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A SHALLOW WATER EQUATION BASED ON DISPLACEMENT AND PRESSURE AND THE ZU-CLASS SYMPLECTIC METHOD

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Abstract. *In this paper, the shallow water problem with sloping water bottom is discussed. By treating the incompressible condition as the constraint, a constrained Hamilton variational principle is presented for the shallow water problem. Based on the constrained Hamilton variational principle, a shallow water equation based on displacement and pressure (SWE-DP) is developed. A hybrid numerical method combining the finite element method for spatial discretization and the Zu-class method for time integration is constructed for the SWE-DP. The correctness and effectiveness of the proposed SWE-DP are verified by using three classical numerical examples. Numerical examples show that the proposed method performs well in the simulation of the shallow water with sloping depths and can correctly deal with wet-dry interfaces.*

1 INTRODUCTION

The theory of shallow water is widely applied in engineering such as the offshore engineering and the environment engineering; because of its importance, it has been studied extensively[1-3]. At present, a great number of theoretical models have been proposed for simulating shallow water problems, such as the KdV equation, Saint-Venant equation (SVE) and Boussinesq equation[2, 4-7]. However, all these equations are based on the Euler coordinate. The water equations constructed using the Euler coordinate consist of the convective and source terms. The existence of the convective term make the numerical scheme should be designed carefully. The most widely used method may be the finite volume method[1], because its discrete scheme can preserve the mass and momentum. However the source term often destroy the conservativeness of the finite volume method[2], which results in the harmonious problem of numerical solutions, i.e., the numerical still water cannot keep motionless[3, 4]. The harmonious problem of numerical solutions is one of the two most difficult problems in computational shallow water dynamics. Another one is the wetting/drying condition for shallow water flow which often lead to some numerical problems such as the mass loss and the negative water depth[5].

Generally, an efficient and modern numerical scheme for the nonlinear shallow water problem has to satisfy two requirements: 1) preserve the motionless steady states; 2) allow the occurrence of dry states[3, 4, 6]. However, in the Euler framework it is very difficult to satisfy these two requirements simultaneously. Fortunately, the shallow water problem can also be studied using the Lagrange coordinate, and these two requirement can be satisfied easily in the Lagrange coordinate. In Ref. [9], Tao applied the Lagrange coordinate to discuss the sudden starting of a floating body in deep water. In Ref. [10], Tao and Shi applied the Lagrange coordinate to discuss the problem of hydrodynamic pressure on a suddenly starting vessel. In Ref. [11], Shi employed the Lagrange coordinate to discuss the nonlinear wave induced by the acceleration of a cylindrical tank. In the Lagrangian method, the displacements are viewed as unknown variables. One advantage of the Lagrangian method is that the nonlinear boundary condition on the free surface can be exactly satisfied[9-11]. However, it should be mentioned that in Refs. [9-11], the authors obtained the hydrodynamics equations based on Newton's law rather than the Hamilton variational principle. Undoubtedly, it is important to find a variational principle for the hydrodynamics problem. And it is very hard to find this variational principle in the Euler description. However, it is easy to obtain a Hamilton variational principle by means of the Lagrangian method. Based on the Hamilton variational principle, numerical methods that have been successfully developed and widely applied in structural dynamics, such as the finite element method[12] and symplectic method[13], can be applied to simulate the shallow water problem. The symplectic method preserves the symplecticity and energy of the system and, hence, performs better than the traditional non-symplectic method, especially for problems that require extensive numerical simulation[13].

Recently, Wu and Zhong[7] proposed a constrained Hamilton variational principle for the shallow water problem. In the constrained Hamilton variational principle, the incompressible condition is treated as the constraint, and the pressure is seen as the Lagrangian multiplier. According to the constrained Hamilton variational principle, a shallow water equation based on displacement and pressure (SWE-DP) is developed. To numerically solve the SWE-DP, they developed a hybrid numerical method combining the finite element method for spatial discretization and the Zu-class method for time integration. Their method is symplectic and can preserve the volume numerically. However, in Ref. [7], authors only considered the shallow water with even bottom. The wetting/drying condition and the sloping water bottom was not discussed. In this study, we extend the method proposed in Ref. [7] to the shallow water with sloping water bottom and wet-dry interface. In Section 2, a SWE-DP for the shallow water flow with sloping bottom and wet-dry interface is constructed. In Section 3, a hybrid numerical method combining the finite element method for spatial discretization and the Zu-class method for time integration is established for the proposed SWE-DP. In Section 4, three classical numerical examples are applied to verify the correction of the proposed SWE-DP and the effectiveness of the proposed numerical method. At last Section, some conclusions are presented.

