control points is called a NURBS-patch.

2.2 Multipatch coupling with Lagrange multipliers

The Lagrange multiplier approach introduces boundary conditions as a weak formulation so that they do not need to be fulfilled implicitly by the base functions. The approach is e.g. a useful method to couple two sub-domains with incompatible displacement discretization on a common boundary. It allows the unavoidable deviations between the two displacement fields minimizing the error resulting from the incompatible displacement fields in means of the energy norm [6]. To couple the displacements $\mathbf{u}^{(1)}$ of sub-domain Ω_1 with the displacements $\mathbf{u}^{(2)}$ of sub-domain Ω_2 at the common boundary Γ_λ a new vector field λ is introduced on this boundary. The coupling conditions are then incorporated into the weak formulation as an additional term Π_λ that is added to the governing equations $\Pi^{(1)}$ and $\Pi^{(2)}$ of the two sub-domains

$$\Pi = \Pi^{(1)} + \Pi^{(2)} + \Pi_\lambda \quad \Pi_\lambda = \int_{\Gamma_\lambda} (\mathbf{u}^{(1)} - \mathbf{u}^{(2)}) \cdot \lambda d\Gamma \quad (5)$$

For equations of elasticity λ is the traction field that needs to be applied on the border to couple the two subregions. Basically there are two different methods of discretizing Π_λ . The first one is to compute λ from the stress field of the subregions and therefore indirectly from the displacement field. This approach is called Nitsche-Method and has the advantage that no additional degrees of freedom (DOF) are introduced [9]. The drawback of this method is that the formulation heavily depends on the governing differential equations on the subregions. Therefore the more general method is used in this work where the Lagrange multipliers are interpolated by basis functions and add new DOF to the system. This leads to a saddle point problem and must be treated by appropriate solving algorithms. The advantage of this approach is that the governing equations of the sub-domains do not matter as long as they use the displacements as primary variables. As basis functions for the Lagrange multipliers the basis functions that interpolate the displacements on the boundary of one of the adjacent sub-domains are used. This choice is quite common even there may be more elaborate solutions available. For example dual Mortar methods, which use orthogonal basis functions to the ones of the coupled regions are subject to ongoing research as they greatly increase efficiency when solving the resulting system of linear equations. An open problem is the construction of such a dual-basis for higher order basis functions that do not compromise the overall convergence rate even there are promising approaches to this problem [5].

As the boundary edge of a stiffener is coupled to a panel the basis functions of the stiffener are used for the Lagrange multipliers λ . The stiffeners basis functions are chosen because the boundary that should be coupled is also a boundary of the parameter space. This makes the extraction of the basis functions easier. The Integral in eq. 5 is evaluated by a gauss type quadrature with the evaluation points defined in the parametric space of the stiffener. Here again the evaluation points lie on the boundary of the stiffeners parameter space and can be evaluated more easily. The corresponding positions on the panel are located by a projection algorithm that uses a gradient descent method. This evaluation method of the integral is used as stiffener and plate are refined uniformly and therefore both have similar element size. For refinements where this is not the case more complex evaluation methods have to be chosen in order not to bypass base functions on the more refined edge as described in [10].

