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EQUATIONS OF MOTION FOR MECHANICAL SYSTEMS SUBJECT TO ACATASTATIC CONSTRAINTS

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Abstract. *This study is focused on a class of discrete mechanical systems subject to equality motion constraints involving time and acatastatic terms. In addition, their original configuration manifold possesses time dependent geometric properties. The emphasis is placed on a proper application of Newton's law of motion. A first step is to consider the corresponding event manifold by adding time to the set of coordinates and then determine its geometric properties. Next, the way of introducing time dependence in these properties through a coordinate transformation in the event manifold or the motion constraints is also investigated. Also, foliation theory is used to clarify some key concepts and provide an accurate definition of a scleronomic manifold, leading to a set of coordinate invariant conditions. The analysis is completed by an appropriate set of equations of motion on the original configuration manifold, appearing as a system of second order ordinary differential equations. Finally, the analytical findings are enhanced and illustrated further by considering selected examples.*

1 INTRODUCTION

Among the many subjects of analytical dynamics, a central place is occupied by those referring to the derivation of the equations of motion of systems subject to equality constraints (e.g., [8, 21, 23]). Over the last decades, it has become apparent that many of the theoretical questions in this area of dynamics can be answered in an illustrative and complete way by employing fundamental concepts of differential geometry [4, 17, 18]. This in turn provides a stronger foundation in solving difficult engineering problems by employing new, more accurate and robust numerical techniques (e.g., [2, 6, 11]). Based on these observations, the main objective of this work is to use such concepts in order to treat a class of mechanical systems. These systems are described by configuration manifolds possessing time dependent geometric properties (i.e., metric and connection) and are subject to equality motion constraints involving an explicit time dependence and acatastatic terms. Besides providing a new and clear interpretation of the key concepts, the theoretic framework chosen allows a systematic derivation of the equations of motion for the class of systems examined.

The present study is an extension of recent work of the authors on scleronomic systems [15, 19]. Again, the emphasis is put on explaining several demanding theoretical aspects, which are of keen interest to the engineering community. In addition, the main philosophical approach adopted is that the safest and deepest principle of Mechanics is Newton's law of motion [7, 9]. Specifically, the new contributions of this work can be summarized as follows. First, the central issue is the consistent application of Newton's law to the class of mechanical systems examined. Due to the presence of time in the properties of the configuration manifold, the need to express Newton's law in an invariant form leads naturally to its application to the event configuration manifold of the system [12, 23]. This step is performed first in a geometrically consistent manner. Moreover, another issue clarified in this work is the way time effects are introduced into the properties of a configuration manifold in an explicit manner. It is shown that these effects appear either by performing a time dependent transformation in the event manifold or by enforcing additional time dependent motion constraints. In addition, it is demonstrated that the former effects are in fact removable. Furthermore, the geometrical picture built by employing the classical engineering approach is enhanced in a remarkable way by utilizing some ideas of foliation theory [3, 14, 22]. Among other things, this theory provides a coordinate invariant way of stating the conditions for a configuration manifold to be scleronomic, even though its geometric properties may exhibit a spurious time dependence. Another important new contribution of the present work is the derivation of the equations of motion for the class of systems examined as a set of second order ordinary differential equations in the original configuration manifold. This presents certain advantages when compared to classical approaches leading to a set of differential algebraic equations, instead. It also exhibits better characteristics than previous methods leading to sets of ordinary differential equations through elimination of motion constraints or Lagrange multipliers [2, 15]. Finally, for the special case of scleronomic systems, the equations of motion derived are shown to become identical with a similar set of equations obtained earlier [15].

The organization of this paper is as follows. First, some useful concepts of differential geometry are briefly summarized in Section 2. In Section 3, the idea of the event manifold is employed and its geometric properties are set up. Then, the equations of motion are derived, firstly on the event and subsequently on the configuration manifold. In Section 4 are discussed effects caused by a special change of coordinates in the event manifold. Likewise, in Section 5 is investigated the effect caused by the presence of time dependent and acatastatic terms in the motion constraints. The results of these effects are identified and separated in Section 6, where a set of scleronomicity conditions is presented by employing basic ideas of foliation theory on manifolds. Then, the equations of motion for the general class of systems examined are derived and presented in Section 7. Finally, results are presented for two mechanical examples, illustrating basic features of the class of systems examined.

2 SOME GEOMETRIC PRELIMINARIES

This work focuses on the dynamics of mechanical systems, whose spatial configuration can be determined at any time by a finite set of local coordinates, say $q^1,\ldots,q^n$ [12, 21]. The motion of such a system can be represented by the motion of a fictitious point p moving along a curve $\gamma=\gamma(s)$ on an n-dimensional manifold M, the configuration space of the system [23]. Then, if φ is a coordinate map acting from a neighborhood of point p to the classical space $\mathbb{R}^n$, the coordinates of p are

$$q=\varphi(p)\,, \tag{1}$$

with $q=(q^1,\ldots,q^n)$. Usually, the parameter s coincides with time t and the quantity $\underline{v}=d\gamma/dt$ represents the velocity vector at p. This vector is tangent to γ and belongs to an n-dimensional vector space T_pM, the tangent space of the manifold at p [4, 10], with components $v^i=\dot{q}^i$ in the holonomic basis $\mathfrak{B}_g=\{\underline{g}_1 \quad \ldots \quad \underline{g}_n\}$ [5]. Adopting the usual summation convention for repeated indices [24], the velocity vector can be written as

$$\underline{v}=\sum\nolimits_{i=1}^{n} v^i\,\underline{g}_i = v^i\,\underline{g}_i\,.$$

In general, any element $\underline{u}$ of the vector space T_pM can be expressed in the form $\underline{u}=u^i\,\underline{e}_i$, where $\mathfrak{B}_e=\{\underline{e}_1 \quad \ldots \quad \underline{e}_n\}$ is an arbitrary basis of T_pM. Also, one needs often to establish a relation between two different bases, say between $\mathfrak{B}_g$ and $\mathfrak{B}_e$, through equations of the form

$$\underline{e}_i=A_i^j\,\underline{g}_j \quad \text{or} \quad \underline{g}_i=B_i^j\,\underline{e}_j \quad (i,j=1,\ldots,n), \tag{2}$$

where A_i^j and B_i^j are elements of $n\times n$ matrices, satisfying the conditions

$$A_i^j\,B_k^i=A_k^i\,B_i^j=\delta_k^j\,, \tag{3}$$

which involve Kronecker's delta δ_k^j [17]. Moreover, any basis of T_pM is characterized by its structure constants c_{jk}^i, defined through the Lie bracket in the form

$$[\underline{e}_j,\underline{e}_k]=c_{jk}^i\,\underline{e}_i\,.$$

A complete description of the motion of a dynamical system can be achieved by introducing the dual space to T_pM at any point of the manifold, denoted by T_p^*M. Its elements are called convectors. Then, for any vector $\underline{u}$, a covector $\underset{\sim}{u}^*$ may be found through the dual product. For example, to each basis $\mathfrak{B}_e$ of T_pM, a dual basis $\mathfrak{B}_e^*$ can be constructed for T_p^*M, with elements $\{\underset{\sim}{e}^i\}$ satisfying

$$\underset{\sim}{e}^i(\underline{e}_j)=\delta_j^i\,, \tag{4}$$

Apart from the set of points, a manifold used in dynamics should possess two additional geometrical objects. One of them is the metric tensor, with components in the basis $\mathfrak{B}_e^*$ given by

$$g_{ij} = \langle \underline{e}_i, \underline{e}_j \rangle\ ,\tag{5}$$

where $\langle \cdot,\cdot \rangle$ denotes the inner product of the tangent space $T_p M$. This provides a dual product by

$$\underset{\sim}{u}^*(\underline{w}) \equiv \langle \underline{u}, \underline{w} \rangle, \quad \forall \underline{w} \in T_p M\ .\tag{6}$$

If the components of the metric tensor are identified with those of mass matrix, through the kinetic energy, then the covector corresponding to the velocity vector $\underline{v}$ is the generalized momentum $\underset{\sim}{p}^*$.

Another geometrical tool which is essential in dynamics is the affine connection, represented by symbol ∇. Its components Λ^i_{jk} in a basis $\mathfrak{B}_e$ of space $T_p M$, known as affinities, are defined by

$$\nabla_{\underline{e}_j} \underline{e}_k = \Lambda^i_{jk} \underline{e}_i\ .\tag{7}$$

Then, the covariant differential of a covector $\underset{\sim}{u}^*$ along a curve on M with tangent vector $\underline{v}$ is

$$\nabla_{\underline{v}} \underset{\sim}{u}^* = (u_{i,j} - \Lambda^\ell_{ji} u_\ell) v^j \underset{\sim}{e}^i\ ,\tag{8}$$

with $u_{i,j} = \partial u_i / \partial q^j$. Then, M is a Newton manifold if determination of the true path of motion on M due to a given set of applied forces $\underset{\sim}{f}^*$ is based on application of Newton's second law with form

$$\nabla_{\underline{v}} \underset{\sim}{p}^* = \underset{\sim}{f}^*\ ,\tag{9}$$

Next, consider a new manifold M_A, with a dimension m smaller than n, generated after imposing a set of motion constraints on M , which are eventually cast in the form

$$v^i = N^i_\alpha v^\alpha \quad (i = 1,\ldots,n\ ;\ \alpha = 1,\ldots,m),\tag{10}$$

where $\underline{v}_A = v^\alpha \underline{e}_\alpha$ is a vector of $T_{p_A} M_A$ and $\underline{v} = v^i \underline{e}_i$ is an element of $T_p M$. Then, it was shown in a previous study that if the original manifold M is Newtonian, the new manifold M_A remains Newtonian, provided that the metric and connection on M_A satisfy the compatibility conditions

$$g_{\alpha\beta} = N^i_\alpha g_{ij} N^j_\beta\tag{11}$$

and

$$(\Lambda^\lambda_{\beta\alpha} g_{\lambda\gamma} - N^i_{\alpha,\gamma} N^j_\beta g_{ij} - N^i_\alpha N^j_\beta N^k_\gamma \Lambda^\ell_{ji} g_{\ell k}) v^\beta v^\gamma = 0\ ,\tag{12}$$

with $N^i_{\alpha,\gamma} = \partial N^i_\alpha / \partial q^\gamma$ [26]. Using Eq. (8), the last condition yields

$$\Lambda^\lambda_{\beta\alpha} = g^{\lambda\gamma} [N^j_\beta (N^i_{\alpha,\gamma} g_{ij} + N^i_\alpha N^k_\gamma \Lambda^\ell_{ji} g_{\ell k}) + \sigma_{\alpha\beta\gamma}]\ ,$$

where $\sigma_{\alpha\beta\gamma} = -\sigma_{\alpha\gamma\beta}$ and the remaining part corresponds to what is known in the literature as "the metric compatibility condition" [4, 24].

3 GEOMETRIC PROPERTIES OF EVENT MANIFOLD - APPLICATION OF NEWTON'S LAW

For a mechanical system, the geometric properties of the configuration manifold M can be determined through enforcement of a set of constraints on the motion of a system of N unconstrained particles, taking place within an archetypal Euclidean manifold E^{3N}, with dimension $3N$ [9, 17]. When inertial Cartesian coordinates are used, the metric of this space involves only constant diagonal elements, while all the affinities vanish [19]. Furthermore, even when general curvilinear coordinates are used, the geometric properties of this manifold may depend on the position on the manifold but they do not depend on time explicitly. In this way, E^{3N} presents a raw model for a scleronomic manifold. Then, for a general scleronomic manifold M, it is true that

$$g_{ij} = g_{ij}(q) \quad \text{and} \quad \Lambda^k_{ij} = \Lambda^k_{ij}(q)\,, \tag{13}$$

where the coordinates q are related to position on the manifold through Eq. (1). However, in several cases, the motion constraints imposed on the original system of particles or the transformations performed are such that the geometric properties of a configuration manifold M appear in the form

$$g_{ij} = g_{ij}(q(t),t) \quad \text{and} \quad \Lambda^k_{ij} = \Lambda^k_{ij}(q(t),t) \tag{14}$$

and the manifold is rheonomic. In fact, this can occur artificially, even in the archetypal scleronomic manifold E^{3N}, by performing a velocity transformation like that of Eq. (2), with elements $A^j_i(q,t)$, depending on time [9, 16]. Another way to introduce time dependence is through enforcement of a set of motion constraints. These ideas will become more concrete in Sections 4 and 5, respectively.