2 HAMILTON VARIATIONAL PRINCIPLE OF SHALLOW WATER WAVES

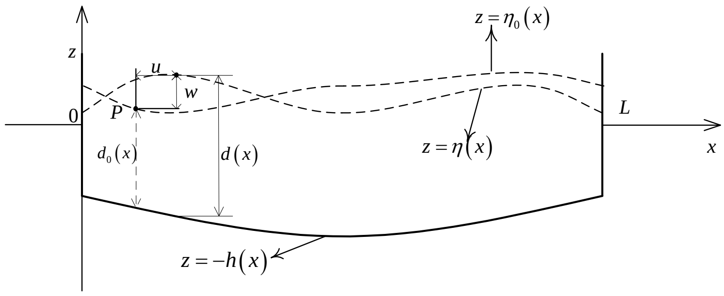

Fig. 1 The considered model

Consider a shallow water domain shown by Fig. 1. The water bed profile is defined by $z=-h(x)$, in which $0\le x\le L$. The initial water surface is $z=\eta_0(x)$. $u(x,z,t)$ and $w(x,z,t)$ represent the displacements of the particle, which is localized at (x,z) initially, at time t in x and z, respectively, and the location of the particle at time t is denoted by (ξ,ζ), in which

$$\xi = x+u(x,z,t),\ \ \zeta = z+w(x,z,t). \tag{1}$$

According to Ref. [7], the action integral of the two-dimensional water can be shown as

$$S=\int_0^t (T-U+R)\mathrm{d}s \tag{2}$$

where T, U and R are kinetic energy, potential energy and constrained term, i.e.,

$$T=\int_0^L\int_{-h(x)}^{\eta_0(x)}\frac{1}{2}\rho\left(\dot{u}^2+\dot{w}^2\right)\mathrm{d}x\mathrm{d}z\,, \tag{3}$$

$$U=\int_0^L\int_{-h(x)}^{\eta_0(x)}\rho g\,(z+w)\mathrm{d}x\mathrm{d}z\,, \tag{4}$$

$$R=\int_0^L\int_{-h(x)}^{\eta_0(x)}p\theta\mathrm{d}x\mathrm{d}z\,, \tag{5}$$

p is the pressure, and θ is the water volume strain, which can be written as

$$\theta = u_x+w_z+u_xw_z-u_zw_x\,. \tag{6}$$

For the shallow water flow, it can be assumed that the horizontal displacement is independent of the vertical coordinate z, i.e, $u(x,t)=u(x,z,t)$, hence the water volume strain can be simplified as

$$\theta = u_x + w_z + u_x w_z . \tag{7}$$

According to the incompressible condition, the water volume strain should be rigorously zero $\theta = 0$, hence we have

$$(1+w_z)(1+u_x)=1, \quad \text{or} \quad w_z(x,z)=\frac{-u_x(x)}{1+u_x(x)}. \tag{8}$$

Equation (8) shows that the vertical displacement distributes linearly along the vertical coordinate z. Let the vertical displacements at the bottom and the surface be denoted by

$$w(x,-h,t)=\varphi(x,t), \quad w(x,0,t)=\eta(x,t), \tag{9}$$

the vertical displacement can be written as:

$$w(x,z,t)=\frac{z+h}{d_0}\eta(x,t)+\frac{\eta_0-z}{d_0}\phi(x,t). \tag{10}$$

Combining Eq. (8) and Eq. (10) gives

$$w_z=-\frac{u_x}{1+u_x}=\frac{\eta(x,t)-\phi}{h}. \tag{11}$$

At the bottom, $z=-h(x)$, $w(x,-h,t)=\varphi(x,t)$, and the horizontal displacement is $u(x,t)$, hence

$$\varphi(x,t)=h(x)-h(x+u). \tag{12}$$

Substituting Eq. (12) into Eq. (10) yields

$$w(x,z,t)=\frac{z+h}{d_0}d-z-h(x+u) \tag{13}$$

where $d(x,t)$ is the water depth which can be expressed as

$$d(x,t)=\eta(x,t)+h(x+u)+d_0(x)-h(x). \tag{14}$$

Substituting Eq. (13) into Eq. (4), the potential energy is obtained:

$$\begin{aligned} U &= \int_0^L \int_{-h(x)}^{\eta_0(x)} \rho g(z+w)\mathrm{d}z\mathrm{d}x \\ &= \frac{1}{2}\int_0^L \rho g d_0 d\mathrm{d}x - \int_0^L \rho g d_0 h(x+u)\mathrm{d}x \end{aligned}. \tag{15}$$

