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BUCKLING OF ANNULAR PLATE JOINT WITH CIRCULAR BEAM

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Abstract. *Buckling of an annular elastic thin plate joint with circular beam under the action of radial stresses is studied. Such plate can be considered as a model of the supporting ring of a cylindrical shell in the case when the ring has a T-shaped cross-section. First the initial stress-resultants are found. Then assuming that the plate is narrow buckling equations and boundary conditions are simplified by means of an asymptotic technique. The approximate eigenvalue problem has analytical solutions for particular cases. In the general case, for its solution the Rayleigh-Ritz method and the shooting procedure are used.*

1 INTRODUCTION

The external pressure acting on a circular cylindrical shell may cause buckling of the shell. Reinforcing of the shell by rings leads to raising its critical pressure. If the width of the rings is sufficiently small the buckling mode is similar to the buckling mode of the shell without rings, and the shell surface is covered by a series of pits stretched along the generatrix of the cylinder. In the case of this typical buckling the rings may be considered as circular beams. Almost in all studies of ring-stiffened shells the beam model of the ring is used (see [1]-[3] and references there).

If the width of a ring grows, the critical pressure increases until the typical buckling mode will be replaced with the buckling mode consisting of small pits formed on the surface of the ring while the cylindrical shell itself does not actually deform (see, [4]). The buckling of the plate occurs under the action of radial stresses arising on the plate edge joined with the cylindrical shell. These stresses are caused by the pressure acting on a circular cylindrical shell. The beam model can not be used for studying the buckling of the wide ring which should be considered as an annular plate. Buckling of annular thin plates is well enough studied (see, for example, [5]-[7]). The aim of this paper is the solution of the buckling problem for the annular plate stiffened by a circular beam.

In [8] by means of an asymptotic analysis the approximate boundary conditions on the plate edge joined with the cylindrical shell are obtained. If the plate is thin these conditions coincide with the conditions on the clamped edge. In the paper [4] for the ring with the rectangular cross-section on the other plate edge the free edge conditions were introduced. The approximate values of the critical pressure found by the solution of the eigenvalue problem for the annular plate are in good agreement with the numerical ones obtained by FEM for the ring-stiffened shell.

In this paper we consider buckling under the action of radial stresses of the ring with the T-shaped cross-section joined with the cylindrical shell and located outside the shell. As a model of such a ring we use a circular plate. We assume that the edge joined with the shell is clamped and the other plate edge is stiffened by the circular beam.

Two cases are analyzed. In the first case the shell is under the internal pressure p (see Figure $1a$) and in the second case the pressure is external (see Figure $1b$).

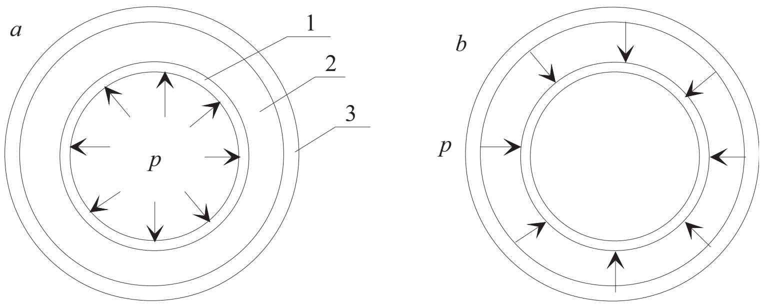

Figure 1: Pressure acting on the shell; 1 — shell, 2 — plate, 3 — beam.

In the work [9] the buckling of a circular plate under the action of radial stresses on the inner edge was considered. For the specific pre-buckling stresses the analytical solution was obtained.

2 BASIC EQUATIONS

Let r be the radial coordinate on the middle surface of the plate and σ_0 the radial stress at the edge $r = r_0$ causing buckling, where r_0 is the radius of the shell. The compressive stresses are regarded as positive. After separation of variables equations describing the buckling of the annular plate can be written in the following form