When the geometric properties of a manifold M are time dependent, the need to keep the form invariance of Newton's law, as expressed by Eq. (9) through the covariant derivative of momentum, makes it convenient to perform the analysis of the motion on the so called event or extended configuration manifold, denoted by $\bar{M}$, with dimension $n+1$ [12, 23]. A natural way to introduce this manifold is through coordinate maps. Specifically, if $\bar{\varphi}$ is a coordinate map of $\bar{M}$, corresponding to a coordinate map φ on M, then the coordinates of point p of $\bar{M}$ can be selected by

$$\bar{q} = \bar{\varphi}(p) = (q,\tau)\,,$$

where τ is a new coordinate, introduced for accommodating time. This means that

$$\bar{q}^i = q^i \quad (i=1,\ldots,n) \quad \text{and} \quad \bar{q}^{n+1} \equiv \bar{q}^0 = \tau\,. \tag{16}$$

Consequently, the tangent vector $\bar{v}$ to the image curve $\bar{\gamma}(s)$ on $\bar{M}$ of a curve $\gamma(s)$ on M at point p belongs to an ($n+1$)-dimensional vector space $T_p\bar{M}$, with components

$$\bar{v}^i = \tfrac{d}{ds}(\bar{\varphi}\circ\bar{\gamma})^i \quad (i=1,\ldots,n) \quad \text{and} \quad \bar{v}^{n+1} \equiv \bar{v}^0 = d\tau/ds \tag{17}$$

in the natural basis $\mathfrak{B}_{\bar{g}} = \{\underline{\bar{g}}_1 \quad \cdots \quad \underline{\bar{g}}_n \quad \underline{\bar{g}}_0\}$ of $T_p\bar{M}$, created by the tangent vectors to the coordinate lines corresponding to the coordinate map defined by Eq. (15). The last result can be put in the form

$$\underline{\bar{v}} = \bar{v}^I \underline{\bar{g}}_I = \bar{v}^i \underline{\bar{g}}_i + \bar{v}^0 \underline{\bar{g}}_0 \quad (I = 1,\ldots,n+1;\ i = 1,\ldots,n). \tag{18}$$

Similar expressions can also be obtained for bases $\mathfrak{B}_{\bar{e}}$ of $T_p\bar{M}$ other than the natural basis $\mathfrak{B}_{\bar{g}}$ but respecting the separation of the temporal from the spatial coordinates induced by Eq. (19). Such bases will be referred to as separable bases of $T_p\bar{M}$ (or $\bar{M}$, for simplicity) in the sequel.

In essence, incorporation of time into the set of coordinates helps in converting the original manifold M into an extended configuration manifold $\bar{M}$, which is scleronomic [17]. Then, a linear velocity transformation similar to that of Eq. (10) can always be established between the two scleronomic manifolds $\bar{E}^{3N} \equiv E^{3N} \times \mathbb{R}$ and $\bar{M}$. Before employing this transformation, it is first necessary to determine the geometric properties of $\bar{E}^{3N}$ and $\bar{M}$. Based on the definition of the former manifold as a Cartesian product of E^{3N} and $\mathbb{R}$, its metric matrix appears in the form

$$\bar{G} = [\bar{G}_{IJ}] = \begin{bmatrix} G(q) & \underline{0} \\ \underline{0}^T & G_{00}(\tau) \end{bmatrix}, \tag{19}$$

where the $n \times n$ submatrix $G = [G_{ij}]$ includes the components of the metric tensor on E^{3N}, while $G_{00}(\tau)$ is a non-negative scalar, which may be a function of the temporal coordinate. In addition, the set of affinities on $\bar{E}^{3N}$, say $\bar{\Gamma}^K_{IJ}$ (with $I, J, K = 1,\ldots,n+1$), includes the following elements

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij}(q) \quad (i,j,k = 1,\ldots,n) \quad \text{and} \quad \bar{\Gamma}^0_{00} = \Gamma^0_{00}(\tau), \tag{20}$$

where Γ^k_{ij} are the affinities on E^{3N}, while all the remaining mixed type affinities (involving any combination of the spatial coordinates and the temporal coordinate) are zero.

Next, knowledge of the set of motion constraints imposed on $\bar{E}^{3N}$ in order to get the motion on $\bar{M}$ permits evaluation of the velocity transformation matrix between the tangent spaces of these manifolds, expressed by Eq. (10) (see Section 5). Then, the geometric properties of $\bar{M}$ can be determined by application of Eqs. (11) and (12). First, the metric matrix on $\bar{M}$ is obtained in the form

$$\bar{g}(\bar{q}) = [\bar{g}_{IJ}] = \begin{bmatrix} \hat{g}(q,\tau) & \underline{\bar{g}}_0(q,\tau) \\ \underline{\bar{g}}_0^T(q,\tau) & \bar{g}_{00}(q,\tau) \end{bmatrix}, \tag{21}$$

where $\hat{g} = [\hat{g}_{ij}]$ is an $n \times n$ matrix, while $\underline{\bar{g}}_0$ is an n-vector and $\bar{g}_{00}$ is a scalar quantity. In addition, the set of affinities becomes complete, in the sense

$$\bar{\Lambda}^K_{IJ}(\bar{q}) = [\bar{\Lambda}^k_{ij}, \bar{\Lambda}^k_{i0}, \bar{\Lambda}^k_{0j}, \bar{\Lambda}^k_{00}, \bar{\Lambda}^0_{ij}, \bar{\Lambda}^0_{i0}, \bar{\Lambda}^0_{0j}, \bar{\Lambda}^0_{00}]. \tag{22}$$

Taking into account the qualitative difference in the role of the temporal and the spatial coordinates and Eq. (19), it is frequently advantageous to select a special basis in $T_p\bar{M}$, in which

$$\underline{\bar{g}}_0(q,\tau)=\underline{0}\,.$$

Such a basis will be called a standard basis of $\bar{M}$ in the sequel. Based on Eq. (5), this means that standard is a special separable basis, with base vector corresponding to the temporal co-ordinate being normal to the base vectors associated with the subspace of $T_p\bar{M}$ spanned by the spatial coordinates. Then, it is easy to show that in a standard basis of $T_p\bar{M}$ it is true that

$$\bar{v}^i = v^i \quad (i=1,\ldots,n), \tag{23}$$

where v^i represents velocity components in the corresponding basis of T_pM . Due to the symmetry of the metric matrix, this choice is always possible through a transformation.

For the problem at hand, the communication between the configuration manifold M and the event manifold $\bar{M}$, after selecting $s=\tau$, is achieved by imposing the single holonomic constraint

$$\bar{q}^0-\tau=0 \quad \Rightarrow \quad \bar{v}^0-1=0\,. \tag{24}$$

Moreover, Eqs. (17) and (23) define an $(n+1)\times n$ transformation between the spaces T_pM and $T_p\bar{M}$. By employing a standard basis in $T_p\bar{M}$, it can be put in the following compact form

$$\bar{v}^I=\delta_i^I v^i+\delta_0^I\bar{v}^0 \quad (I=1,\ldots,n+1\,;\; i=1,\ldots,n). \tag{25}$$

Based on this transformation, the components g_{ij} and Λ_{ij}^k of the metric and the affinities on manifold M can be related to the components $\bar{g}_{IJ}$ and $\bar{\Lambda}_{IJ}^K$ of the metric and the affinities on manifold $\bar{M}$ through the compatibility conditions (11) and (12). More specifically, employing a separable basis on $\bar{M}$, the linear part of the velocity transformation expressed by Eq. (10) is given in the present case by

$$N_i^I=\delta_i^I\,. \tag{26}$$

Therefore, it is straightforward to verify that Eq. (11) is satisfied when

$$g_{ij}=\bar{g}_{ij} \quad (i,j=1,\ldots,n), \tag{27}$$

while application of Eq. (14) yields the following condition on the affinities

$$(\Lambda_{ji}^{\ell}g_{\ell k}-\bar{\Lambda}_{ji}^{\ell}\bar{g}_{\ell k}-\bar{\Lambda}_{ji}^{0}\bar{g}_{0k})v^j v^k=0 \quad (i,j,k,\ell=1,\ldots,n). \tag{28}$$

In a standard basis, the last condition is satisfied by the simpler choice $\bar{\Lambda}_{ij}^k=\Lambda_{ij}^k$.

The above help in establishing the interrelation between manifolds M and E^{3N} , through the interrelation between the event manifolds $\bar{M}$ and $\bar{E}^{3N}$, as shown schematically in Fig. 1.

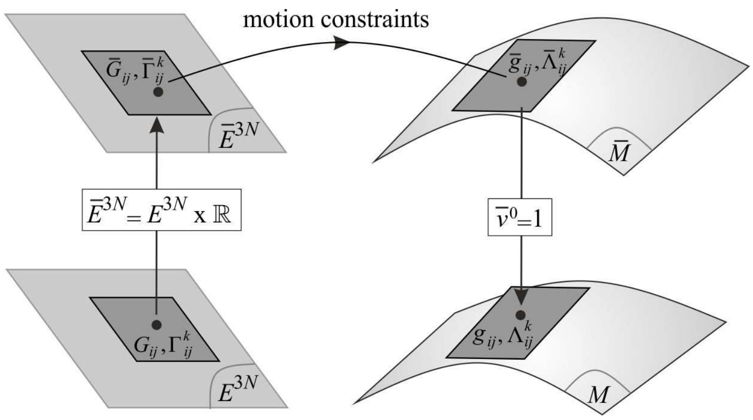

Fig. 1 Relation between manifolds M and E^{3N}, through their relation to $\bar{M}$ and $\bar{E}^{3N}$

Using the metric matrix in the form of Eq. (21), the kinetic energy of the system on the extended configuration manifold $\bar{M}$ can be put in the form

$$\bar{T} = \tfrac{1}{2}\bar{v}^I \bar{g}_{IJ} \bar{v}^J \equiv \bar{T}_2 + \bar{T}_1 + \bar{T}_0 \quad (I, J = 1, \ldots, n+1),$$

where, using Eqs. (18) and (27), the three individual terms are found in a separable basis as follows

$$\bar{T}_2 = \tfrac{1}{2}\bar{v}^i g_{ij} \bar{v}^j, \quad \bar{T}_1 = \bar{v}^i \bar{g}_{i0} \bar{v}^0 \quad \text{and} \quad \bar{T}_0 = \tfrac{1}{2}\bar{v}^0 \bar{g}_{00} \bar{v}^0 .$$

Therefore, all these terms are in fact quadratic in velocity. Moreover, the generalized momenta in the event manifold $\bar{M}$ can be determined in the form

$$\bar{p}_I = \bar{g}_{IJ} \bar{v}^J \quad (I, J = 1, \ldots, n+1),$$

with

$$\bar{p}_i = \bar{g}_{iJ} \bar{v}^J = g_{ij} \bar{v}^j + \bar{g}_{i0} \bar{v}^0 \quad (i = 1, \ldots, n) \quad \text{and} \quad \bar{p}_{n+1} \equiv \bar{p}_0 = \bar{g}_{0J} \bar{v}^J = \bar{g}_{0j} \bar{v}^j + \bar{g}_{00} \bar{v}^0 .$$

Finally, since the extended configuration manifold $\bar{M}$ is scleronomic, the true path of motion on it can be determined by application of Newton's second law in the generalized form

$$\nabla_{\underset{\sim}{\bar{v}}} \underset{\sim}{p}^*_{\bar{M}} = \underset{\sim}{f}^*_{\bar{M}}, \tag{29}$$

where $\underset{\sim}{f}^*_{\bar{M}}$ represents the applied forces on $\bar{M}$, while

$$\nabla_{\underset{\sim}{\bar{v}}} \underset{\sim}{p}^*_{\bar{M}} = (\bar{p}_{I,K} \bar{v}^K - \bar{\Lambda}^L_{JI} \bar{p}_L \bar{v}^J) \underset{\sim}{\bar{e}}^I, \tag{30}$$

where $\bar{p}_{I,J} \equiv \partial \bar{p}_I / \partial \bar{q}^J$ [10, 19].

