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WAVE PROPAGATION IN THIN PRETWISTED ANISOTROPIC STRIPS

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Abstract. *This paper presents an asymptotically exact cross-sectional framework coupled with spectrally formulated one-dimensional (1-D) finite element model for the study of wave propagation in thin pre-twisted anisotropic strips. The cross-sectional model used here is based on the dimensional reduction of laminated shell theory to nonlinear 1-D theory using the variational asymptotic method (VAM). A linearized version of this model (zeroth-order asymptotic analysis) is used in the development of a 1-D spectral finite element (SFE) for a pre-twisted strip. Within this modeling framework, an exact dynamic stiffness matrix is derived, as the governing equations are solved using the exact interpolating functions in frequency-wavenumber domain. Compared to regular three-dimensional (3-D) finite element (FE) analysis, this model requires less computation cost since single element is sufficient to capture the frequency response of the strips. For numerical validation of the model, the natural frequencies of the strip without delamination are compared to the data available in literature. Finally, the developed framework is used to simulate wave propagation due to modulated sinusoidal pulse input.*

1 INTRODUCTION

Thin pre-twisted beams, made of laminated composites, are increasingly being used in helicopter rotor blades, wind turbine blades and propellers. In the case of helicopter rotor blades they are used to accommodate the centrifugal force along with flapping, lead-lag, and torsional motion using a flexible structural component known as flexbeams. This allows for the design of a rotor system that is independent of bearings making the design bearing-less [1]. However, the design of flexbeams and flexbeam like structures found in other applications is challenging owing to the nonlinearity that arises because of the large displacements and moderate rotations. This geometrical nonlinearity problem coupled with the anisotropy at the material scale adds to the complexity of the design. This challenge is further compounded when the design has to account for damages typical of laminated composites, e.g., delaminations [2]. Generally, it is desirable to develop a structural health monitoring (SHM) strategy, preferably online, which can aid in the early diagnosis of damages. Over the last few decades, SHM techniques based on dynamic identification techniques (see [3]) have gained prominence. In particular, wave propagation based technique is being extensively used owing to its ability to identify smaller defects, and potential to be used for online SHM [4]. The primary objective of the present work is towards the development of a computationally efficient model, which can not only accurately describe the overall deformation characteristics of the pre-twisted strip but also capture its dynamic behavior. It is envisaged that such a model will be an ideal framework based on which SHM system for damage detection in pre-twisted anisotropic strips can be developed.

For structures with one of it's dimensions much larger than the other two, simple beam models are often used in the preliminary design stage because they can provide valuable insight into the behavior of the structures with much less effort. These models, in general, are developed on $adhoc$ assumptions. These traditional 'ordering scheme' based methodologies can be circumvented when the exact kinematics of the beam reference line displacement and cross-sectional rotation is accounted for in the development of the geometrically-exact equations of motion. $Adhoc$ methodologies have been replaced by rigorous beam modeling (easily extendable to plates and shells) technique, incorporating the kinematics as described above, through the variational asymptotic method (VAM) [5, 6]. It has also been used to model pre-twisted anisotropic strips both in the absence and presence of delaminations [2, 7]. The present work extends the scope of these studies to investigate vibration and wave propagation in pre-twisted strips.

Owing to the importance of pre-twisted anisotropic strips, considerable work has been done on studying the vibration and dynamic characteristics of pre-twisted strips. For brevity, only few important works are reported here. Rosard [8], Anliker and Troesch [9], Gupta and Rao [10] investigated the free vibration characteristics of pre-twisted beams. Carnegie and Thomas extended these works to investigate the coupled bending vibration of pre-twisted beam [11]. More recently, Filiz et al. [12] developed a 3-D model to investigate the dynamics of pre-twisted beam using the spectral-Tchebychev solution. Even though much work has been done on vibration and dynamic behavior of pre-twisted strips, not much work has been done on wave propagation in these structures, in particular.