$$
\begin{gathered}
Q_1' + \frac{1}{r}Q_1 + \frac{m}{r}Q_2 = T_1 w'' + T_2\left(\frac{w'}{r} - \frac{m^2}{r^2}w\right), \\
Q_1 = M_1' + \frac{1}{r}(M_1 - M_2) + 2\frac{m}{r}H, \quad Q_2 = -\frac{m}{r}M_2 + \frac{2}{r}H, \\
M_1 = D(\kappa_1 + \nu\kappa_2), \quad M_2 = D(\kappa_2 + \nu\kappa_1), \quad H = D(1-\nu)\kappa_{12}, \quad D = \frac{Eh^3}{12(1-\nu^2)}, \\
\kappa_1 = -w'', \quad \kappa_2 = \frac{m^2}{r^2}w - \frac{w'}{r}, \quad \kappa_{12} = \left(\frac{m}{r}w\right)',
\end{gathered}
\tag{1}
$$

where ($'$) denotes the derivative with respect to the radial coordinate r, m is the circumferential wave number, w is the normal deflection, Q_1, Q_2 and M_1, M_2, H are the stress-resultants and stress-couples, T_1 and T_2 are the pre-buckling stress-resultants, E is Young's modulus, h is the plate thickness, ν is Poisson's ratio, κ_1, κ_2, κ_{12} are the changes of curvature.

We assume that the loaded edge of the plate $r = r_0$ is clamped:

$$
w = w' = 0, \quad r = r_0. \tag{2}
$$

At the other plate edge $r = r_1$ stiffened by the circular beam the boundary conditions have the form

$$
M_1 = \frac{EJ_r}{r_0^2}w', \quad Q_1 = T_1 w', \quad r = r_1, \tag{3}
$$

where J_r is the moment of inertia of the beam cross-section. For the beam with rectangular cross-section $J_r = a^3 b/12$ (see Figure 2).

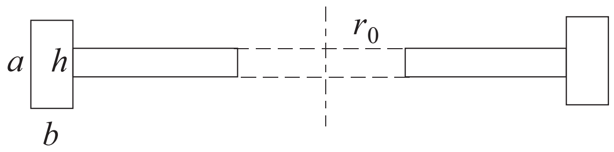

Figure 2: The cross-section of the annular plate joint with the beam

System (1) can be reduced to the equation

$$
\frac{d^4w}{ds^4} + \frac{2}{s}\frac{d^3w}{ds^3} - \frac{2m^2 + 1 - \beta t_1}{s^2}\frac{d^2w}{ds^2} + \frac{2m^2 + 1 + \beta t_2}{s^3}\frac{dw}{ds} + \frac{m^2(m^2 - 4 - \beta t_2)}{s^4}w = 0, \tag{4}
$$

where

$$
\beta = \frac{h\sigma_0 r_0^2}{D}
$$

is the buckling stress parameter,

$$
s = \frac{r}{r_0}, \quad t_k = \frac{r^2 T_k}{h\sigma_0 r_0^2}, \quad k = 1, 2. \tag{5}
$$

3 PRE-BUCKLING STRESS-RESULTANTS

To find the pre-buckling stress-resultants T_1 and T_2 we use equations of axisymmetric tangential deformation of the plate:

$$T_1' + \frac{1}{r}(T_1 - T_2) = 0, \quad T_1 = B\left(u' + \nu\frac{u}{r}\right), \quad T_2 = B\left(\frac{u}{r} + \nu u'\right), \tag{6}$$

where u is the radial displacement and $B = Eh/(1-\nu^2)$. The general solution of equations (6) has the form

$$u = C_1 r + \frac{C_2}{r}, \quad T_{1,2} = B\left(\gamma_0 C_1 \mp \delta_0 \frac{C_2}{r^2}\right), \tag{7}$$

where $\gamma_0 = 1+\nu$, $\delta_0 = 1-\nu$.

In the first case corresponding to the Figure $1a$ the boundary conditions for equations (6) are

$$T_1(r_0) = h\sigma_0, \quad T_1(r_1) = -\frac{ES}{r_1^2}u(r_1), \tag{8}$$

where S is the area of the ring cross-section. Substituting solution (7) into boundary conditions (8) we get

$$(\gamma_0 + \delta_0 S_0)C_1 - \frac{\delta_0(1-S_0)}{r_1^2}C_2 = 0, \quad \gamma_0 C_1 - \frac{\delta_0}{r_0^2}C_2 = \frac{(1-\nu^2)\sigma_0}{E}, \tag{9}$$

where $S_0 = (1+\nu)S/(hr_1)$. The solution of equations (9) has the form

$$C_1 = \frac{(1-S_0)(1-\nu^2)\sigma_0}{dr_1^2 E}, \quad C_2 = \frac{(\gamma_0+\delta_0 S_0)(1-\nu^2)\sigma_0}{d\delta_0 E}, \quad d = \frac{(1-S_0)\gamma_0}{r_1^2} - \frac{\gamma_0 + \delta_0 S_0}{r_0^2}.$$