In terms of Eq. (13), the vertical velocity is

$$\dot{w}(x,z,t)=\frac{z+h}{d_0}\dot{d}(x,t)-h_x(x+u)\dot{u}. \tag{16}$$

For the shallow water problem, the nonlinear effect of the vertical velocity can be ignored, hence we have

$$\dot{w}(x,z,t)=\frac{z+h}{d_0}\dot{d}(x,t)-h_x\dot{u}. \tag{17}$$

Then substituting Eq. (17) into Eq. (3) yields

$$\begin{aligned} T &= \frac{1}{2}\int_0^L \int_{-h(x)}^{\eta_0(x)} \rho\left(\dot{u}^2+\dot{w}^2\right)\mathrm{d}x\mathrm{d}z \\ &= \frac{1}{2}\int_0^L \left(\rho d_0 \dot{u}^2\right)\mathrm{d}x + \frac{1}{2}\int_0^L \rho\left(\frac{d_0}{3}\dot{d}^2 - d_0 h_x \dot{u}\dot{d} + d_0 h_x^2 \dot{u}^2\right)\mathrm{d}x \end{aligned}. \tag{18}$$

The pressure at the water surface is zero. Let the pressure at the bottom be denoted by

$$p\left(x,-h(x),t\right)=\beta(x,t). \tag{19}$$

We assume that the pressure distributes linearly along the vertical coordinate z, i.e.,

$$p(x,z,t)=\frac{\eta_0 - z}{d_0}\beta(x,t). \tag{20}$$

Substituting Eq. (13) into Eq. (7), the water volume strain can be written as

$$\theta = u_x + (1+u_x)\left(\frac{d}{d_0}-1\right). \tag{21}$$

Substituting Eqs. (20) and (21) into Eq. (5) yields

$$R=\frac{1}{2}\int_0^L \beta\left[d(1+u_x)-d_0\right]\mathrm{d}x. \tag{22}$$

In terms of Eqs. (15), (18) and (22), the action integral of the shallow water wave can be written as

$$\begin{aligned} S &= \frac{1}{2}\int_0^t\int_0^L \rho\left(d_0\dot{u}^2+\frac{d_0}{3}\dot{d}^2 - d_0 h_x \dot{u}\dot{d} + d_0 h_x^2 \dot{u}^2\right)\mathrm{d}x\mathrm{d}s \\ &-\int_0^t \frac{1}{2}\int_0^L \rho g d_0 d\,\mathrm{d}x\mathrm{d}s + \int_0^t\int_0^L \rho g d_0 h(x+u)\mathrm{d}x\mathrm{d}s \\ &+\frac{1}{2}\int_0^t\int_0^L \beta\left[d(1+u_x)-d_0\right]\mathrm{d}x\mathrm{d}s \end{aligned}. \tag{23}$$

Taking the first variation of the action integral, i.e., $\delta S = 0$, gives

$$\begin{aligned} \delta S &= \frac{1}{2}\int_0^t\int_0^L \rho\left(-2\delta u d_0 \ddot{u} - 2\delta d\frac{d_0}{3}\ddot{d} + d_0 h_x \delta u \ddot{d} + d_0 h_x \ddot{u}\delta d - 2\delta u d_0 h_x^2 \ddot{u}\right)\mathrm{d}x\mathrm{d}s \\ &-\int_0^t \frac{1}{2}\int_0^L \rho g d_0 \delta d\,\mathrm{d}x\mathrm{d}s + \int_0^t\int_0^L \rho g d_0 h_x(x+u)\delta u\,\mathrm{d}x\mathrm{d}s \\ &+\frac{1}{2}\int_0^t\int_0^L \beta\delta d(1+u_x)\mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_0^t\int_0^L \delta\beta\left[d(1+u_x)-d_0\right]\mathrm{d}x\mathrm{d}s \\ &-\frac{1}{2}\int_0^t\int_0^L \frac{\partial(\beta d)}{\partial x}\delta u\,\mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_0^t \beta d\,\delta u\big|_{x=0}^{x=L}\mathrm{d}s \end{aligned}. \tag{24}$$