Then, a special class of covectors, known as Newton covectors[21], can be defined on manifold $\bar{M}$ by

$$\underset{\sim}{h}^*_{\bar{M}} \equiv \nabla_{\underline{\bar{v}}}\, \underset{\sim}{p}^*_{\bar{M}} - \underset{\sim}{f}^*_{\bar{M}}\,, \tag{31}$$

with components in a general basis of $T_p^*\bar{M}$ given by

$$\bar{h}_I = (\bar{g}_{IJ}\bar{v}^J)_{,K}\bar{v}^K - \bar{\Lambda}^L_{JI}\bar{g}_{LK}\bar{v}^J\bar{v}^K - \bar{f}_I\,. \tag{32}$$

By definition of these covectors, the law of motion on manifold $\bar{M}$ is expressed in the form

$$\underset{\sim}{h}^*_{\bar{M}} = \underset{\sim}{0} \quad \Rightarrow \quad \bar{h}_I = 0\,, \tag{33}$$

in the absence of any motion constraints.

Next, based on Eq. (26), the components of the Newton covectors on the original manifold M are found by employing the following mapping from a separable basis of $T_p^*\bar{M}$ to T_p^*M

$$h_i = \delta_i^I\,\bar{h}_I\,. \tag{34}$$

Using Eqs. (32), (27) and performing simple mathematical operations, it eventually turns out that

$$h_i = (g_{ij}\bar{v}^j + \bar{g}_{i0})_{,k}\bar{v}^k + (g_{ij}\bar{v}^j + \bar{g}_{i0})_{,0} - \bar{\Lambda}^L_{Ji}\bar{g}_{LK}\bar{v}^J\bar{v}^K - f_i\,, \tag{35}$$

with

$$\begin{aligned} \bar{\Lambda}^L_{Ji}\bar{g}_{LK}\bar{v}^J\bar{v}^K &= (\bar{\Lambda}^\ell_{ji}g_{\ell k} + \bar{\Lambda}^0_{ji}\bar{g}_{0k})\bar{v}^j\bar{v}^k \\ &\quad + (\bar{\Lambda}^\ell_{ji}\bar{g}_{\ell 0} + \bar{\Lambda}^0_{ji}\bar{g}_{00} + \bar{\Lambda}^\ell_{0i}g_{\ell j} + \bar{\Lambda}^0_{0i}\bar{g}_{0j})\bar{v}^j + (\bar{\Lambda}^\ell_{0i}\bar{g}_{\ell 0} + \bar{\Lambda}^0_{0i}\bar{g}_{00}). \end{aligned} \tag{36}$$

Again, in the special case of a standard basis of $\bar{M}$, the last expression is simplified considerably to

$$\bar{\Lambda}^L_{Ji}\bar{g}_{LK}\bar{v}^J\bar{v}^K = \Lambda^\ell_{ji}g_{\ell k}v^jv^k + (\bar{\Lambda}^\ell_{0i}g_{\ell j} + \bar{\Lambda}^0_{ji}\bar{g}_{00})v^j + \bar{\Lambda}^0_{0i}\bar{g}_{00}\,. \tag{37}$$

Furthermore, more terms disappear for scleronomic systems, since then all the mixed affinities vanish. Then, Newton's law on manifold M is written in the simpler form

$$h_i = (g_{ij}v^j)_{,k}v^k - \Lambda^\ell_{ji}g_{\ell k}v^jv^k - f_i = 0\,, \tag{38}$$

resembling fully that of Eq. (32) on $\bar{M}$. The extra terms in Eq. (35) are inertial and come either from terms involving the off-diagonal components $\bar{g}_{i0}$ of the metric of a non-standard basis or by mixed affinities, defined on the event manifold only and related to differentiation with respect to time.

4 APPLICATION OF A SPECIAL COORDINATE TRANSFORMATION ON THE EVENT MANIFOLD

Next, let ψ be an alternate coordinate map in the vicinity of point p on M, similar to that defined by Eq. (1). Then, the corresponding set of coordinates of p on $\bar{M}$ can be selected by

$$\bar{x} = \bar{\psi}(p) = (x, \omega)\,,$$

through a new coordinate map $\bar{\psi}$. The last equation can be combined with Eq. (15) to yield

$$\bar{x} = \bar{\chi}(\bar{q}) = (x(q,\tau), \omega(q,\tau))\,, \tag{39}$$

with composite map $\bar{\chi} = \bar{\psi} \circ \bar{\varphi}^{-1}$, so that

$$\bar{x}^i = x^i(q,\tau) \quad (i = 1,\dots,n) \quad \text{and} \quad \bar{x}^{n+1} \equiv \bar{x}^0 = \omega(q,\tau)\,. \tag{40}$$

In fact, for the purposes of the present study, the last coordinate can be selected by

$$\omega(q,\tau) = \tau\,. \tag{41}$$

Then, the Jacobian matrix $\bar{J} = [\bar{J}^I_{\hat{I}}]$ (with $I, \hat{I} = 1,\dots,n+1$) corresponding to this particular type of coordinate transformation is given by

$$\bar{J}(\bar{q}) = \frac{\partial \bar{x}}{\partial \bar{q}} = \begin{bmatrix} J(q,\tau) & \underline{c}(q,\tau) \\ \underline{0}^T & 1 \end{bmatrix}, \tag{42}$$

where the $n \times n$ matrix $J = [J^i_{\hat{i}}]$ and the n-vector $\underline{c}$ are determined by

$$J(q,\tau) = \partial x / \partial q \quad \text{and} \quad \underline{c}(q,\tau) = \partial x / \partial \tau\,. \tag{43}$$

Based on the above, the class of transformations considered are similar to Eq. (2), but have the special structure exhibited by Eq. (42). Then, the new and old base vectors in the corresponding bases of $T_p\bar{M}$ and $T^*_p\bar{M}$ are related by

$$\underline{\bar{e}}_{\hat{I}} = \bar{J}^I_{\hat{I}}(\bar{q})\underline{e}_I \quad \text{and} \quad \underline{e}^I = \bar{J}^I_{\hat{I}}(\bar{q})\underline{\bar{e}}^{\hat{I}}\,, \tag{44}$$

while the components of the vectors and covectors in these bases are related by

$$\bar{v}^I = \bar{J}^I_{\hat{I}}(\bar{q})\bar{v}^{\hat{I}} \quad \text{and} \quad \bar{v}_{\hat{I}} = \bar{J}^I_{\hat{I}}(\bar{q})\bar{v}_I\,, \tag{45}$$

respectively [17]. Finally, the components of the metric tensor and the affinities in the new basis of $T_p\bar{M}$ are found by

$$\bar{g}_{\hat{I}\hat{J}} = \bar{J}^I_{\hat{I}}\bar{g}_{IJ}\bar{J}^J_{\hat{J}} \quad \text{and} \quad \bar{\Lambda}^{\hat{K}}_{\hat{I}\hat{J}} = \bar{J}^I_{\hat{I}}\bar{J}^J_{\hat{J}}\bar{B}^{\hat{K}}_K\bar{\Lambda}^K_{IJ} + \bar{J}^K_{\hat{J},\hat{I}}\bar{B}^{\hat{K}}_K\,,$$

respectively. These relations demonstrate that the new metric components and affinities are time dependent, when at least one element of matrix J or vector $\underline{c}$ is time dependent, even if the original components do not depend on time. Specifically, the new metric matrix appears in the form

$$\hat{\bar{g}} = [\bar{g}_{\hat{I}\hat{J}}] = \bar{J}^T\bar{g}\bar{J} = \begin{bmatrix} \hat{g} & \underline{g}_{\hat{0}} \\ \underline{g}^T_{\hat{0}} & g_{\hat{0}\hat{0}} \end{bmatrix} \tag{47}$$

on the extended configuration manifold $\bar{M}$. Taking Eq. (21) into account, it turns out that

$$\hat{g} = [g_{\hat{i}\hat{j}}] = J^T\bar{g}J\,, \quad \underline{g}_{\hat{0}} = J^T\bar{g}\underline{c} + J^T\underline{\bar{g}}_0 \quad \text{and} \quad g_{\hat{0}\hat{0}} = \underline{c}^T\bar{g}\underline{c} + 2\underline{c}^T\underline{\bar{g}}_0 + \bar{g}_{00}\,. \tag{48}$$

The above results hold even in cases where the elements of matrix J and vector $\underline{c}$ in Eq. (42) do not satisfy Eq. (43). Then, the new basis, defined by Eq. (44), will be anholonomic in general. However, it is easy to verify that it will remain separable. In this respect, it is noteworthy that the set of matrices expressed by Eq. (42) presents similarities with the Galilean group of transformations applied to a particle in order to preserve the form of Newton's law in Cartesian coordinate systems [1, 23]. Also, even when a standard basis is employed in $T_p\overline{M}$ originally (i.e., when $\underline{\overline{g}}_0 = \underline{0}$), the new basis will turn into a separable basis eventually, if $\underline{c} \neq \underline{0}$. Finally, when the transformation is such that $\underline{c} = \underline{0}$, a standard basis yields a new standard basis, since then Eqs. (47) and (48) lead to

$$\hat{\overline{g}} = \begin{bmatrix} J^T \hat{g} J & \underline{0} \\ \underline{0}^T & \overline{g}_{00} \end{bmatrix}. \tag{49}$$

Now, based on Eq. (45b), the components of the Newton covectors in the new basis of $T_p^*\overline{M}$ are

$$\overline{h}_{\hat{I}} = \overline{J}_{\hat{I}}^{I}\,\overline{h}_I\,. \tag{50}$$

Substituting Eq. (32) in the right hand side of the last equation and performing simple operations leads to

$$\overline{h}_{\hat{I}} = (\overline{J}_{\hat{I}}^{I}\overline{g}_{IJ}\overline{J}_{\hat{J}}^{J}\overline{v}^{\hat{J}})_{,\hat{K}}\overline{v}^{\hat{K}} - (\overline{J}_{\hat{I},\hat{K}}^{I}\overline{J}_{\hat{J}}^{J}\overline{g}_{IJ} + \overline{J}_{\hat{I}}^{I}\overline{J}_{\hat{J}}^{J}\overline{J}_{\hat{K}}^{K}\overline{\Lambda}_{JI}^{L}\overline{g}_{LK})\overline{v}^{\hat{J}}\overline{v}^{\hat{K}} - \overline{J}_{\hat{I}}^{I}\,\overline{f}_I\,. \tag{51}$$

Next, following an application of the compatibility conditions expressed by Eqs. (11) and (12) in the more appropriate form

$$\overline{g}_{\hat{I}\hat{J}} = \overline{J}_{\hat{I}}^{I}\overline{g}_{IJ}\overline{J}_{\hat{J}}^{J} \quad \text{and} \quad (\overline{\Lambda}_{\hat{J}\hat{I}}^{\hat{L}}\overline{g}_{\hat{L}\hat{K}} - \overline{J}_{\hat{I},\hat{K}}^{I}\overline{J}_{\hat{J}}^{J}\overline{g}_{IJ} - \overline{J}_{\hat{I}}^{I}\overline{J}_{\hat{J}}^{J}\overline{J}_{\hat{K}}^{K}\overline{\Lambda}_{JI}^{L}\overline{g}_{LK})\overline{v}^{\hat{J}}\overline{v}^{\hat{K}} = 0\,, \tag{52}$$

it eventually turns out that

$$\overline{h}_{\hat{I}} = (\overline{g}_{\hat{I}\hat{J}}\overline{v}^{\hat{J}})_{,\hat{K}}\overline{v}^{\hat{K}} - \overline{\Lambda}_{\hat{J}\hat{I}}^{\hat{L}}\overline{g}_{\hat{L}\hat{K}}\overline{v}^{\hat{J}}\overline{v}^{\hat{K}} - \overline{f}_{\hat{I}}\,. \tag{53}$$