3 Investigation of a simple stiffened panel

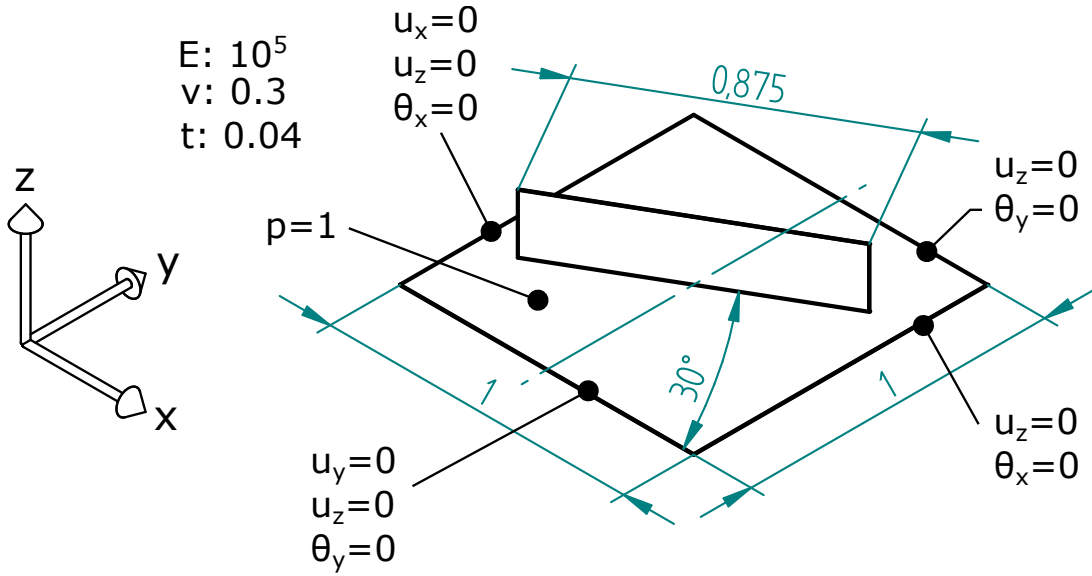


Figure 1: Model description of the test problem

To test the method a simple example is investigated where a rectangular stiffener is coupled to a square panel as shown in fig. 1. The panel is loaded by a pressure in positive z -Direction. The angle between the stiffener and the panel edge is 30° so that the foot of the stiffener is crossing elements of the panel at different locations. At the coarsest IGA-mesh panel and stiffener are modeled as one element each of degree $p = 1$. To investigate performance they are refined uniformly. The deformation and the von Mises stresses for one refinement step are shown in fig. 2.

To investigate the overall performance of the method the results are compared to a FEM-model with conforming meshes of quadratic shell elements. The model uses a structured mesh of $n_{elem} = 20 \times 87 = 1740$ elements for the stiffener and $n_{elem} = 12157$ elements for the base panel with a total of $n_{DOF} = 250848$ DOF. The error is measured as the relative difference of the total strain energy Δe .

$$\Delta e = \frac{0.5 \mathbf{U}^{FE} \cdot \mathbf{F}^{FE} - 0.5 \mathbf{U}^{IGA} \cdot \mathbf{F}^{IGA}}{0.5 \mathbf{U}^{FE} \cdot \mathbf{F}^{FE}} \quad (6)$$

with \mathbf{U} being the displacement vector and \mathbf{F} the force vector of the discrete models. To investigate how well the displacement coupling conditions are fulfilled the L_2 -norm of the relative differences between the displacements of both subregions at the common boundary Γ_λ is used.

$$\|e_U\|_{L_2} = \int_{\Gamma_\lambda} \frac{|\mathbf{u}^{(1)} - \mathbf{u}^{(2)}|}{0.5(|\mathbf{u}^{(1)} + \mathbf{u}^{(2)}|)} d\Gamma \quad (7)$$