At the boundaries $x=0$ and L, the displacement are often known or the water depths are rigorously zeros, hence, we have

$$d(x)=0 \text{ or } \delta u(x)=0, \quad x=0,L\ . \tag{25}$$

Combining Eq. (24) with Eq.(25) yields

$$\begin{cases} \rho d_0 \ddot{u} - \frac{1}{2}\rho d_0 h_x \ddot{d} + \rho d_0 h_x^2 \ddot{u} - \rho g d_0 h_x \left(x+u\right) + \frac{1}{2}\frac{\partial\left(\beta d\right)}{\partial x} = 0 \\ \rho \frac{d_0}{3}\ddot{d} - \frac{1}{2}\rho d_0 h_x \ddot{u} + \rho g d_0 - \frac{1}{2}\beta\left(1+u_x\right) = 0 \\ d\left(1+u_x\right) = d_0 \end{cases} \tag{26}$$

which is the shallow water equation based on displacement and pressure (SWE-DP).

3 NUMERICAL TREATMENT

It is necessary to develop a numerical method for the SWE-DP. In this section, a hybrid method combining the finite element for the spatial discretization and Zu-class method for the time integration is presented in detail.

3.1 Discretization in space

The proposed SWE-DP (26) is derived in terms of the constrained Hamilton variational principle. Hence, it is a natural choice to use the finite element method for spatial discretization. Let the region $\left[0,L\right]$ be divided into N_e basic elements with N_d nodes; see Fig. 2.

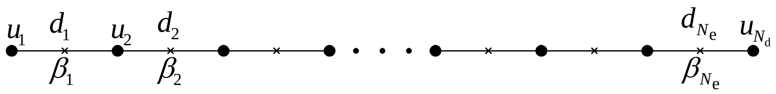

Fig. 2 The finite element mesh

On the nth element, the horizontal displacement $u\left(x\right)$ is approximated by the linear function, and $d\left(x\right)$ and $\beta\left(x\right)$ are treated as constant values, i.e.,

$$u\left(x\right) = \frac{x_{n+1}-x}{\Delta x_n}u_n + \frac{x-x_n}{\Delta x_n}u_{n+1},\ d\left(x\right) = d_n,\ \beta\left(x\right) = \beta_n, \tag{27}$$

where x_n is the node location in x, Δx_n is the length of the nth element, u_n is the node horizontal displacement, d_n is the water depth evaluated at the mid-point of the nth element, and β_n is the pressure of the water bottom evaluated at the mid-point of the nth element. In terms of Eq. (27), the incompressible condition on the nth element can be approximated as

$$d_n \Delta x_n \left(1 + \frac{u_{n+1}-u_n}{\Delta x_n}\right) = \Delta x_n d_{0,n}. \tag{28}$$

The kinetic energy can be approximated as

$$T = \frac{1}{2}\dot{\boldsymbol{u}}^\mathrm{T}\boldsymbol{M}_u\dot{\boldsymbol{u}} + \frac{1}{2}\dot{\boldsymbol{d}}^\mathrm{T}\boldsymbol{M}_d\dot{\boldsymbol{d}} - \dot{\boldsymbol{d}}^\mathrm{T}\boldsymbol{M}_{du}\dot{\boldsymbol{u}} \tag{29}$$

in which

$$\rho_u = \rho d_0\left(1 + h_x^2\right), \quad \rho_d = \frac{\rho d_0}{3}, \quad \rho_{ud} = \rho h_x d_0, \tag{30}$$

$$\begin{aligned}
&\boldsymbol{u} = \begin{pmatrix} u_1 & u_2 & \cdots & u_{N_\mathrm{d}} \end{pmatrix}^\mathrm{T}, \quad \boldsymbol{d} = \begin{pmatrix} d_1 & d_2 & \cdots & d_{N_\mathrm{e}} \end{pmatrix}^\mathrm{T} \\
&\boldsymbol{M}_u = \mathrm{diag}\left\{ M_{u,1} \quad \cdots \quad M_{u,n} \quad \cdots \quad M_{u,N_\mathrm{d}} \right\} \\
&\boldsymbol{M}_d = \mathrm{diag}\left\{ \rho_{d,1}\Delta x_1 \quad \rho_{d,2}\Delta x_2 \quad \cdots \quad \rho_{d,N_\mathrm{e}}\Delta x_{N_\mathrm{e}} \right\} \\
&\boldsymbol{M}_{du} = \frac{1}{4}\begin{bmatrix} \Delta x_1\rho_{ud,1} & \Delta x_1\rho_{ud,1} & 0 & \cdots & 0 \\ 0 & \Delta x_2\rho_{ud,2} & \Delta x_2\rho_{ud,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Delta x_{N_\mathrm{e}}\rho_{ud,N_\mathrm{e}} & \Delta x_{N_\mathrm{e}}\rho_{ud,N_\mathrm{e}} \end{bmatrix},
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
&M_{u,1} = \frac{\Delta x_1\rho_{u,1}}{2}, \quad M_{u,n} = \frac{\Delta x_{n-1}\rho_{u,n-1} + \Delta x_n\rho_{u,n}}{2}, \quad M_{u,N_\mathrm{d}} = \frac{\Delta x_{N_\mathrm{e}}\rho_{u,N_\mathrm{e}}}{2} \\
&\rho_{u,n} = \rho_u\left(\frac{x_n + x_{n+1}}{2}\right), \quad \rho_{d,n} = \rho_d\left(\frac{x_n + x_{n+1}}{2}\right), \quad \rho_{ud,n} = \rho_{ud}\left(\frac{x_n + x_{n+1}}{2}\right).
\end{aligned} \tag{32}$$