Note that the form of the last equation is identical to the fundamental form of Eq. (32). Finally, by employing the mapping from a separable basis of $T_p^*\overline{M}$ to T_p^*M defined by

$$h_{\hat{i}} = \delta_{\hat{i}}^{\hat{I}}\,\overline{h}_{\hat{I}} \tag{54}$$

and performing simple mathematical operations it eventually turns out that

$$h_{\hat{i}} = (g_{\hat{i}\hat{j}}\overline{v}^{\hat{j}} + \overline{g}_{\hat{i}\hat{0}})_{,\hat{k}}\overline{v}^{\hat{k}} + (g_{\hat{i}\hat{j}}\overline{v}^{\hat{j}} + \overline{g}_{\hat{i}\hat{0}})_{,\hat{0}} - \overline{\Lambda}_{\hat{J}\hat{i}}^{\hat{L}}\overline{g}_{\hat{L}\hat{K}}\overline{v}^{\hat{J}}\overline{v}^{\hat{K}} - f_{\hat{i}}\,, \tag{55}$$

with

$$\begin{aligned} \overline{\Lambda}_{\hat{J}\hat{i}}^{\hat{L}}\overline{g}_{\hat{L}\hat{K}}\overline{v}^{\hat{J}}\overline{v}^{\hat{K}} &= (\overline{\Lambda}_{\hat{j}\hat{i}}^{\hat{\ell}}\overline{g}_{\hat{\ell}\hat{k}} + \overline{\Lambda}_{\hat{j}\hat{i}}^{\hat{0}}\overline{g}_{\hat{0}\hat{k}})\overline{v}^{\hat{j}}\overline{v}^{\hat{k}} \\ &\quad + (\overline{\Lambda}_{\hat{j}\hat{i}}^{\hat{\ell}}\overline{g}_{\hat{\ell}\hat{0}} + \overline{\Lambda}_{\hat{j}\hat{i}}^{\hat{0}}\overline{g}_{\hat{0}\hat{0}} + \overline{\Lambda}_{\hat{0}\hat{i}}^{\hat{\ell}}\overline{g}_{\hat{\ell}\hat{j}} + \overline{\Lambda}_{\hat{0}\hat{i}}^{\hat{0}}\overline{g}_{\hat{0}\hat{j}})\overline{v}^{\hat{j}} + (\overline{\Lambda}_{\hat{0}\hat{i}}^{\hat{\ell}}\overline{g}_{\hat{\ell}\hat{0}} + \overline{\Lambda}_{\hat{0}\hat{i}}^{\hat{0}}\overline{g}_{\hat{0}\hat{0}}). \end{aligned} \tag{56}$$

Eq. (55) is similar to Eq. (35), while Eq. (56) is similar to Eq. (36). Moreover, in a standard basis, the last equation can be put in a form similar to Eq. (37). Obviously, the last observations reveal that the form of the components of the Newton covectors remains invariant under the special transformation performed. This confirms the form invariance of Newton's law when expressed in different bases.

To complete the picture, consider a companion change of basis in the configuration manifold M, relating the new and old base vectors in T_pM and T_p^*M by

$$\underline{e}_{\hat{i}} = J^i_{\hat{i}}(q,t)\underline{e}_i \quad \text{and} \quad \underline{\overline{e}}^i = J^i_{\hat{i}}(q,\tau)\underline{\overline{e}}^{\hat{i}},$$

respectively, where matrix $J = [J^i_{\hat{i}}]$ is selected to coincide with the $n \times n$ matrix covering the upper left part of matrix $\bar{J}$ defined by Eq. (42). Then, the components of the vectors and covectors in T_pM and T_p^*M are related by

$$v^i = J^i_{\hat{i}}(q,t)v^{\hat{i}} \quad \text{and} \quad v_{\hat{i}} = J^i_{\hat{i}}(q,t)v_i, \tag{57}$$

respectively, while the components of the metric and the affinities in the new basis of M are given by

$$g_{\hat{i}\hat{j}} = J^i_{\hat{i}} g_{ij} J^j_{\hat{j}} \quad \text{and} \quad \Lambda^{\hat{k}}_{\hat{i}\hat{j}} = J^i_{\hat{i}} J^j_{\hat{j}} B^{\hat{k}}_k \Lambda^k_{ij} + J^k_{\hat{j},\hat{i}} B^{\hat{k}}_k, \tag{58}$$

respectively, with

$$J^i_{\hat{i}} B^{\hat{i}}_j = \delta^i_j \quad \text{and} \quad B^{\hat{j}}_i J^i_{\hat{i}} = \delta^{\hat{j}}_{\hat{i}}. \tag{59}$$

Consequently, the components of the Newton covectors in the transformed basis of the original manifold M are found in the form

$$h_{\hat{i}} = J^i_{\hat{i}} h_i. \tag{60}$$

Employing Eq. (35) and performing lengthy but simple mathematical operations in the right hand side of the last equation, which are similar to those involved in Eqs. (51) and (52), it eventually turns out that $h_{\hat{i}}$ regains the form of Eq. (55). This verifies the existence of commutativity between the special type of coordinate transformation examined and the transformation between the original manifold M and the extended manifold $\bar{M}$.

Finally, using Eq. (23) in conjunction with Eq. (57a) yields

$$\bar{v}^i = v^i = J^i_{\hat{i}}(q,t)v^{\hat{i}} \quad (i, \hat{i} = 1, \ldots, n),$$

where $\bar{v}^i$ is the velocity component in the corresponding standard basis of $\bar{M}$. Taking into account Eq. (59), the last equation leads to

$$v^{\hat{i}} = \bar{v}^{\hat{i}} + \hat{v}^{\hat{i}}, \tag{61}$$

with

$$\hat{v}^{\hat{i}} = B^{\hat{i}}_i J^i_{\hat{0}} \bar{v}^{\hat{0}} = B^{\hat{i}}_i c^i. \tag{62}$$

Eq. (61) shows that $v^{\hat{i}} \neq \bar{v}^{\hat{i}}$, in general, even when $v^i = \bar{v}^i$. Substitution of the term $\bar{v}^{\hat{i}}$ from Eq. (61) into Eq. (55) yields the equations of motion on manifold M in terms of the velocity components $v^{\hat{i}}$ in T_pM and $\hat{v}^{\hat{i}}$. By their definition, Eq. (62), the terms $\hat{v}^{\hat{i}}$ are known and due to the presence of the n-vector $\underline{c}$ in the Jacobian $\bar{J}(\bar{q})$, defined by Eq. (42). Based on Eq. (43), it is clear that this vector becomes zero when a time independent chart (i.e., coordinate system) is selected on $\bar{M}$, defined by

$$x = x(q). \tag{63}$$

This means that the terms $\hat{v}^{\hat{i}}$ can be eliminated by a proper selection of the chart on manifold $\bar{M}$. Therefore, only such charts may have meaningful counterparts on M, which clarifies some of the terminology employed in the literature on "time dependent" charts.

5 APPLICATION OF A SPECIAL COORDINATE TRANSFORMATION ON THE EVENT MANIFOLD

In the present section, it is assumed that the motion of the system examined on configuration manifold M is subject to an additional set of k constraints with form

$$A(q,t)\underline{v} + \underline{a}(q,t) = \underline{0}\,, \tag{64}$$

where $A=[a_i^R]$ is a known $k\times n$ matrix. Then, the motion of the system takes place on a curve $\gamma_A(t)$ of another manifold M_A, with dimension $m=n-k$ [19]. Next, let $\{\theta^\alpha\}$ ($\alpha=1,\ldots,m$) be a set of generalized coordinates in a neighborhood of a point p_A of M_A, related to point p of M through Eq. (64). Furthermore, let $\{\underline{e}_\alpha\}$ and and $\{\underset{\sim}{e}^\alpha\}$ be a basis of $T_{p_A}M_A$ and $T^*_{p_A}M_A$, respectively.

The new fundamental item in Eq. (64) is not so much the explicit appearance of time but rather the acatastatic term $\underline{a}$ [21]. This term destroys the linearity and provides an affine nature to the constraint equations. In principle, Eq. (64) can be used in order to establish a mapping between the tangent spaces $T_{p_A}M_A$ and T_pM. Namely, Eq. (64) can easily be recast in the form

$$\underline{v} = N(q,t)\underline{\dot{\theta}} + \underline{\upsilon}(q,t)\,, \tag{65}$$

where the $n\times m$ matrix N and the n-vector $\underline{\upsilon}$ are determined in terms of the elements of the $k\times n$ matrix A and the k-vector $\underline{a}$ [12, 19]. To overcome the loss of linearity, due to the presence of the term $\underline{\upsilon}$, the last equation is put in the form of a linear transformation between the tangent spaces $T_{p_A}\bar{M}_A$ and $T_p\bar{M}$ of the corresponding event manifolds. This permits conversion of Eq. (65) into the following linear relation between the components of tangent vectors

$$\bar{v}^I = \bar{N}^I_\Phi(\bar{q})\bar{v}^\Phi\,, \tag{66}$$

with $\Phi=1,\ldots,m+1$ and $I=1,\ldots,n+1$, where the extended matrix $\bar{N}$ appears in the form

$$\bar{N} = [\bar{N}^I_\Phi(\bar{q})] = \begin{bmatrix} N(q,\tau) & \underline{\upsilon}(q,\tau) \\ \underline{0}^T & 1 \end{bmatrix}. \tag{67}$$

This $(n+1)\times(m+1)$ matrix has a similar structure with the Jacobian matrix defined by Eq. (42).

Next, the metric matrix on manifold $\bar{M}$ is assumed to have the general form expressed by Eq. (21), while the set of affinities is complete, in the sense of Eq. (22). Then, based on the velocity transformation represented by Eq. (66), the components of the metric matrix and the affinities on the new manifold are determined by employing Eqs. (11) and (12) in the form

$$\bar{g}_{\Phi\Psi} = \bar{N}^I_\Phi \bar{g}_{IJ}\bar{N}^J_\Psi \quad \text{and} \quad (\bar{\Lambda}^{\mathrm{X}}_{\Psi\Phi}\bar{g}_{\mathrm{X}\Omega} - \bar{N}^I_{\Phi,\Omega}\bar{N}^J_\Psi\bar{g}_{IJ} - \bar{N}^I_\Phi\bar{N}^J_\Psi\bar{N}^K_\Omega\bar{\Lambda}^L_{JI}\bar{g}_{LK})\bar{v}^\Psi\bar{v}^\Omega = 0\,. \tag{68}$$

Obviously, any time dependence of matrix N and vector $\underline{\upsilon}$ is transferred into the geometric properties of manifold $\bar{M}_A$. In particular, the metric matrix on the new Newtonian manifold $\bar{M}_A$ is

$$\bar{g}_A = [\bar{g}_{\Phi X}] = \bar{N}^T \bar{g}\, \bar{N} = \begin{bmatrix} N^T \hat{g} N & N^T(\hat{g}\underline{\upsilon} + \underline{\bar{g}}_0) \\ (\underline{\upsilon}^T \hat{g} + \underline{\bar{g}}_0^T) N & \underline{\upsilon}^T \hat{g} \underline{\upsilon} + 2\underline{\upsilon}^T \underline{\bar{g}}_0 + \bar{g}_{00} \end{bmatrix}.$$

In general, this $(m+1)\times(m+1)$ matrix is full. Therefore, unless

$$N^T(\hat{g}\underline{\upsilon} + \underline{\bar{g}}_0) = \underline{0}\,, \tag{70}$$

the new basis obtained in $T_{p_A}\bar{M}_A$ is not standard, even when the basis of $T_p\bar{M}$ is standard.