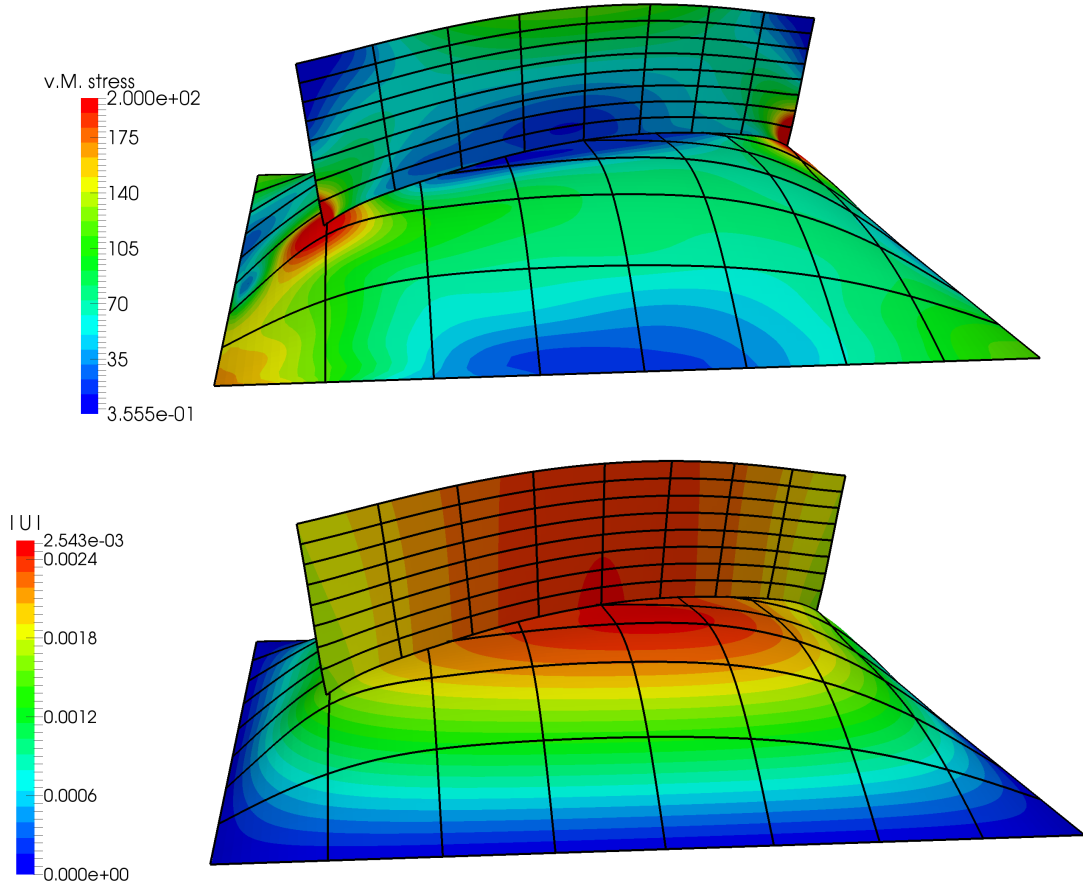


Figure 2: Von Mises stress and displacement in deformed state (warped by factor 70. Stresses clipped at 200). Panel and stiffener have $n_{elem} = 64$ each of order $p = 7$

The results are shown in fig. 3. The difference between the IGA and FEM solution by means of strain energy is smaller than 1% for the finest IGA-mesh. The results show that the convergence rate does not improve with higher approximation orders. Degrees of $p > 3$ do not improve the results. As seen in the stress plot the stiffener introduces two stress singularities at the end-points as these are reentrant corners. Furthermore the Lagrange multipliers introduce a line load to the panel that is modeled as a shell. This line load results in a discontinuity of the σ_{xz} and σ_{yz} stress components of the panel. The component $\sigma_{zz} = 0$ as the shell is based on a plane stress assumption. Therefore the tractions in z-direction are coupled into the transversal shear stresses resulting in steep gradients in the discretized model. As seen in fig. 2 this does not lead to oscillation phenomena in the stress field even with high order basis functions but impairs the convergence speed at least locally around the stiffener foot. The displacement coupling condition can be fulfilled very accurate even in the end-points with the stress singularities.

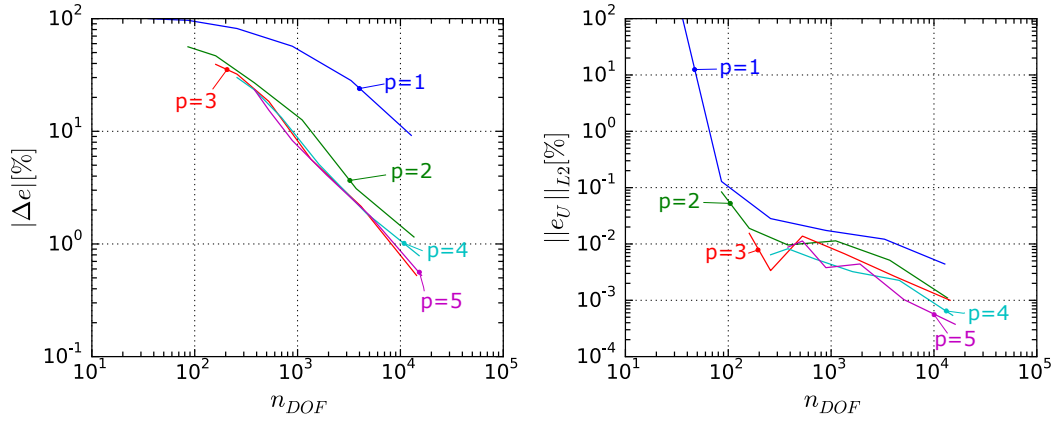


Figure 3: Convergence against FEM-solution and relative error of displacement coupling condition in L_2 -norm