Taking into account formulas (7) we obtain

$$T_{1,2} = \frac{h\sigma_0 r_0^2}{r^2}\left(\frac{r^2(1-S_0)\gamma_0 \mp r_1^2(\gamma_0+\delta_0 S_0)}{r_0^2(1-S_0)\gamma_0 - r_1^2(\gamma_0+\delta_0 S_0)}\right). \tag{10}$$

In the second case corresponding to the Figure $1b$ the stresses are tensile. We get the pre-buckling stress-resultants for the second case substituting $-\sigma_0$ instead σ_0 in formulas (10).

If $S_0 = 1$ then

$$T_{1,2} = \pm\frac{h\sigma_0 r_0^2}{r^2}. \tag{11}$$

The analytical solution of equations (1) for the pre-buckling stress-resultants (11) was obtained in [9], however the condition $S_0 = 1$ is unlikely in a real structure.

Taking $S_0 = 0$ in formulas (10), we get the pre-buckling stress-resultants for the plate without the beam which corresponds to a ring with the rectangular cross-section.

4 APPROXIMATE SOLUTION FOR THE FIRST CASE

Consider the buckling of the annular plate under the action of compressive radial stresses applied to the inner clamped plate edge (see Figure $1a$). The outer plate edge is joined with circular beam.

Let us suppose that the plate is narrow, i.e. the non-dimensional width of the plate is $\varepsilon = r_1/r_0 - 1 \ll 1$. Usually this condition holds for the rings supporting a shell.

Assume that $m \sim 1$. Then after replacing the variable

$$s = 1 + \varepsilon x \tag{12}$$

in equations (4) and neglecting small terms we get the approximate equation

$$\frac{d^4 w}{dx^4} + \beta\varepsilon^2 t_1 \frac{d^2 w}{dx^2} + \beta\varepsilon^3 t_2 \frac{dw}{dx} = 0. \tag{13}$$

Replacing variable in relations (2) and (3) according to (12) we obtain the following approximate boundary conditions

$$\begin{gathered} w = w' = 0, \quad x = 0, \\ w'' + cw' = 0, \quad w''' + \beta\varepsilon^2 t_1 w' = 0, \quad x = 1, \end{gathered} \tag{14}$$

where

$$c = \frac{12\varepsilon(1-\nu^2)J_n}{h_n^3}, \quad J_n = \frac{J_r}{r_0^4}, \quad h_n = \frac{h}{r_0}.$$

In the case $S_0 = 1$ it follows from (11) and (5) that $t_{1,2} = \pm 1$. Hence in the first approximation equation (13) and boundary conditions (14) can be written as

$$\frac{d^4 w}{dx^4} + \beta_0^2 \frac{d^2 w}{dx^2} = 0, \quad \beta_0^2 = \beta\varepsilon^2 \tag{15}$$

$$\begin{gathered} w = w' = 0, \quad x = 0, \\ w'' + cw' = 0, \quad w''' + \beta_0^2 w' = 0, \quad x = 1. \end{gathered} \tag{16}$$

Substituting the general solution of equation (15)

$$w(x) = C_1 \sin\beta_0 x + C_2 \cos\beta_0 x + C_3 x + C_4$$

into boundary conditions (16) we obtain that eigenvalue problem (15), (16) has a nontrivial solution if

$$\beta_0 \cos\beta_0 + c\sin\beta_0 = 0. \tag{17}$$

The non-dimensional critical load β_c may be found by means of the formula

$$\beta_c = \beta_0^2/\varepsilon^2, \tag{18}$$

where β_0 is the minimal positive root of equation (17).