As usual, application of Eq. (66) means that the base vectors in $T_{p_A}\bar{M}_A$ and $T_p\bar{M}$ are related by

$$\underline{e}_\Phi = \bar{N}^I_\Phi(\bar{q})\underline{e}_I \quad (\Phi = 1,\ldots,m+1;\ I = 1,\ldots,n+1),$$

while the components of the covectors are related by

$$\bar{v}_\Phi = \bar{N}^I_\Phi(\bar{q})\bar{v}_I\,. \tag{71}$$

Therefore, the components of the Newton covectors in the basis of $T^*_{p_A}\bar{M}_A$ are found through

$$\bar{h}_\Phi = \bar{N}^I_\Phi \bar{h}_I\,. \tag{72}$$

Substituting Eq. (32) in the right hand side of the last equation and performing simple operations yields first the following relation

$$\bar{h}_\Phi = (\bar{N}^I_\Phi \bar{g}_{IJ} \bar{N}^J_\Psi \bar{v}^\Psi)_{,\Omega}\bar{v}^\Omega - (\bar{N}^I_{\Phi,\Omega}\bar{N}^J_\Psi \bar{g}_{IJ} + \bar{N}^I_\Phi \bar{N}^J_\Psi \bar{N}^K_\Omega \bar{\Lambda}^L_{JI}\bar{g}_{LK})\bar{v}^\Psi \bar{v}^\Omega - \bar{N}^I_\Phi \bar{f}_I\,. \tag{73}$$

Next, application of the compatibility conditions expressed by Eqs. (68), leads to the final form

$$\bar{h}_\Phi = (\bar{g}_{\Phi\Psi}\bar{v}^\Psi)_{,\Omega}\bar{v}^\Omega - \bar{\Lambda}^X_{\Psi\Phi}\bar{g}_{X\Omega}\bar{v}^\Psi\bar{v}^\Omega - \bar{f}_\Phi\,. \tag{74}$$

Once again, the form of the last equation is similar to Eq. (32). Also, in accordance with Eq. (34), the transformation between separable bases from $T^*_{p_A}M_A$ to $T^*_p\bar{M}$ appears in the form

$$h_\alpha = \delta^\Phi_\alpha \bar{h}_\Phi\,. \tag{75}$$

Consequently, by performing simple mathematical operations one can originally obtain the components of the Newton covectors on the manifold M_A in the form

$$\begin{aligned} h_\alpha = {} & (\bar{g}_{\alpha\beta}\bar{v}^\beta + \bar{g}_{\alpha 0})_{,\gamma}\bar{v}^\gamma + (\bar{g}_{\alpha\beta}\bar{v}^\beta + \bar{g}_{\alpha 0})_{,0} - (\bar{\Lambda}^\gamma_{\beta\alpha}\bar{g}_{\gamma\delta} + \bar{\Lambda}^0_{\beta\alpha}\bar{g}_{0\delta})\bar{v}^\beta\bar{v}^\delta \\ & - (\bar{\Lambda}^\gamma_{\beta\alpha}\bar{g}_{\gamma 0} + \bar{\Lambda}^0_{\beta\alpha}\bar{g}_{00} + \bar{\Lambda}^\gamma_{0\alpha}\bar{g}_{\gamma\beta} + \bar{\Lambda}^0_{0\alpha}\bar{g}_{0\beta})\bar{v}^\beta - (\bar{\Lambda}^\gamma_{0\alpha}\bar{g}_{\gamma 0} + \bar{\Lambda}^0_{0\alpha}\bar{g}_{00}) - f_\alpha = 0. \end{aligned} \tag{76}$$

Some simplifications apply to the last equation when the transition from manifold $\bar{M}_A$ to M_A is performed by using a standard basis in $T_{p_A}\bar{M}_A$, so that

$$\bar{v}^\alpha = v^\alpha, \quad \bar{g}_{\alpha\beta} = g_{\alpha\beta}, \quad \bar{\Lambda}^\gamma_{\alpha\beta} = \Lambda^\gamma_{\alpha\beta} \quad (\alpha,\beta,\gamma = 1,\ldots,m)$$

and the new metric matrix is block diagonal. In addition, some more terms drop out when the system is scleronomic. Then, Eq. (76) can eventually be put in the form

$$h_\alpha = (g_{\alpha\beta} v^\beta)_{,\gamma} v^\gamma - \Lambda^\gamma_{\beta\alpha} g_{\gamma\delta} v^\beta v^\delta - f_\alpha ,$$

which is identical to that of Eq. (38), obtained for the original manifold M in Section 3.

6 APPLICATION OF FOLIATION THEORY AND DEFINITION OF A SCLERONOMIC MANIFOLD

The material presented in this section completes the geometrical picture created in Section 3 for the event manifold $\bar{M}$. First, the attention is focused on creating the extended configuration manifold $\bar{M}$ based on the configuration manifold M at a fixed time $\tau = t$. This is done by considering the flow

$$p = \psi(p_0, t; \tau) \equiv \psi_{\tau,t} p_0 , \tag{77}$$

taking points $p_0 \in M \subset \bar{M}$ at time t to points $p \in \bar{M}$ at a later time τ, as shown in Fig. 2, so that a local coordinate chart on M and a time independent chart on $\bar{M}$ share the same spatial coordinates at each point. Using terminology of foliation theory, the local coordinates (q, τ) on $\bar{M}$, where q are also coordinates on manifold M for each τ, are called distinguished coordinates, while the temporal coordinate τ is known as a transverse coordinate [14]. In this setting, the chart corresponding to Eqs. (39) and (41) is a foliated or distinguished chart, while manifold M is considered to be a leaf of $\bar{M}$ through point p under this foliation. The specific foliation examined here has codimension 1 and partitions the event manifold $\bar{M}$ into manifolds M of dimension n, one at each time τ. Moreover, T denotes a transverse submanifold of $\bar{M}$, which is single dimensional and is described by the temporal coordinate. In fact, it is a total transversal since it meets all the leaves of the foliation, by construction [22]. In addition, the collection of the subspaces of $T_p\bar{M}$ possessing vectors with projection on T_pM only, at all points of $\bar{M}$, forms the special distribution D_M on $\bar{M}$. This is known as the structural distribution of $\bar{M}$, while its complement D_T is called the transversal distribution on $\bar{M}$ [3]. Finally, the tangent space of the event manifold $\bar{M}$ at point p can be split in the form

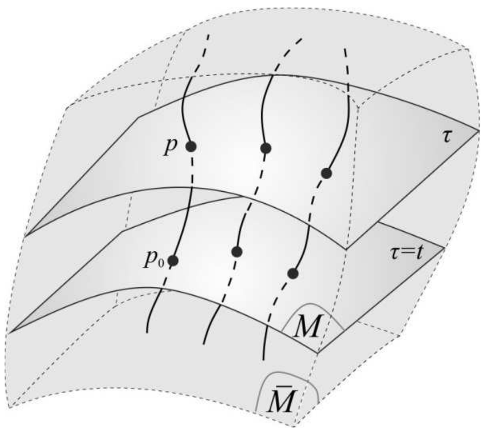

Fig. 2 Construction of manifold $\bar{M}$ by M and the time flow

$$T_p\bar{M} = H_p \oplus V_p , \tag{78}$$

where H_p and V_p are subspaces of $T_p\bar{M}$, forming the distribution D_M and D_T, respectively. Among other things, Eq. (78) implies that any vector of $T_p\bar{M}$ can be split uniquely in the form

$$\underline{u} = \underline{h} + \underline{v}\,, \tag{79}$$

with $\underline{h} \in H_p$ and $\underline{v} \in V_p$, where $\underline{v}$ is a tangent vector along the time flow defined by Eq. (77).

Next, it is assumed that manifold M is a fixed set of points but its metric components and affinities may depend on time. The objective is to provide conditions for M to be scleronomic. To achieve this goal, it is first necessary to define two special sets of curves on $\bar{M}$. The first set includes "leaf curves" $\gamma_M(\tau;s)$, lying entirely on M for each fixed τ, with tangent vector $\underline{h} = d\gamma_M/ds$ belonging to D_M, as depicted in Fig. 3. Likewise, the second set includes "transverse curves" $\gamma_T(q;\sigma)$, with tangent vector $\underline{v} = d\gamma_T/d\sigma$ belonging entirely to the transversal distribution D_T.

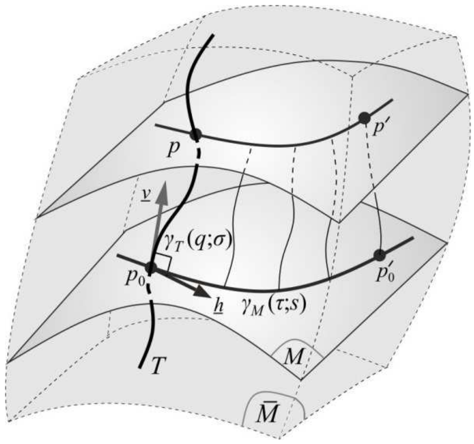

Fig. 3 Definition of two special families of curves on $\bar{M}$

By definition, a scleronomic manifold M can be viewed as a fixed surface within a Euclidean space of a bigger dimension [7, 23]. This means that its geometric properties may depend on its position in M but not on time. Then, the corresponding event manifold $\bar{M}$ can be seen as a product

$$\bar{M} = M \times T\,. \tag{80}$$

When a manifold possesses properties depending on time in an explicit manner, it can be viewed as a surface varying with time within a Euclidean space possessing a sufficiently bigger dimension [4, 17]. However, time dependence may also be introduced in the geometric properties by performing a transformation in $\bar{M}$ even in inherently scleronomic manifolds, as was shown in Section 4. The key to resolving this problem comes from the product form expressed by Eq. (80), which is true for a scleronomic manifold only. This is crucial in revealing and stating its geometric properties, since it implies that the event manifold $\bar{M}$ carries two complementary foliations [3]. Specifically, in such a case, a second foliation exists, which has codimension n and partitions the event manifold $\bar{M}$ into manifolds T of dimension 1, one for each combination of the spatial coordinates q. Based on this, for a configuration manifold M to be scleronomic, it is required that an autoparallel curve starting at a point of M and ending on another point of M, for a fixed τ, should always remain on M. Likewise, an autoparallel curve starting on T and ending on T, for a fixed set of spatial coordinates q, should always remain on T. Taking into account the definition of the special families of curves γ_M and γ_T, these statements are expressed by the following mathematical conditions

$$\nabla_{\underline{h}}\underline{h} \in D_M \quad \text{and} \quad \nabla_{\underline{v}}\underline{v} \in D_T . \tag{81}$$

Furthermore, the Lie derivative of the vector fields created by extending the vector $\nabla_{\underline{h}}\underline{h}$ on all of $\bar{M}$, along a vector field on $\bar{M}$ with tangent vector $\underline{v}$ (i.e., the time flow represented by Eq. (77)), should be zero. Likewise, the Lie derivative of the vector fields created by extending the vector $\nabla_{\underline{v}}\underline{v}$ on all of $\bar{M}$, along a vector field on $\bar{M}$ with tangent vector $\underline{h}$, should also be zero. Mathematically, these statements are equivalent to

$$L_{\underline{v}}(\nabla_{\underline{h}}\underline{h}) = \underline{0} \quad \text{and} \quad L_{\underline{h}}(\nabla_{\underline{v}}\underline{v}) = \underline{0}, \tag{82}$$

where the above Lie derivatives can be determined through the corresponding Lie brackets by

$$L_{\underline{v}}(\nabla_{\underline{h}}\underline{h}) = [\underline{v}, \nabla_{\underline{h}}\underline{h}] \quad \text{and} \quad L_{\underline{h}}(\nabla_{\underline{v}}\underline{v}) = [\underline{h}, \nabla_{\underline{v}}\underline{v}],$$

respectively. Moreover, the parallel transfer of the tangent vector $\underline{h}$ of a leaf curve $\gamma_M(\tau;s)$ along a transverse curve $\gamma_T(q;\sigma)$ and vice versa is preserved on a scleronomic manifold. This means that

$$\nabla_{\underline{v}}\underline{h} = \underline{0} \quad \text{and} \quad \nabla_{\underline{h}}\underline{v} = \underline{0}.$$

On a scleronomic manifold M, conditions should also be imposed on the metric tensor of $\bar{M}$. First, based on the product form of $\bar{M}$, expressed by Eq. (80), its metric matrix in the holonomic basis defined by the foliated coordinates can be decomposed in the special form

$$\bar{g} = \begin{bmatrix} g_M(q) & \underline{0} \\ \underline{0}^T & g_T(\tau) \end{bmatrix}.$$

Then, the following conditions must also be true for preserving the form of the metric tensor along the vector fields created by the special vectors $\underline{v}$ and $\underline{h}$ on a scleronomic manifold

$$L_{\underline{v}}\bar{g}_M = 0 \quad \text{and} \quad L_{\underline{h}}\bar{g}_T = 0, \tag{84}$$

where $\bar{g}_M$ and $\bar{g}_T$ represent the restriction of metric $\bar{g}$ on the structural and transverse distribution of $\bar{M}$, respectively. Among other things, the last conditions guarantee that the metric $\bar{g}$ is bundle-like and any autoparallel of $\bar{M}$, which is orthogonal to a leaf of the foliation, is also orthogonal to every other leaf it meets [22].