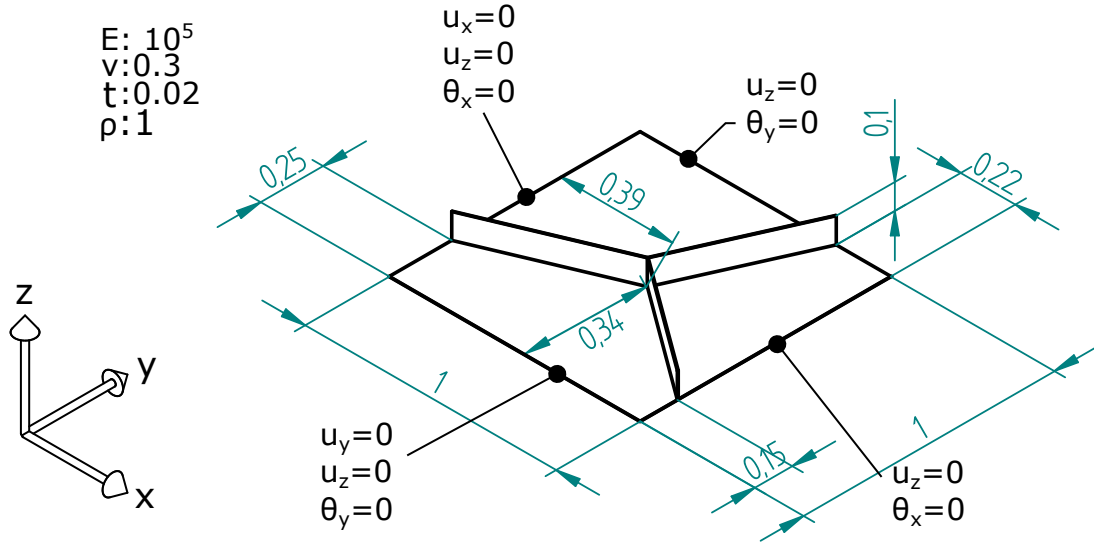


Figure 4: Geometry for test problem with multiple stiffeners

4 Results for multiple stiffeners

As a second example a panel with multiple stiffeners is investigated. The test geometry is depicted in fig. 4. It is modeled with 3 Patches. One for the base panel and two for the stiffeners where one stiffener has a kink. The two stiffeners are connected by Lagrange multipliers in the same way as the stiffeners and the base panel. The results are again validated against a FEM-model with conforming meshes by comparing the strain energy. The FE-model consists of quadratic shell elements with $n_{elem} = 1730$ elements for the stiffeners and $n_{elem} = 12127$ elements for the panel with a total of $n_{DOF} = 250848$ DOF. The base panel is loaded by a pressure of $p = 1$ for the static test case. Beside the strain energy of the static case the first 4 eigenfrequencies are compared where a density of $\rho = 1$ is assumed. The results are given in fig. 6. Fig. 7 shows the 4th eigenform obtained by the IGA method. All eigenfrequencies can be obtained with less than 1% difference to the FE-results. In the static and the modal analyses the higher orders basis functions tend to better accuracy with same number of DOF but again the convergence speed is limited by the stress singularities.