Conventional finite element (FE) analysis of pre-twisted anisotropic strips is difficult due to lack of computational resources. This takes us to the realm of frequency domain analysis and in particular, spectral analysis. This spectral analysis is used to construct a frequency domain based matrix methodology proposed by Doyle [13], which is called spectral finite element (SFE) method. The first steps in the development of SFE involves the transformation of the governing wave equations from time domain to frequency-wavenumber domain using discrete Fourier transform (DFT). This process converts the governing coupled partial differential equations (PDEs) into a set of ordinary differential equations (ODEs) in the spatial domain which can be solved by standard methods. The resulting complex shape function matrix is a linear superposition of all the wave modes. Following the conventional FE method, the complex dynamic stiffness matrix is then formed which is exact. As shown in [13], this process leads to an efficient model suitable for use within the framework of automated FE method. The primary difference between conventional FE and SFE methods is that at the elemental nodes spectral amplitudes of the nodal variables are determined at each frequency steps in SFE method. On the contrary, pseudo-static variables are evaluated at each time steps or eigen frequencies in standard FE technique.

In the current paper, SFE has been developed to find the natural frequencies of pre-twisted anisotropic strips based on dimensional reduction of laminated shell theory to a one dimensional non-linear beam theory, and validated with the data available in literature. Later, it is used to simulate wave propagation using narrow banded modulated sinusoidal pulse as input. As mentioned earlier, though SFE has been used extensively for vibration and wave propagation study of composite beams and laminates [14], it has not been developed for pre-twisted beams.

The manuscript is organized as follows. Section 2 provides the mathematical formulation of the governing equations using VAM. In Section 3, The SFE formulation is described briefly. The numerical results are presented in Section 4. The paper ends with a section on important conclusions.

2 MATHEMATICAL FORMULATION

The development of the kinematics for this problem follows the procedure enumerated in [7]. Here, the details are omitted and only the general procedure is enumerated. The overall deformation of a cross-section contains all forms of warping and location moderate rotations. Following the classical laminated shell theory, the 3-D strain measures are related to 2-D strain measures by $\Gamma_{\alpha\beta} = \epsilon_{\alpha\beta} + x_3\rho_{\alpha\beta}$, where, $\epsilon_{\alpha\beta}$ are membrane strains and $\rho_{\alpha\beta}$ are middle surface bending curvatures. Starting from the kinematics, then applying VAM in an iterative manner, relations between membrane strains and beam 1-D strains are obtained. The strip, being considered a 2-D elastic body, it's strain energy density can be written as $U_{2D} = \dfrac{1}{2}\epsilon^T[\mathbf{AB}|\mathbf{BD}]\epsilon$. Here, $\mathbf{A}$, $\mathbf{D}$, $\mathbf{B}$ are the familiar membrane, bending, and coupling stiffness matrices, respectively. The 1-D strain energy density can be obtained by integrating the 2-D strain energy density along the width direction. However, the unknown warping terms, which are functions of x_2 have to be predetermined to carry out the integration which can be done at any given order employing variational minimization principles. In doing so, a 9×9 stiffness matrix is obtained, where the first 4×4 terms represent the linear stiffness matrix (obtained from zeroth order approximation). The above described procedure can also be followed to determine the cross-sectional stiffness terms in the presence of delamination [2].

In the present work, focus is limited to linear analysis. Hence, the kinetic energy of the beam, including three translation degrees of freedom (DOF) and rotation about x_1, excluding the other variables for contributing to higher order terms, can be determined as

$$K = \frac{1}{2}\left\{\begin{array}{c}\dot{u}_1\\ \dot{\theta}_1\\ \dot{u}_3\\ \dot{u}_2\end{array}\right\}^T \left[\begin{array}{cccc}\rho A & 0 & 0 & 0\\ 0 & \rho I_p & 0 & 0\\ 0 & 0 & \rho A & 0\\ 0 & 0 & 0 & \rho A\end{array}\right]\left\{\begin{array}{c}\dot{u}_1\\ \dot{\theta}_1\\ \dot{u}_3\\ \dot{u}_2\end{array}\right\} \tag{1}$$