In the case of arbitrary S_0 we get the following approximate expressions for the narrow plate taking into account formulas (10), (5) and (12):

$$t_1 = 1 - \frac{\varepsilon h_n x}{\varepsilon h_n + S_n}, \quad t_2 = \frac{\nu S_n - h_n}{\varepsilon h_n + S_n}, \quad S_n = \frac{S}{r_0^2} = \frac{h_n r_{1n} S_0}{1+\nu}. \tag{19}$$

Equation (13) and boundary conditions (14) take the form

$$\frac{d^4 w}{dx^4} + \beta_0^2 t_1(x) \frac{d^2 w}{dx^2} + \varepsilon\beta_0^2 t_2 \frac{dw}{dx} = 0. \tag{20}$$

$$
\begin{gathered}
w = w' = 0, \quad x = 0, \\
w'' + cw' = 0, \quad w''' + \beta_0^2 t_1(1) w' = 0, \quad x = 1.
\end{gathered} \tag{21}
$$

Eigenvalue problem (20), (21) does not have an analytical solution. To find an approximate value of β_0 we use the Rayleigh-Ritz method. Multiplying equations (20) by w and integrating by parts over the interval $[0, 1]$ we obtain

$$
[w'''w - w''w']_0^1 + \int_0^1 (w'')^2\, dx + \beta_0^2 \int_0^1 t_1 w'' w\, dx + \varepsilon \beta_0^2 t_2 \int_0^1 w' w\, dx = 0. \tag{22}
$$

Taking into account boundary conditions (21) from equation (22) we find

$$
\beta_0^2 = \frac{G_1 + I_1}{G_2 + I_2 + I_3}, \tag{23}
$$

where

$$
\begin{gathered}
G_1 = c[w'(1)]^2, \quad G_2 = t_1(1) w'(1) w(1), \quad I_1 = \int_0^1 (w'')^2\, dx, \\
I_2 = -\varepsilon t_2 \int_0^1 w' w\, dx, \quad I_3 = -\int_0^1 t_1 w'' w\, dx.
\end{gathered} \tag{24}
$$

The substituting in formulas (23), (24) instead exact solution $w(x)$ an arbitrary function $W(x)$ satisfying boundary conditions $W(0) = W'(0) = 0$ get an approximate value of β_0. Accuracy of the approximation will be high if we take the function $W(x)$ close to $w(x)$. To choose the function $W(x)$ we consider solutions of eigenvalue problem (15), (16). If $c = 0$ then the minimal positive root of equation (17) is $\beta_1 = \pi/2$. Corresponding to $\beta_1 = \pi/2$ buckling mode has the form $W_1(x) = 1 - \cos(\pi x/2)$. In the case $c = \infty$ the buckling mode is $W_2(x) = 1 - \cos(\pi x)$. Let for any dimensionless stiffness c

$$
W(x) = 1 - \cos kx, \tag{25}
$$

where $k(c) = \pi(c + 1)/(c + 2)$, $k(0) = \pi/2$, $k(\infty) = \pi$.

Formula (23) for eigenvalue problem (15), (16) after sustituting function (25) in relation (24) and neglecting small terms may be written in the form

$$
\beta_0^2 = \frac{4ck \sin k + k^2(2k + \sin 2k)}{2k - \sin 2k}. \tag{26}
$$

Comparison of the minimal positive roots of equation (17) and the approximate values of β_0 obtained from formula (26) for $c \in [0, 10]$ shows that the relative error of approximate results is less than 0.1%.

We use function (25) also for the calculation eigenvalues of buckling problem (20), (21). Substituting this function instead $w(x)$ in relations (24), we obtain

$$
\begin{gathered}
G_1 = c(k \sin k)^2, \quad G_2 = \frac{S_n k \sin k (1 - \cos k)}{S_n + \varepsilon}, \\
I_1 = \frac{k^3}{4}(2k + \sin 2k), \quad I_2 = \varepsilon \frac{1 - \nu S_n}{2(S_n + \varepsilon)}(1 - \cos k)^2, \\
I_3 = \frac{2k^2(2S_n + \varepsilon) - 2kS_n(4\sin k - \sin 2k) - \varepsilon(7 - 8\cos k + \cos 2k)}{8(S_n + \varepsilon)}.
\end{gathered} \tag{27}
$$

Formulas (23) and (27) get an approximate minimal positive eigenvalue β_0 of problem (20), (21). The non-dimensional critical load β_c may be found by means of relation (18).