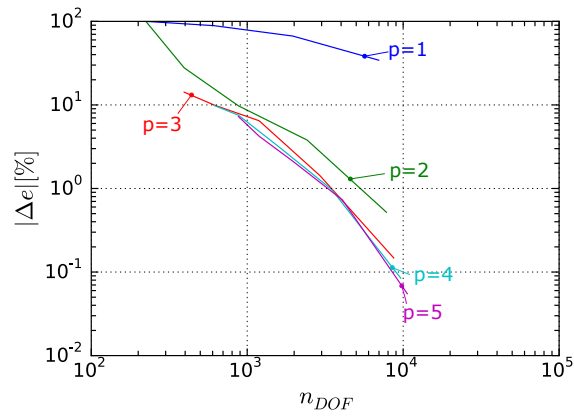


Figure 5: Convergence against FEM-solution for test case 2

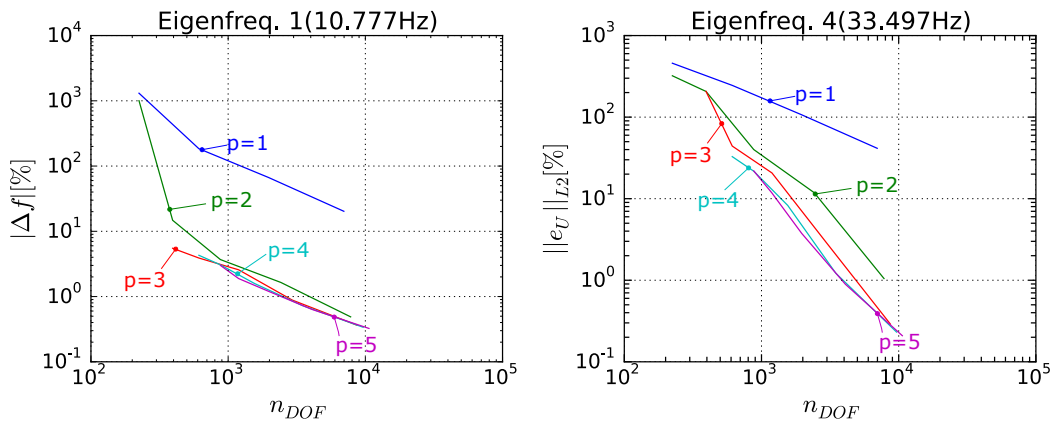


Figure 6: Convergence of eigenfrequencies no. 1 and 4 against FEM-solution

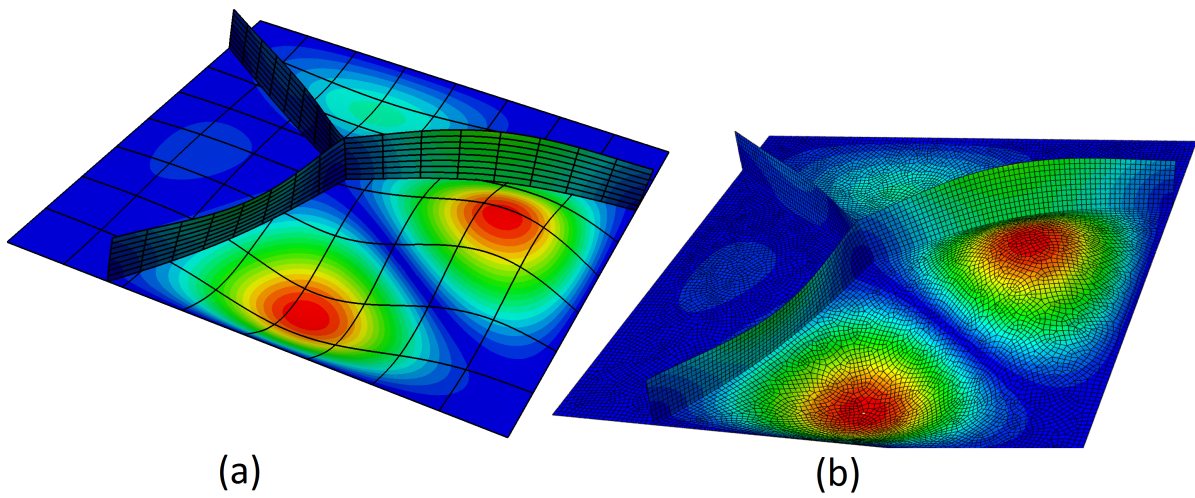


Figure 7: Eigenmode no. 4 obtained by IGA-model of order $p = 7$ and $n_{elem} = 64$ for panel and short stiffener and $n_{elem} = 128$ for long stiffener (a) and obtained by FEM-reference model (b)

5 Conclusion

A method to model stiffened structures in IGA based on Lagrange multipliers is implemented. With the Lagrange multiplier approach stiffeners and base panel can be modeled as separate domains and coupled afterwards. Even though NURBS are limited to structured meshes arbitrary shaped stiffeners can be coupled to a base panel without subdividing it into multiple parts with the proposed method. It could be shown that the results are of similar accuracy compared to standard FEM approaches with conforming meshes for static and modal analysis. The displacement coupling conditions can be fulfilled very accurate especially by higher order base functions. Stress singularities at the end of stiffeners does not lead to stress oscillations even in cases with high order base functions.

The convergence speed for higher order base functions is limited by the stress discontinuities at the stiffeners foot. This problem is not introduced by the Lagrange multiplier approach but by the way of modeling the stiffener and also occurs on conforming meshes. A way to evade the problem is local refinement what can be accomplished by T-Splines or adaptive IGA methods as described in [11] where the latter even the order p can be lowered locally. Both approaches seem promising for future improvements of the method.

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