Applying Hamilton's principles, the governing equations can be written as,

$$S_{11}\frac{\partial^2 u_1}{\partial x_1^2} + S_{12}\frac{\partial^2 \theta_1}{\partial x_1^2} - S_{13}\frac{\partial^3 u_3}{\partial x_1^3} + S_{14}\frac{\partial^3 u_2}{\partial x_1^3} = \rho A\frac{\partial^2 u_1}{\partial t^2} \tag{2}$$

$$S_{12}\frac{\partial^2 u_1}{\partial x_1^2} + S_{22}\frac{\partial^2 \theta_1}{\partial x_1^2} - S_{23}\frac{\partial^3 u_3}{\partial x_1^3} + S_{24}\frac{\partial^3 u_2}{\partial x_1^3} = \rho I_p\frac{\partial^2 \theta_1}{\partial t^2} \tag{3}$$

$$S_{14}\frac{\partial^3 u_1}{\partial x_1^3} + S_{24}\frac{\partial^3 \theta_1}{\partial x_1^3} - S_{34}\frac{\partial^4 u_3}{\partial x_1^4} + S_{44}\frac{\partial^4 u_2}{\partial x_1^4} = -\rho A\frac{\partial^2 u_2}{\partial t^2} \tag{4}$$

$$-S_{13}\frac{\partial^3 u_1}{\partial x_1^3} - S_{23}\frac{\partial^3 \theta_1}{\partial x_1^3} + S_{33}\frac{\partial^4 u_3}{\partial x_1^4} - S_{34}\frac{\partial^4 u_2}{\partial x_1^4} = -\rho A\frac{\partial^2 u_3}{\partial t^2} \tag{5}$$

where, S_{ij} are the stiffness terms, which can be readily determined for healthy and delaminated pre-twisted strips [2, 7], ρ is the material mass density, A cross-sectional area and I_p is the polar moment of inertia. Since the sectional resultants are conjugates to the 1-D beam strains in terms of the 1-D strain energy density, the stiffness constitutive law is obtained, which serve as boundary conditions. For this problem, they can be written in compact form as,

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{22} & S_{23} & S_{24} \\ S_{13} & S_{23} & S_{33} & S_{34} \\ S_{14} & S_{24} & S_{34} & S_{44} \end{bmatrix} \begin{Bmatrix} \gamma_{11} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} \tag{6}$$

where, F_1 is the resultant axial force, M_1 being the twist, M_2, M_3 are the bending moments about 2 and 3 axes respectively.

2.1 SPECTRAL FINITE ELEMENT (SFE) FORMULATION

The SFE formulation begins by assuming a solution for the displacement field. In particular, using discrete Fourier transformation (DFT), the displacement vectors can be represented as,

$$\mathbf{u}(x_1, t) = \sum_{n=1}^{N} \hat{\mathbf{u}}(x_1, \omega_n) e^{-i\omega_n t} \tag{7}$$

where, $i = \sqrt{-1}$, ω_n is the frequency at n^{th} sampling point and N is the Nyquist point in the fast Fourier transform (FFT) used for numerical implementation. Substituting Eq. 7 in Eq. 2-5, a set of ODEs are obtained for the Fourier coefficients, $\hat{u_1}(x_1)$, $\hat{\theta_1}(x_1)$, $\hat{u_3}(x_1)$ and $\hat{u_2}(x_1)$, and can be solved, since it contains constant coefficients, by writing $\tilde{u_1} e^{-ikx_1}$, $\tilde{\theta_1} e^{-ikx_1}$, $\tilde{u_3} e^{-ikx_1}$ and $\tilde{u_2} e^{-ikx_1}$, respectively. Here, k is the wave number in x_1 direction, yet to be determined and $\tilde{u_1}$, $\tilde{\theta_1}$, $\tilde{u_3}$ and $\tilde{u_2}$ are unknown constants. Substituting these forms in the ODEs, a polynomial eigenvalue problem (PEP) is posed to find $(k, \boldsymbol{v})$ as below