Consider an annular plate with the parameters $\varepsilon = 0.1$, $\nu = 0.3$, $h_n = h_p/r_0 = 0.01$, which outer edge is stiffened by the circular beam of a rectangular cross-section. The width and height of the beam cross-section are $a = a_n r_0$ and $b = b_n r_0$ correspondingly (see Figure 2). Then $S_n = a_n b_n / h_n$, $J = a^3 b/12$. Values of the dimensionless stiffness c and the non-dimensional critical load β_c for different values of a_n and b_n are given in the Table 1. The last column of

a_n	b_n	c	Values of β_c	
			Ritz method	Shooting
0	0.0	0	830	784
0.01	0.01	0.001	685	663
0.04	0.01	0.058	520	515
0.05	0.02	0.227	448	447
0.06	0.03	0.590	469	468
0.07	0.04	1.248	542	541
0.08	0.05	2.330	640	639
0.10	0.07	6.370	816	816
0.12	0.09	14.15	915	915

Table 1: Results of calculation for various dimensions of beam cross-section.

the Table 1 contains values β_c obtained by the numerical solution of eigenvalue problem (20), (21). These results are found by means of the initial-value or shooting procedure represented in [10]. The approximate results calculated from formula (23) are given in the penultimate column. Comparison of the numerical and the approximate results shows that the relative error of the approximate results is less than 6%.

In Fig. 3, for the plate stiffened by the beam of a square cross-section the parameter β_0 vs. $a_n = b_n$ is plotted.

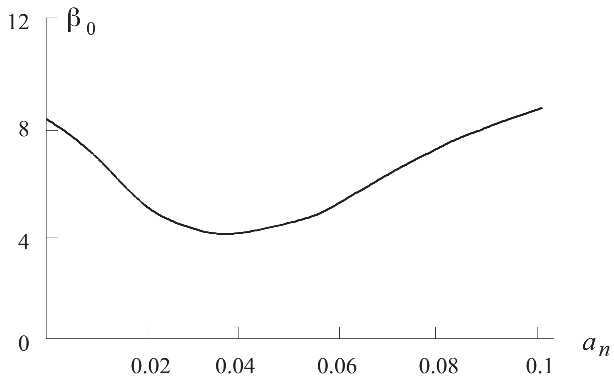

Figure 3: Parameter β_0 vs. dimension of beam cross-section a_n.

At increase in the sizes of the beam cross-section section the parameter of the critical load $\beta_c = \beta_0^2/\varepsilon^2$ decreases, and then increases. Decrease of the critical load β_c with growth of the sizes a_n and b_n is connected with increase in the pre-buckling stress-resultants t_1 at the external edge of the plate, and its increase is caused by increase in stiffness of a beam c. Hence, by the local buckling under an internal pressure the ring with the rectangular cross-section for which $a_n = b_n = 0$ is more effective than the ring with the T-shaped cross-section.

5 APPROXIMATE SOLUTION FOR THE SECOND CASE

The second case differs from the first case only in that the radial stress at the edge $r = r_0$ is tensile i.e. its has opposite sign. Therefore to obtain in the second case an approximate equation and boundary conditions for the narrow plate and $m \sim 1$ we should change β to $-\beta$ in formulas (13) and (14). Then for $m \sim 1$ by means of relation (23) we obtain that $\beta < 0$, and plate buckling does not occur.

Consider the case of the large circumferential wave number: $m \sim 1/\varepsilon$. Then for a narrow plate the approximate buckling equation and the boundary conditions are

$$\frac{d^4w}{dx^4} - (2m_0^2 + \beta_0 t_1)\frac{d^2w}{dx^2} + m_0^2(m_0^2 + \beta_0 t_2)w = 0, \tag{28}$$

$$\begin{gathered} w = w' = 0, \quad x = 0, \\ w'' + cw' - \nu m_0^2 w = 0, \quad w''' - [(2-\nu)m_0^2 + \beta_0 t_1]w' = 0, \quad x = 1. \end{gathered} \tag{29}$$

where

$$m_0 = \varepsilon m, \quad \beta_0 = \varepsilon^2\beta, \quad t_1 = 1 - \frac{\varepsilon h_n x}{\varepsilon h_n + S_n}, \quad t_2 = \frac{\nu S_n - h_n}{\varepsilon h_n + S_n}, \quad S_n = \frac{h_n r_{1n} S_0}{1+\nu}.$$

Equation (28) has analytical solutions only in the cases $S_0 = 0$ and $S_0 = 1$ (see [4] and [9]). In the general case we search approximate value of β_0 by means of the Rayleigh-Ritz method. We multiply equations (28) by w, integrate by parts over the interval $[0, 1]$ and obtain

$$[w'''w - w''w']_0^1 + I_2 + 2m_0(I_1 - [w'w]_0^1) - \beta_0 I_3 + m_0^2(m_0^2 + \beta_0 t_2)I_0 = 0. \tag{30}$$