In summary, the manifold scleronomicity conditions are expressed by Eqs. (81)-(84). These conditions do not depend on the basis selected on $T_p\bar{M}$. However, they take a very explicit and useful form when they are expressed in the special coordinate basis corresponding to the foliated coordinates of $\bar{M}$. In fact, taking into account that the metric is Riemannian, the two complementary foliations of $\bar{M}$ are now orthogonal, that is $D_T = D_M^{\perp}$ [3]. Therefore, the base vector $\underline{\bar{g}}_0$, corresponding to the temporal coordinate, is orthogonal to all the base vectors $\underline{\bar{g}}_i$ ($i = 1, \ldots, n$), corresponding to the spatial coordinates. At this point, it is clear that a foliated basis is a holonomic separable basis, while a foliated basis with $\underline{\bar{g}}_0$ orthogonal to $\underline{\bar{g}}_i$

corresponds to a holonomic standard basis. Also, the special vectors which are tangent to the leaf and transverse family of curves can be expressed in the form

$$\underline{h} = h^i(q)\overline{\underline{g}}_i \quad \text{and} \quad \underline{v} = v^0(\tau)\overline{\underline{g}}_0 ,$$

respectively, in a holonomic standard basis. Next, employing the first of Eqs. (81) and performing the necessary mathematical operations it turns out originally that

$$\nabla_{\underline{h}}\underline{h} = (h^K_{,I} + \overline{\Lambda}^K_{IJ} h^J) h^I \, \overline{\underline{g}}_K = (h^k_{,i} + \overline{\Lambda}^k_{ij} h^j) h^i \overline{\underline{g}}_k + \overline{\Lambda}^0_{ij} h^i h^j \, \overline{\underline{g}}_0 .$$

Then, a simultaneous application of Eq. (82a) leads eventually to the conditions

$$\overline{\Lambda}^k_{ij} = \overline{\Lambda}^k_{ij}(q) \quad \text{and} \quad \overline{\Lambda}^0_{ij} = 0 . \tag{85}$$

Treating Eqs. (81b) and (82b) in a similar manner yields

$$\overline{\Lambda}^0_{00} = \overline{\Lambda}^0_{00}(\tau) \quad \text{and} \quad \overline{\Lambda}^i_{00} = 0 . \tag{86}$$

Likewise, application of the two conditions in Eq. (83) leads to

$$\overline{\Lambda}^K_{0j} = 0 \quad \Rightarrow \quad \overline{\Lambda}^k_{0j} = \overline{\Lambda}^0_{0j} = 0 \quad \text{and} \quad \overline{\Lambda}^K_{i0} = 0 \quad \Rightarrow \quad \overline{\Lambda}^k_{i0} = \overline{\Lambda}^0_{i0} = 0 , \tag{87}$$

respectively. Finally, substituting the appropriate vector and tensor quantities in Eqs. (84) and performing the necessary mathematical operations, taking into account that the Lie derivative of the metric tensor along the flow defined by Eq. (77) is given by

$$L_{\underline{v}}\, g = [v^I g_{JK,I} + (v^I_{,J} - c^I_{LJ} v^L) g_{IK} + (v^I_{,K} - c^I_{LK} v^L) g_{IJ}]\, \underline{e}^J \otimes \underline{e}^K ,$$

leads eventually to

$$\tfrac{\partial}{\partial t}(g_M)_{ij} = 0 \quad \Rightarrow \quad g_M = g_M(q) \quad \text{and} \quad \tfrac{\partial}{\partial q} g_T = 0 \quad \Rightarrow \quad g_T = g_T(\tau) . \tag{88}$$

Thus, Eqs. (85)-(88) provide a set of scleronomicity conditions in component form, with respect to a standard basis. Obviously, these conditions are identical to the conditions stated earlier in the form of Eq. (13) or, alternatively, Eqs. (19) and (20).

7 EQUATIONS OF MOTION ON THE CONFIGURATION MANIFOLD

In Section 5, a set of equations of motion was derived on the constrained manifold M_A, with respect to a minimal set of coordinates. Such an approach is inconvenient to apply, especially for complex systems, since it requires the selection of a specific set of independent coordinates and elimination of the dependent coordinates through the constraint equations at every time instant. In general, this is not an easy task to achieve. Moreover, it requires differentiation of the constraint equations, leading to their violation at the lower kinematic level [2, 11]. In addition, the geometric properties of the constrained manifold need to be re-determined at every step, through Eqs. (11) and (12). For these reasons, it is frequently advantageous to derive the equations of motion on the original manifold M, for which the geometric properties are known and not affected by the additional constraints.

Next, it is assumed that the original configuration manifold M of the class of systems examined possesses time dependent geometric properties. Moreover, the system is subject to an additional set of k time dependent acatastatic motion constraints, in the form of Eq. (64), or equivalently in the form

$$\dot{\psi}^R(q, \underline{v}, t) \equiv a^R_i(q,t) v^i + a^R_0(q,t) = 0 \quad (R = 1, \ldots, k). \tag{89}$$

When a constraint is holonomic, Eq. (89) can be integrated and cast in the form

$$\phi^R(q,t)=0\,.$$

The derivation of the equations of motion is based on a consistent application of Newton's law of motion, similar to that performed in an earlier study [15]. One of the basic ideas in that study was to consider the configuration manifold M as the total space of a fiber bundle with base space M_A and fibers consisting of the Cartesian product manifold $M_C=M_1\times\cdots\times M_k$, where the single dimensional manifolds M_R, $R=1,\ldots,k$, are related to the action of the R-th motion constraint. Since that study is applicable to scleronomic systems, some modifications are necessary before applying it here. First, based on Eq. (89), the velocity components of a vector in $T_p\bar{M}$ and $T_{p_R}\bar{M}_R$ are related by

$$\bar{v}^i=c^i_R\bar{v}^R+c^i_{R0}\bar{v}^0_R\quad (i=1,\ldots,n)\quad\text{and}\quad \bar{v}^0=\bar{v}^0_R,$$

which can be combined and put in the form of Eq. (65). Moreover, the components c^i_R and c^i_{R0} correspond to special vectors of $T_p\bar{M}$, for each constraint R, satisfying the conditions

$$a^R_i\,c^i_R=1\quad\text{and}\quad a^R_i\,c^i_{R0}=-a^R_0\quad (R=1,\ldots,k\,;\ i=1,\ldots,n). \tag{90}$$

In addition, a transformation similar to that represented by Eqs. (66) and (67) can be established between $T\bar{M}$ and $T\bar{M}_R$, through the $(n+1)\times 2$ matrix

$$\bar{N}_R=\begin{bmatrix}\underline{c}_R(q,\tau) & \underline{c}_{R0}(q,\tau)\\ 0 & 1\end{bmatrix}.$$

Therefore, the components of the metric and the affinities on the event manifold $\bar{M}_R$ can be determined by an equation similar to Eq. (68), while the Newton covectors on the dual space $T^*_{p_R}M_R$ are obtained by application of an equation similar to Eq. (76), with form

$$\begin{aligned}h_R=&(g_{RR}\bar{v}^R+\bar{g}_{R0})_{,R}\bar{v}^R+(g_{RR}\bar{v}^R+\bar{g}_{R0})_{,0}-(\bar{\Lambda}^R_{RR}g_{RR}+\bar{\Lambda}^0_{RR}\bar{g}_{0R})\bar{v}^R\bar{v}^R\\&-(\bar{\Lambda}^R_{RR}\bar{g}_{R0}+\bar{\Lambda}^0_{RR}\bar{g}_{00}+\bar{\Lambda}^R_{0R}g_{RR}+\bar{\Lambda}^0_{0R}g_{0R})\bar{v}^R-\bar{\Lambda}^R_{0R}\bar{g}_{R0}-\bar{\Lambda}^0_{0R}\bar{g}_{00}-f_R.\end{aligned}$$

Employing the connection compatibility condition and performing simple mathematical operations, the last expression can eventually be put in the form

$$h_R=(\bar{m}_{RR}\dot{\bar{\lambda}}^R+\bar{m}_{R0})^{\boldsymbol{\cdot}}+\bar{c}_{RR}\dot{\bar{\lambda}}^R+\bar{k}_{RR}\bar{\lambda}^R-\bar{f}_R\,. \tag{91}$$

The quantity $\bar{\lambda}^R$ represents the spatial coordinate of the two dimensional manifold $\bar{M}_R$. Also, in the last equation and in the sequel, the convention on repeated indices does not apply to index R, while

$$\bar{m}_{RR}\equiv g_{RR}=c^i_Rg_{ij}c^j_R\ ,\quad \bar{m}_{R0}\equiv\bar{g}_{R0}=c^i_R(g_{ij}c^j_{R0}+\bar{g}_{i0})\ ,\quad \bar{c}_{RR}=-c^i_R\frac{\partial f_i}{\partial v^j}c^j_R-c^i_{R,0}g_{ij}c^j_R\ ,\quad \bar{k}_{RR}=-c^i_R\,f_{i,j}c^j_R$$

and

$$\bar{f}_R=c^i_R\,f_i+c^i_{R,0}(g_{ij}c^j_{R0}+\bar{g}_{i0})\,. \tag{92}$$

Following the analysis presented in [15], by incorporating the above modifications into the constraint equations and omitting the details, the equations of motion can be put in the form

$$\underset{\sim}{h}^*_M=\underset{\sim}{h}^*_C \tag{93}$$

on manifold M, with

$$\underset{\sim}{h}_M^* = h_i \underset{\sim}{e}^i ,$$

where h_i is given by Eqs. (35) and (36), while

$$\underset{\sim}{h}_C^* = \sum_{R=1}^{k} h_R a_i^R \underset{\sim}{e}^i ,$$

with h_R given by Eq. (91). Next, substitution of the last two equations in Eq. (93) leads eventually to the following set of equations of motion on the original configuration manifold M

$$\begin{aligned}(g_{ij}\bar{v}^j + \bar{g}_{i0})^{\cdot} - (\bar{\Lambda}^{\ell}_{ji}\bar{g}_{\ell k} + \bar{\Lambda}^{0}_{ji}\bar{g}_{0k})\bar{v}^j\bar{v}^k + (\bar{\Lambda}^{\ell}_{ji}\bar{g}_{\ell 0} + \bar{\Lambda}^{0}_{ji}\bar{g}_{00} + \bar{\Lambda}^{\ell}_{0i}\bar{g}_{\ell j} + \bar{\Lambda}^{0}_{0i}\bar{g}_{0j})\bar{v}^j \\ + \bar{\Lambda}^{\ell}_{0i}\bar{g}_{\ell 0} + \bar{\Lambda}^{0}_{0i}\bar{g}_{00} - f_i = \sum_{R=1}^{k} a_i^R[(\bar{m}_{RR}\dot{\bar{\lambda}}^R + \bar{m}_{R0})^{\cdot} + \bar{c}_{RR}\dot{\bar{\lambda}}^R + \bar{k}_{RR}\bar{\lambda}^R - \bar{f}_R],\end{aligned} \tag{94}$$

where

$(g_{ij}\bar{v}^j)^{\cdot} = (g_{ij}\bar{v}^j)_{,\ell}\bar{v}^{\ell} + (g_{ij}\bar{v}^j)_{,0}$ and $(\bar{m}_{RR}\dot{\bar{\lambda}}^R)^{\cdot} = (\bar{m}_{RR}\dot{\bar{\lambda}}^R)_{,R}\dot{\bar{\lambda}}^R + (\bar{m}_{RR}\dot{\bar{\lambda}}^R)_{,0}$.