$$\boldsymbol{\Psi}(k)\boldsymbol{v} = \left(k^4 \boldsymbol{A}^4 + k^3 \boldsymbol{A}^3 + k^2 \boldsymbol{A}^2 + k\boldsymbol{A} + \boldsymbol{A}_0\right) \boldsymbol{v} = 0, \qquad \boldsymbol{v} \neq 0, \tag{8}$$

where, $\boldsymbol{A}^i \epsilon\ \mathbb{C}^{4\times 4}$, k is an eigenvalue and $\boldsymbol{v}$ is the right eigenvector. For brevity, the explicit expressions are not given in the paper. Since $\boldsymbol{A}^4$ is singular [16], $\boldsymbol{\Psi}(k)$ is not regular and admits infinite eigenvalues [15]. This PEP can be solved to find the wavenumbers by several methods available. The one adopted in this paper is linearization of PEP method.

Following the solution of the PEP problem given by Eq. 8, the displacements presented in Eq. 7 can also be written in a compact form in frequency-wavenumber domain, as,

$$\boldsymbol{u}(x, t) = \sum_{n=1}^{N} \hat{\boldsymbol{u}}(x_1, \omega_n) e^{-i\omega_n t} = \sum_{n=1}^{N} \left(\sum_{j=1}^{12} \tilde{\boldsymbol{u}}_j e^{-ik_j x} \right) e^{-i\omega_n t} \tag{9}$$

where, k_j is the wavenumber associated with the j^{th} mode of wave at ω_n. The generic displacement vector at any x_1 location can be given explicitly as,

$$\hat{\boldsymbol{u}}(x_1,\omega_n) = \left\{ \begin{array}{c} \hat{u}_1(x_1,\omega_n) \\ \hat{\theta}_1(x_1,\omega_n) \\ \hat{u}_3(x_1,\omega_n) \\ \hat{u}_2(x_1,\omega_n) \end{array} \right\} = \left[\begin{array}{ccccc} R_{11} & R_{12} & \dots & R_{1\,11} & R_{1\,12} \\ R_{21} & R_{22} & \dots & R_{2\,11} & R_{2\,12} \\ R_{31} & R_{32} & \dots & R_{3\,11} & R_{3\,12} \\ R_{41} & R_{42} & \dots & R_{4\,11} & R_{4\,12} \end{array} \right] [\boldsymbol{D}] \left\{ \begin{array}{c} c_1 \\ c_2 \\ \vdots \\ c_{11} \\ c_{12} \end{array} \right\} \tag{10}$$

where, $[\boldsymbol{D}]$ is diagonal matrix of order 12×12 and can be expressed as,

$$[D] = \left\{ \begin{array}{ll} e^{-ik_jx} & j \text{ odd} \\ e^{-ik_j(L-x)} & \text{otherwise} \end{array} \right. \tag{11}$$

where, L is length of the element. The entries associated with $j = 1, 3, ...9, 11$ and $j = 2, 4, ...10, 12$ contribute respectively to forward and backward propagating waves. It is to be noted that all these expressions are evaluated at a particular value of ω_n. The $\boldsymbol{R}_{4\times12}$ matrix is called amplitude ratio or eigenvector matrix, obtained as the solution of the PEP given by Eq. 8. $\boldsymbol{c} = \{c_1, c_2, \dots, c_{12}\}$ are unknowns constants, to be found out by applying boundary conditions. These unknowns are written in terms of nodal displacements, namely, $x = 0$ and $x = L$.