Here

$$I_2 = \int_0^1 (w'')^2\,dx, \quad I_1 = \int_0^1 (w')^2\,dx, \quad I_0 = \int_0^1 w^2\,dx, \quad I_3 = \int_0^1 t_1 w'' w\,dx.$$

Using boundary conditions (29) from equation (30) we find

$$\beta_0 = \frac{I_2 + m_0^2 I_1 + m_0^4 I_0 - 2\nu m_0^2 w(1)w'(1) + c(w'(1))^2}{I_3 - t_2 m_0^2 I_0 - t_1(1)w(1)w'(1)}. \tag{31}$$

As a Ritz function $W(x)$ we use the solution of following eigenvalue problem

$$\frac{d^4w}{dx^4} - (2m_0^2 + \beta_0)\frac{d^2w}{dx^2} + m_0^2(m_0^2 + \beta_0 t_2)w = 0, \tag{32}$$

$$\begin{gathered} w = w' = 0, \quad x = 0, \\ w'' + cw' - \nu m_0^2 w = 0, \quad w''' - [(2-\nu)m_0^2 + \beta_0 t_1]w' = 0, \quad x = 1. \end{gathered} \tag{33}$$

Substituting $w = e^{\lambda x}$ into equation (32) we obtain the biquadratic equation

$$\lambda^4 - (2m_0^2 + \beta_0)\lambda^2 + m_0^2(m_0^2 + \beta_0 t_2) = 0$$

which has the following roots:

$$\lambda_{1,2} = \pm\sqrt{b_0 - a_0}, \quad \lambda_{3,4} = \pm\sqrt{b_0 + a_0},$$

where

$$a_0 = \sqrt{(1-t_2)m_0^2\beta_0 + \beta_0^2/4}, \quad b_0 = m_0^2 + \beta_0/2.$$

In the case $\beta_0 \le m_0^2/t_2$ all roots are real and eigenvalue problem (32), (33) does not have a nontrivial solution. Let us search $\beta_0 > m_0^2/t_2$. Then equation (32) has the general solution

$$w = C_1 \sin\alpha x + C_2 \cos\alpha x + C_3 \sinh\gamma x + C_4 \cosh\gamma x,$$

where C_k, $k = 1, 2, 3, 4$ are arbitrary constants,

$$\alpha = \sqrt{a_0 - b_0}, \quad \gamma = \sqrt{a_0 + b_0}.$$

To obtain the constants C_k we substitute the general solution into boundary conditions (33) and get a system of linear homogeneous algebraic equations. The determinant of this system may be written as

$$D(\beta_0) = \begin{vmatrix} 0 & 1 & 0 & 1 \\ \alpha & 0 & \gamma & 0 \\ (A-\alpha^2)s_1 + c\alpha c_1 & (A-\alpha^2)c_1 - c\alpha s_1 & (A+\gamma^2)s_2 + c\gamma c_2 & (A+\gamma^2)c_2 + c\gamma s_2 \\ \alpha(B-\alpha^2)c_1 & -\alpha(B-\alpha^2)s_1 & \gamma(B+\gamma^2)c_2 & \gamma(B+\gamma^2)s_2 \end{vmatrix}$$

where

$$A = -\nu_0^2, \quad B = m_0^2(\nu - 2) - \beta_0 t_1(1),$$

$$s_1 = \sin\alpha, \quad s_2 = \sinh\gamma, \quad c_1 = \cos\alpha, \quad c_2 = \cosh\gamma.$$

After transformations, we get

$$D(\beta_0) = F s_1 s_2 + G c_1 c_2 + H + c\alpha\gamma(\alpha^2 + \gamma^2)(\alpha s_1 c_2 + \gamma c_1 s_2).$$

Here

$$\begin{gathered} F = \gamma^2(A-\alpha^2)(B+\gamma^2) - \alpha^2(A+\gamma^2)(B-\alpha^2), \\ G = -\alpha\gamma[(A-\alpha^2)(B+\gamma^2) + (A+\gamma^2)(B-\alpha^2)], \\ H = \alpha\gamma[(A+\gamma^2)(B+\gamma^2) + (A-\alpha^2)(B-\alpha^2). \end{gathered}$$

If $\beta_0(m_0)$ is the minimal positive root of equation $D(\beta_0) = 0$ then we can find the parameter of the critical load β_c from the following formulas

$$\beta_c = \beta_m/\varepsilon^2, \quad \beta_m = \min_{m_0} b_0(m_0). \tag{34}$$