The additional information needed for a complete mathematical formulation is obtained by incorporating an appropriate form of the k equations of the constraints. In particular, proceeding in a manner similar to that followed in previous work [15], a second order ordinary differential equation (ODE) is obtained for each holonomic constraint, with form

$$(\bar{m}_{RR}\dot{\phi}^R)^{\cdot} + \bar{c}_{RR}\dot{\phi}^R + \bar{k}_{RR}\phi^R = 0 , \tag{95}$$

which forces both $\dot{\phi}^R$ and ϕ^R to become zero eventually. Likewise, each nonholonomic constraint gives rise to a scalar ODE with form

$$(\bar{m}_{RR}\dot{\psi}^R)^{\cdot} + \bar{c}_{RR}\dot{\psi}^R = 0 , \tag{96}$$

causing $\dot{\psi}^R$ to become zero.

Next, by introducing the matrix notation

$\underline{\bar{q}} = (\bar{q}^1 \ \cdots \ \bar{q}^n)^T$, $\underline{\bar{\lambda}} = (\bar{\lambda}^1 \ \cdots \ \bar{\lambda}^k)^T$, $\underline{\bar{v}} = (\bar{v}^1 \ \cdots \ \bar{v}^n)^T$ and $M = [g_{ij}]$,

Eq. (94) can be put in the following general form

$$(M(\underline{\bar{q}},t)\underline{\bar{v}})^{\cdot} + \underline{h}(\underline{\bar{q}},\underline{\bar{v}},t) = A^T(\underline{\bar{q}},t)[(\bar{M}\dot{\underline{\bar{\lambda}}})^{\cdot} + \underline{\bar{h}}(\underline{\bar{\lambda}},\dot{\underline{\bar{\lambda}}},t)] . \tag{97}$$

The array $\underline{h}(\underline{\bar{q}},\underline{\bar{v}},t)$ includes all the terms in Eq. (94) multiplied by the affinities or originating from the components $\bar{g}_{i0}$ of the metric on the event manifold $\bar{M}$, together with the applied forces f_i . Likewise, the elements of the diagonal matrix $\bar{M} = diag(\bar{m}_{11} \ \cdots \ \bar{m}_{kk})$ and the array

$\underline{\bar{h}} \equiv \bar{C}\dot{\underline{\bar{\lambda}}} + \bar{K}\underline{\bar{\lambda}} + \dot{\underline{\bar{m}}}_0 - \underline{\bar{f}}$,

including the diagonal matrices $\bar{C} = diag(\bar{c}_{11} \ \cdots \ \bar{c}_{kk})$ and $\bar{K} = diag(\bar{k}_{11} \ \cdots \ \bar{k}_{kk})$ and the arrays $\dot{\underline{\bar{m}}}_0 = (\dot{\bar{m}}_{10} \ \cdots \ \dot{\bar{m}}_{k0})^T$ and $\underline{\bar{f}} = (\bar{f}_1 \ \cdots \ \bar{f}_k)^T$, are determined through application of Eq. (92).

In summary, Eq. (94) furnishes a set of n second order ODEs. These equations together with Eqs. (95) and (96) form a set of $n+k$ second order ODEs in the $n+k$ unknown coordinates q^i and λ^R, describing the behavior of mechanical systems with an arbitrary (but finite) number of coordinates, possessing a time dependent original configuration manifold and being subject to time dependent and acatastatic motion constraints. In general, solution of these equations can only be achieved by numerical means, after applying a suitable numerical discretization [20, 25]. For the scleronomic case, it is easy to verify that these equations are simplified considerably and become identical with those presented in [15]. The ODE form of the set of equations derived is associated with several advantages over formulations leading to sets of algebraic-differential equations (DAEs) [2, 11]. It also presents advantages over previous formulations leading to an ODE form after elimination of the redundant coordinates or the Lagrange multipliers from the equations of motion, since this is done at the expense of violating the motion constraints at the lower kinematical levels [2, 15]. Finally, another advantage of the present approach is that the affinities are independent and not derived from the metric components (i.e., the connection is not necessarily metric compatible). This allows for the most general and consistent derivation of the equations of motion [19].

8 EXAMPLES

Two examples are presented in this section, illustrating theoretical aspects investigated in the previous sections. The first refers to motion of a constrained particle, with attention on recognizing when a constraint is scleronomic or rheonomic. Then, the problem of the rolling motion of a sphere over a rotating table is re-examined, by employing the scleronomicity conditions developed in Section 6.

8.1 Spherical pendulum with a moving end

Consider the spatial motion of a particle with mass m. The unconstrained motion is described by three coordinates x^i with respect to an inertial Cartesian coordinate frame $\mathbb{X}$. Then, the components of the metric tensor and the affinities can be easily determined for both the configuration manifold $M = E^3$ and the corresponding event manifold $\bar{E}^3 = E^3 \times \mathbb{R}$. The particle motion is constrained by a rigid bar with negligible mass and length L, as shown in Fig. 5. The particle is located at one end of the bar, while the other end (point P) moves along the axis Ox^3 with a known displacement history

$$x_P(t) = \hat{x}_P \cos\omega t \, .$$

This is represented by a holonomic constraint with equation

$$\phi^1(x,t) = (x^1)^2 + (x^2)^2 + (x^3 - x_P(t))^2 - L^2 = 0 \, , \tag{98}$$

which by differentiation with respect to time can be put in the general form of Eq. (89), with

$$\underset{\sim}{a}^1(x,t) = (2x^1 \quad 2x^2 \quad 2(x^3 - x_P(t))) \quad \text{and} \quad a_0^1(x,t) = -2(x^3 - x_P(t))\dot{x}_P(t) \, .$$

Next, consider the motion of the particle with respect to a new set of coordinates, corresponding to a new frame $\mathbb{S}$. This frame has origin at point P and axes remaining parallel to the axes of frame $\mathbb{X}$. Due to the constraint by Eq. (98), the position of the particle can be fully determined by the angular spherical coordinates θ^1 and θ^2, as depicted in Fig. 5. Specifically, the following relations are established between the original Cartesian coordinates and these spherical coordinates

$$x^1 = L\sin\theta^1\cos\theta^2 \, , \quad x^2 = L\sin\theta^1\sin\theta^2 \quad \text{and} \quad x^3 = L\cos\theta^1 + \hat{x}_P\cos\omega t \, . \tag{99}$$

Then, the velocity transformation (66) is set up with extended matrix $\bar{N}$ in the form of Eq. (67), with

$$N = \begin{bmatrix} L\cos\theta^1\cos\theta^2 & -L\sin\theta^1\sin\theta^2 \\ L\cos\theta^1\sin\theta^2 & L\sin\theta^1\cos\theta^2 \\ -L\sin\theta^1 & 0 \end{bmatrix} \quad \text{and} \quad \underline{\upsilon} = \begin{pmatrix} 0 \\ 0 \\ -\omega\hat{x}_P\sin\omega t \end{pmatrix} .$$

Substituting the above into Eq. (69), the metric matrix is determined in the form

$$\bar{g}_A = \begin{bmatrix} mL^2 & 0 & m\omega L\hat{x}_P \sin\theta^1 \sin\omega t \\ 0 & mL^2 \sin^2\theta^1 & 0 \\ m\omega L\hat{x}_P \sin\theta^1 \sin\omega t & 0 & g_{00} + m(\omega\hat{x}_P \sin\omega t)^2 \end{bmatrix}.$$

This result shows that the basis obtained in $\bar{M}_A$ is not standard. In addition, it is found that there exist both pure and mixed affinities, which are expressions of coordinates θ^1, θ^2 and t.

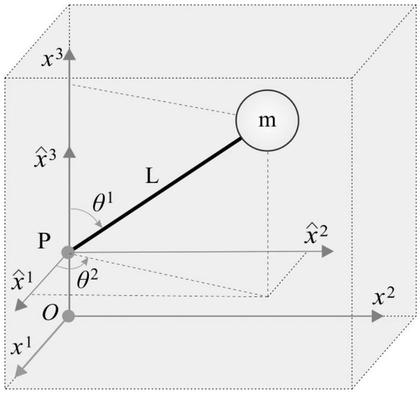

Fig. 5 A spherical pendulum with a translating end

Next, the emphasis was placed in investigating whether these effects are removable by a change of basis in $\bar{M}_A$. Using integrability conditions, the following change of coordinates

$$\theta^1 = 2\tan^{-1}(e^{\hat{\theta}^1 + \cos\omega\hat{\tau}}), \quad \theta^2 = \hat{\theta}^2 \quad \text{and} \quad \tau = \hat{\tau}, \tag{100}$$

was found to lead to a coordinate basis where the new metric matrix is block diagonal. In fact, the metric matrix in the new coordinate system is obtained in the 3×3 diagonal form

$$\hat{\bar{g}}_A = diag(mL^2 \sin\eta(\hat{\theta}^1,\hat{\tau}) \quad mL^2 \sin\eta(\hat{\theta}^1,\hat{\tau}) \quad g_{00} + m[\omega\hat{x}_P \cos\eta(\hat{\theta}^1,\hat{\tau})\sin\omega t]^2),$$

with

$$\eta(\hat{\theta}^1,\hat{\tau}) = 2\tan^{-1}(e^{\hat{\theta}^1 + \cos\omega\hat{\tau}}).$$

However, all its elements have a mixed dependence on both the spatial coordinate $\hat{\theta}^1$ and the temporal coordinate $\hat{\tau}$. Similar findings resulted in evaluating the corresponding affinities. These imply that conditions (13), or alternatively Eqs. (85)-(88), do not hold in the case examined. Therefore, the manifold M_A is rheonomic. Since the original manifold M is scleronomic, the motion constraint expressed by Eq. (98) is rheonomic, in accordance with the classical definitions [12].

Next, consider the motion of the particle with respect to a new Cartesian coordinate frame $\mathbb{Y}$, with coordinates y^i $(i=1,2,3)$, in the same configuration manifold M. In particular, the origin of this frame is point P, while its axes remain parallel to the corresponding axes of frame $\mathbb{X}$, as shown in Fig. 5. Then, the two sets of coordinates are related by

$$x^1 = y^1, \quad x^2 = y^2 \quad \text{and} \quad x^3 = y^3 + \hat{x}_P \cos\omega t. \tag{101}$$

In the new coordinate system, the motion constraint appears also in a holonomic form

$$\phi^1(y,t) = (y^1)^2 + (y^2)^2 + (y^3)^2 - L^2 = 0. \tag{102}$$

However, in contrast to Eq. (98), this now appears in a scleronomic catastatic form, according to common terminology [12, 23]. In order to explain the discrepancy with Eq. (98), it is useful to view Eq. (101), together with $t=\hat{t}$, as a coordinate transformation in the original event manifold $\bar{M}=\bar{E}^3$, similar to that expressed by Eq. (39). Then, the Jacobian is obtained in the form of Eq. (42), with

$$J = I_3 \quad \text{and} \quad \underline{c} = (0 \quad 0 \quad -\omega\hat{x}_P \sin\omega t)^T .$$

Consequently, evaluation of the metric in the new basis of $\bar{E}^3$ through Eq. (48) yields

$$\hat{g} = g = mI_3 , \quad \underline{g}_{\hat{0}} = (0 \quad 0 \quad -m\omega\hat{x}_P \sin\omega t)^T \quad \text{and} \quad g_{\hat{0}\hat{0}} = g_{00} + m(\omega\hat{x}_P \sin\omega t)^2 .$$

As expected, based on the fact that $\underline{c} \neq \underline{0}$, the new basis in $T_p\bar{M}$ is not standard, since $\underline{g}_{\hat{0}} \neq \underline{0}$ for $\omega\hat{x}_P \neq 0$. Moreover, all the affinities on $\bar{M}$ are found to be equal to zero, exception for

$$\bar{\Lambda}^{\hat{3}}_{\hat{0}\hat{0}} = \omega^2 \hat{x}_P \cos\omega t .$$

Therefore, despite the fact that it does not involve time in an explicit manner, the motion constraint expressed by Eq. (102) is in fact rheonomic, since the configuration manifold M is scleronomic, while the constrained manifold M_A is rheonomic, as shown above. This result resolves the difference observed in the time dependence appearing in the constraint equations (98) and (102). Furthermore, it demonstrates that the classical classification of constraints on a mechanical system with configuration manifold possessing time dependent geometric properties is accurate only when the components involved in the constraint equations are expressed with respect to standard bases in both $\bar{M}$ and $\bar{M}_A$.