Eq. 10 can be written as,

$$\hat{\boldsymbol{u}}(x_1,\omega_n) = [\boldsymbol{R}]\,[\boldsymbol{D}]\boldsymbol{c} = [\boldsymbol{T}_1(x_1,\omega_n)]\boldsymbol{c} \tag{12}$$

Eq. 12 at the two nodes, *i.e.*, $x_1 = 0$ and $x_1 = L$ gives

$$\hat{\boldsymbol{u}}_e = \left[\begin{array}{c} \boldsymbol{T}_1(0,\omega_n) \\ \boldsymbol{T}_1(L,\omega_n) \end{array} \right] \boldsymbol{c} = [\boldsymbol{T_2}]\boldsymbol{c} \tag{13}$$

where, $[\boldsymbol{T}_2]$ is a non-singular 12×12 complex matrix and represents the local wave characteristics of the displacement field. Further, Eq. 7 can be used to eliminate unknown constants $\boldsymbol{c}$ from Eq. 13 as,

$$\hat{\boldsymbol{u}}(x_1,\omega_n) = [\boldsymbol{T}_1(x_1,\omega_n)][\boldsymbol{T}_2]^{-1}\hat{\boldsymbol{u}}_e = [\boldsymbol{\zeta}(x_1,\omega_n)]_e\hat{\boldsymbol{u}}_e, \tag{14}$$

where, $[\boldsymbol{\zeta}(x_1,\omega_n)]_e$ is the exact element shape function matrix that connects the nodal displacement vector to the generic displacement vector at arbitrary location along the length of the beam.

Next, using the forced boundary conditions given in Eq. 6, and adding the additional shear or transverse forces $F_2 = \frac{dM_3}{dx_1}$ and $F_3 = \frac{dM_2}{dx_1}$ for the additional DOF, the force vector can be computed in the terms of unknown constants $\boldsymbol{c}$ as,

$$\hat{\boldsymbol{f}} = [\boldsymbol{P}(x_1,\omega_n)]\,\boldsymbol{c} \tag{15}$$

Eq. 15 is evaluated at two nodes, and the nodal force vectors are obtained as,

$$\hat{\boldsymbol{f}}_e = \left\{ \begin{array}{c} \hat{\boldsymbol{f}}_1 \\ \hat{\boldsymbol{f}}_2 \end{array} \right\} = \left[\begin{array}{c} \boldsymbol{P}(0, \omega_n) \\ \boldsymbol{P}(L, \omega_n) \end{array} \right] \boldsymbol{c} = \left[\begin{array}{c} \boldsymbol{P}(0, \omega_n) \\ \boldsymbol{P}(L, \omega_n) \end{array} \right] [\boldsymbol{T}_2]^{-1} \hat{\boldsymbol{u}}_e = [\boldsymbol{K}_e]\, \hat{\boldsymbol{u}}_e \qquad (16)$$

where, $[\boldsymbol{K}_e]$ is the exact dynamic stiffness matrix at ω_n relating the nodal displacements and forces in frequency domain.

3 NUMERICAL RESULTS AND DISCUSSION

First, the natural frequencies of the pre-twisted anisotropic strips are extracted from the the exact dynamic stiffness matrix given in the formulation. A structurally coupled $AS4/3501-6$ graphite-epoxy strip is considered for obtaining the natural frequencies of the beam. The beam material and geometrical properties are as given below in Table 1 and 2, respectively.

Property	Value
E_{11}(GPa)	142
E_{22}/E_{33}(GPa)	9.8
G_{12}(GPa)	6.0
G_{13}/G_{23}(GPa)	4.8
ν_{12}	0.3
ν_{13}/ν_{23}	0.34
ρ (kg/m^3)	1580

Table 1: Material properties of AS4/3501-6 graphite epoxy composite.

Dimension	Value
Length (m)	0.56
Breadth (m)	0.03
Thickness (m)	0.0147

Table 2: Geometrical properties of AS4/3501-6 graphite epoxy composite test specimen.