Since $D(\beta_m) = 0$, the system of linear homogeneous algebraic equations with the unknowns C_k has nontrivial solution for $\beta_0 = \beta_m$. As such solution we take the minors of elements of the third row of a determinant $D(\beta_m)$:

$$\begin{gathered} C_1^* = \alpha\gamma(B-\alpha^2)s_1 + \gamma^2(B+\gamma^2)s_2, \quad C_2^* = \alpha\gamma(B-\alpha^2)c_1 - \alpha\gamma(B+\gamma^2)c_2, \\ C_3^* = -\alpha^2(B-\alpha^2)s_1 - \alpha\gamma(B+\gamma^2)s_2, \quad C_4^* = -\alpha\gamma(B-\alpha^2)c_1 + \alpha\gamma(B+\gamma^2)c_2, \end{gathered}$$

Hence, the buckling mode of the problem (32), (33) is

$$W = C_1^* \sin\alpha x + C_2^* \cos\alpha x + C_3^* \sinh\gamma x + C_4^* \cosh\gamma x.$$

Substituting in formula (30) instead the exact solution $w(x)$ the function $W(x)$ we obtain the approximate value of the non-dimesion critical load of problem (28), (29):

$$\beta_0 = \frac{I_2 + m_0^2 I_1 + m_0^4 I_0 - 2\nu m_0^2 W(1)W'(1) + c(W'(1))^2}{I_3 - t_2 m_0^2 I_0 - t_1(1)W(1)W'(1)}, \tag{35}$$

where

$$I_2 = \int_0^1 (W'')^2\,dx, \quad I_1 = \int_0^1 (W')^2\,dx, \quad I_0 = \int_0^1 W^2\,dx, \quad I_3 = \int_0^1 t_1 W'' W\,dx.$$

The relative error of approximate results, obtain from formula (28), in comparison with numerical results, found by means of the shooting procedure, is less than 0.1%.

In Fig. 4, for the plate with the parameters $\varepsilon = 0.1$, $\nu = 0.3$, $h_n = 0.01$ stiffened by the beam of a square cross-section the critical load β_0 vs. the dimensionless sizes of the beam cross-section a_n is plotted.

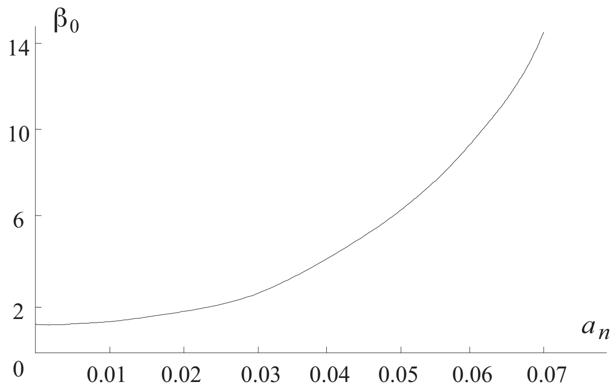

Figure 4: Parameter β_0 vs. sizes of beam a_n.

At increase in the sizes of the beam cross-section the parameter of critical load $\beta_c = \beta_0^2/\varepsilon^2$ increases. Therefore, by the local buckling under an external pressure the ring with the T-shaped cross-section is more effective than the ring with rectangular cross-section.

6 CONCLUSIONS

We discussed buckling of rings with the T-shaped cross-section supporting a circular cylindrical shell. The cases of internal and external uniform pressures acting on the shell are analyzed. As a model of the ring the annular thin narrow plate joint with a circular beam is used. Buckling equations and boundary conditions are simplified by means of an asymptotic technique. To find the approximate critical load the Rayleigh-Ritz method is used. The solutions of the approximate eigenvalue problems are obtained in closed form. Comparison of the numerical results found by means of the shooting procedure and the approximate results shows that the relative error of the approximate results is small.

The dependence of the critical load on the dimensions of the beam cross-section is studied. In case of the internal pressure with the increase in these dimensions the critical load decreases, and then increases. Therefore, by the local buckling under an internal pressure the ring with the rectangular cross-section is more effective than the ring with the T-shaped cross-section. If the pressure is external then the increase in the dimensions of beam cross-section caused the increase of critical load, and the ring with the T-shaped cross-section is more effective than the ring with rectangular cross-section.

The results obtained in the current paper may be used for optimal design of the cylindrical shell stiffened by rings.

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