8.2 Rolling of a sphere on a rotating table

Next, consider the motion of a rigid sphere, rolling without slipping on a horizontal table [9]. This table rotates with an angular speed $\Omega(t)$ with respect to a fixed vertical axis Ox^3, as shown in Fig. 6. The sphere has radius r, mass m and a known mass moment of inertia J with respect to any axis passing through its center of mass C. The coordinate system $\mathbb{F}$, shown in Fig. 6, is fixed (inertial) and the angular velocity of the table is

$$\underline{\omega}_T(t) = \Omega(t)\underline{e}_3 . \tag{103}$$

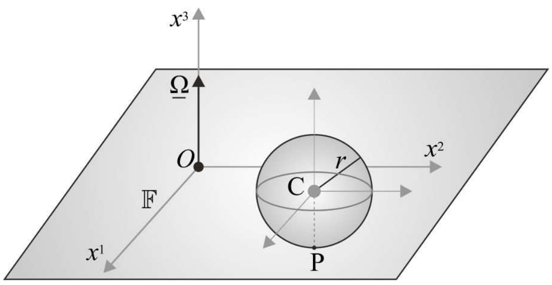

Fig. 6 Rigid sphere rolling over a rotating table

First, the position of the sphere is determined at any time t by the generalized coordinates

$$\underline{q}(t) = (\underline{x}_C^T \quad \underline{\theta}^T)^T , \tag{104}$$

where the arrays

$$\underline{x}_C = (x^1 \quad x^2 \quad x^3)^T \quad \text{and} \quad \underline{\theta} = (\theta^1 \quad \theta^2 \quad \theta^3)^T$$

represent the position of the center of mass and the orientation of the sphere, respectively, with respect to the inertial reference frame $\mathbb{F}$. Due to the symmetry of the sphere, the classical Cartesian rotation vector is selected for the description of its rotational motion [11]. The coordinates $\underline{q}(t)$ represent a point on a product configuration manifold $M = \mathbb{R}^3 \times M(3)$, with dimension n=6 [18]. Then, the motion of the system can be viewed as the motion of a point along a specific path $\gamma(t)$ on manifold M, with tangent vector given by

$$\underline{v}(t) = (\underline{v}_C^T \quad \underline{\omega}^T)^T ,$$

where

$$\underline{v}_C = (v^1 \quad v^2 \quad v^3)^T \quad \text{and} \quad \underline{\omega} = (\omega^1 \quad \omega^2 \quad \omega^3)^T ,$$

with $v^i = \dot{x}^i$ ($i = 1,2,3$). Moreover, the quasi-velocities $\underline{\omega}$ are related to the true angular velocities $\underline{\dot{\theta}}$ through the corresponding tangent operator $T(\underline{\theta})$ at $\underline{\theta}$ [11], by

$$\underline{\omega} = T^T(\underline{\theta})\underline{\dot{\theta}} .$$

Then, based on the kinetic energy of the sphere, the metric of M is found in the block diagonal form

$$g = [g_{ij}] = \begin{bmatrix} mI_3 & 0 \\ 0 & JI_3 \end{bmatrix} . \tag{105}$$

Moreover, since the manifold M examined is a product manifold, its affinities appear in product form as well. In particular, all the affinities related to the translational part can be selected to be zero, while the non-zero affinities related to the rotational part of the motion can be chosen to take the following constant values in a body frame

$${}_R\Lambda^1_{23} = -{}_R\Lambda^1_{32} = {}_R\Lambda^2_{31} = -{}_R\Lambda^2_{13} = {}_R\Lambda^3_{12} = -{}_R\Lambda^3_{21} = 1 . \tag{106}$$

Moreover, all the mixed affinities are zero [18]. Then, the affinities can be evaluated in the coordinate system used by employing the rotation matrix $R = \exp(\tilde{\theta})$ as a transformation matrix in Eq. (46).

Next, due to the constraints imposed on the motion of the sphere, there appear $k = 3$ constraint equations, so that the dimension of manifold M_A is just $m = 3$. Specifically, the first constraint is holonomic and guarantees that the center of the sphere stays on a constant height

$$\phi^1 = x^3 - r = 0 \quad \Rightarrow \quad \dot{\phi}^1 = v^3 = 0 . \tag{107}$$

In addition, the rolling condition of the sphere leads to two more constraints, namely

$$\dot{\psi}^2 = \dot{x}^1 - r\omega^2 + \Omega x^2 = 0 \quad \text{and} \quad \dot{\psi}^3 = \dot{x}^2 + r\omega^1 - \Omega x^1 = 0 , \tag{108}$$

which are nonholonomic and acatastatic. Therefore, the 3×6 matrix of constraints A and the vector $\underline{a}$ appearing in Eq. (64) take the form

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -r & 0 \\ 0 & 1 & 0 & r & 0 & 0 \end{bmatrix} \quad \text{and} \quad \underline{a}(\underline{q},t) = \begin{pmatrix} 0 \\ \Omega x^2 \\ -\Omega x^1 \end{pmatrix} . \tag{109}$$

Based on these results, one can determine the components of the special constraint vectors $\underline{c}_R$ and $\underline{c}_{R0}$ by Eq. (90). For instance, a possible choice for these vectors is the following

$$\underline{c}_1 = (0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0)^T , \quad \underline{c}_2 = (0 \quad 0 \quad 0 \quad 0 \quad -1/r \quad 0)^T , \quad \underline{c}_3 = (0 \quad 0 \quad 0 \quad 1/r \quad 0 \quad 0)^T ,$$

$$\underline{c}_{10} = \underline{0} , \quad \underline{c}_{20} = (0 \quad 0 \quad 0 \quad 0 \quad \Omega x^2/r \quad 0)^T \quad \text{and} \quad \underline{c}_{30} = (0 \quad 0 \quad 0 \quad \Omega x^1/r \quad 0 \quad 0)^T .$$

Consequently, application of the first two relations in Eq. (92) leads to determination of the elements of the diagonal mass matrix $\bar{M}$ and the vector $\underline{\bar{m}}_0$ in the form

$$\bar{m}_{11} = m , \quad \bar{m}_{22} = \bar{m}_{33} = J/r^2 , \quad \bar{m}_{10} = 0 , \quad \bar{m}_{20} = -\Omega x^2 J/r^2 \quad \text{and} \quad \bar{m}_{30} = \Omega x^1 J/r^2 .$$

Likewise, given a set of applied forces acting on the sphere, which can be represented by a resultant force $\underline{f}(\underline{q},\underline{\dot{q}},t)$ and moment $\underline{m}_C(\underline{q},\underline{\dot{q}},t)$ with respect to its center C [21], the elements $\bar{c}_{RR}$ and $\bar{k}_{RR}$ of the diagonal damping and stiffness matrices $\bar{C}$ and $\bar{K}$, as well as the corresponding forcing terms $\bar{f}_R$ ($R = 1,\ldots,3$) are also evaluated by applying Eq. (92).

Collecting all the above and substituting in Eq. (94) leads to the equations of motion of the constrained system examined on the original configuration manifold in matrix form. The set of coordinates $\underline{q}$ is given by Eq. (104), while the Lagrange multipliers $\underline{\lambda} = (\lambda^1 \quad \lambda^2 \quad \lambda^3)^T$ correspond to the set of constraints (107) and (108). In the present formulation, the six ODEs represented by Eq. (97) are accompanied by a constraint equation in the form of Eq. (95) and two constraint equations in the form of Eq. (96), where the terms ϕ^1, $\dot{\phi}^1$, $\dot{\psi}^2$ and $\dot{\psi}^3$ are taken from Eqs. (107) and (108). This leads to a system of 9 second order ODEs for the 9 unknowns, $\underline{q}$ and $\underline{\lambda}$, of the problem.

Finally, a simple inspection of Eqs. (105) and (106) reveals that the original configuration manifold M is scleronomic. Moreover, based on Eq. (109), an appropriate 7×4 transformation matrix $\bar{N}$ is first determined in the form of Eq. (67), so that the 4×4 metric matrix corresponding to the event constrained manifold $\bar{M}_A$ is determined by application of Eq. (69) in the diagonal form

$$\bar{g}_A = diag((m + J/r^2) \quad (m + J/r^2) \quad J \quad g_{00} + mJ\Omega^2[(\theta^1)^2 + (\theta^2)^2]/(mr^2 + J)) ,$$

where θ^α (with $\alpha = 1,\ldots,4$) is a coordinate set on $\bar{M}_A$. This corresponds to a standard basis. Evaluation of the corresponding structure constants shows that they are all zero, except for

$$c^1_{02} = -c^1_{20} = J\Omega/(mr^2 + J) = c^2_{10} = -c^2_{01} .$$

This indicates that the basis selected in $\bar{M}_A$ is not holonomic. Therefore, it is more convenient here to apply the scleronomicity checks in the coordinate invariant form presented in Section 6 in order to judge safely whether the constrained manifold is rheonomic or not. Indeed, application of Eq. (84) to the example considered reveals that the first condition on the metric is satisfied identically. However, evaluation of the Lie derivative required by the second condition in the same equation shows that there appears the following non-zero term

$$(L_{\underline{h}}\bar{g}_T)_{00} = 2mJ\,\Omega^2/(mr^2 + J) ,$$

which becomes zero only when $\Omega = 0$, corresponding to a stationary table. This demonstrates that Eq. (84) is violated when $\Omega \neq 0$ and reveals that manifold M_A is rheonomic, indeed.

9 SYNOPSIS

This study was devoted to mechanical systems possessing configuration manifolds with time dependent geometric properties and being subject to additional acatastatic equality motion constraints involving time explicitly. This time dependence, in conjunction with the need to keep invariant the form of Newton's law in different manifolds, led to an introduction and consideration of the corresponding event manifold. These time dependent terms were shown to be introduced by either a basis transformation in the event manifold or by additional motion constraints. Moreover, the time terms introduced by a basis transformation are artificial and can be removed from the equations of motion. These results were reinforced and enhanced further by exploiting some concepts of foliation theory. Apart from providing useful clarifications on the geometry of the motion, this theory was also employed in establishing a set of coordinate invariant conditions for judging whether a configuration manifold is scleronomic. Finally, the equations of motion were obtained in the original configuration manifold, in the presence of constraints. Also, some of the analytical findings were illustrated by examining two systems exhibiting basic features of the class of systems examined.

The results presented in this work are applicable and cover a wide range of engineering systems. To obtain them, some useful tools from differential geometry were employed. In return, apart from providing a nice interpretation of the key concepts, these tools helped in performing a consistent application of Newton's law to manifolds with general time dependent geometric properties. Eventually, this led to a system of second order ODEs, which are much easier to handle than sets of DAEs [2, 11]. In addition, the final set of equations of motion was obtained on the original configuration manifold, whose geometric properties are not affected by the presence of the additional motion constraints. Also, these equations involve properties of the corresponding event manifold. These provide fertile ground for developing new advanced formulations, leading to more accurate, effective and robust numerical methods for solving these equations, by enhancing available techniques [6, 13, 20].

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