The natural frequencies of the beam are plotted with different pretwist values and are compared with the data from [17] and are shown in the Figure 1. The first nine modes shown are in the order of first, second, and third flapwise bending (F1, F2, F3), first

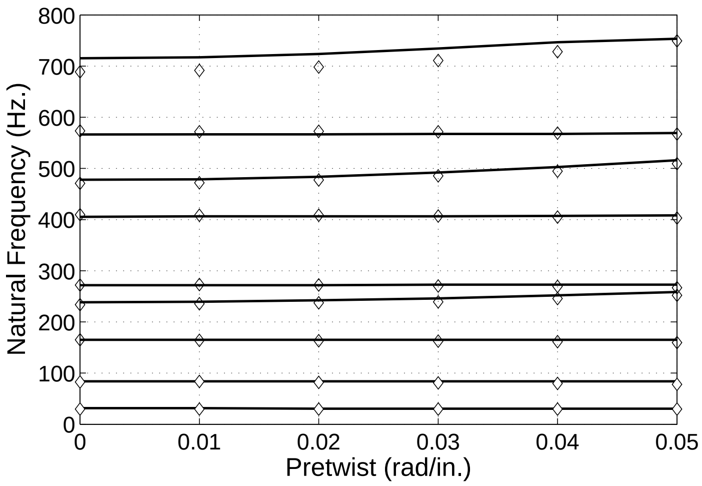

Figure 1: Natural frequencies of the beam marked by the sharp peaks. Data points show are from Ref. [17].

torsion (T1), fourth flapwise bending (F4), second torsion (T2), fifth and sixth flapwise bending (F5, F6), and third torsion (T3). It is observed that pretwist affects the free vibration characteristics of the strip and the current model predictions agree well with those presented in [17].

Next, $AS4/3501-6$ graphite epoxy beam is considered with a pre-twist of $k_1 = 0.05\,rad/in;$ for wave propagation analysis. To get rid of the reflected wave forms from the boundaries a semi-infinite beam is considered for the wave propagation analysis. This configuration will result in bending-twist coupling. A torsional loading of sinusoidal pulse modulated at $5\,kHz$ is applied at the free end. Figure 2 shows the torsional response at a distance from the free end under the torsional loading. The bigger blob represents the torsional mode and the smaller blob represents the flap-wise flexural mode.

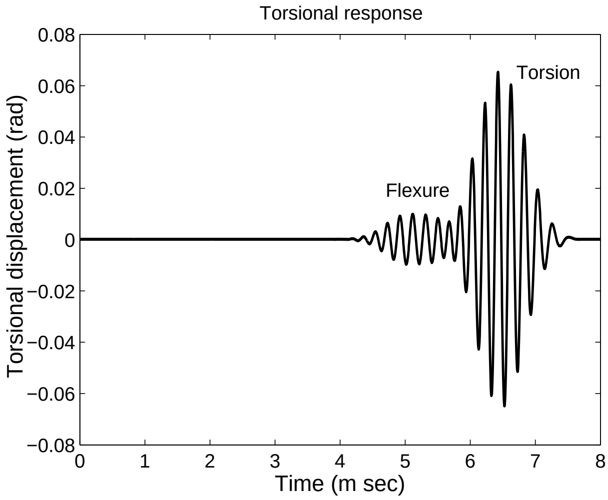

Figure 2: Torsion response due to modulated sinusoidal input

4 CONCLUSIONS

Wave propagation in pre-twisted anisotropic strip is modeled using spectral finite element method. The framework developed combines the advantages of beam modeling using variational asymptotic method and capturing the dynamics from spectral formulation. The model developed is validated by determining the natural frequency of the strip as a function of pretwist and comparing with results available in the literature. Model predictions are in good agreement with literature. Wave response simulated using the framework is presented to highlight the capability of the technique to investigate wave characteristics with application to structural health monitoring.

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