The potential energy can be approximated as

$$U = \boldsymbol{g}_d^\mathrm{T}\boldsymbol{d} - \boldsymbol{g}_h^\mathrm{T}\boldsymbol{h} \tag{33}$$

where

$$\begin{aligned}
&\boldsymbol{g}_h^\mathrm{T} = \left(\frac{\Delta x_1}{2} g_{u,1}, \left(\frac{\Delta x_n}{2} + \frac{\Delta x_{n-1}}{2} \right) g_{u,n}, \frac{\Delta x_{N_e}}{2} g_{u,N_d} \right)^\mathrm{T} \\
&\boldsymbol{g}_d^\mathrm{T} = \left(\frac{\left(g_{d,1} + g_{d,2}\right)\Delta x_1}{2} \quad \frac{\left(g_{d,2} + g_{d,3}\right)\Delta x_2}{2} \quad \cdots \quad \frac{\left(g_{d,N_\mathrm{d}-1} + g_{d,N_\mathrm{d}}\right)\Delta x_{N_\mathrm{e}}}{2} \right)^\mathrm{T}. \\
&\boldsymbol{h}^\mathrm{T} = \left(h\left(x_1 + u_1\right), h\left(x_n + u_n\right), h\left(x_{N_d} + u_{N_d}\right) \right)^\mathrm{T} \\
&g_{d,n} = \frac{\rho g}{2} d_0\left(x_n\right), \quad g_{u,n} = \rho g d_0\left(x_n\right)
\end{aligned} \tag{34}$$

The Lagrangian modified term can be approximated as：

$$R = \boldsymbol{\lambda}^\mathrm{T}\boldsymbol{\Delta d} + \boldsymbol{d}^\mathrm{T}\boldsymbol{\Lambda C u} - \boldsymbol{\lambda}^\mathrm{T}\boldsymbol{s} = \boldsymbol{\lambda}^\mathrm{T}\boldsymbol{\Delta d} + \boldsymbol{\lambda}^\mathrm{T}\boldsymbol{DCu} - \boldsymbol{\lambda}^\mathrm{T}\boldsymbol{s} \tag{35}$$

in which

$$\begin{aligned} &\boldsymbol{s}^{\mathrm{T}} = \left(\Delta x_1 d_{0,1} \quad \Delta x_n d_{0,2} \quad \Delta x_{N_e} d_{0,N_e} \right) \\ &\boldsymbol{D} = \operatorname{diag}\left\{ \boldsymbol{d} \right\}, \quad \boldsymbol{\varDelta} = \operatorname{diag}\left\{ \Delta x_1, \Delta x_2, \cdots \right\} \\ &\boldsymbol{\beta}^{\mathrm{T}} = \left(\beta_1 \quad \beta_2 \quad \ldots \quad \beta_{N_e} \right), \quad \boldsymbol{B} = \operatorname{diag}\left\{ \boldsymbol{\beta} \right\} \\ &\boldsymbol{C} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & -1 & 1 \end{bmatrix} \end{aligned} . \tag{36}$$

In terms of Eqs. (29), (33) and (35), the action integral can be approximated as

$$S = \int_0^t \begin{bmatrix} \frac{1}{2}\dot{\boldsymbol{u}}^{\mathrm{T}} \boldsymbol{M}_u \dot{\boldsymbol{u}} + \frac{1}{2}\dot{\boldsymbol{d}}^{\mathrm{T}} \boldsymbol{M}_d \dot{\boldsymbol{d}} - \dot{\boldsymbol{d}}^{\mathrm{T}} \boldsymbol{M}_{du} \dot{\boldsymbol{u}} - \boldsymbol{g}_d^{\mathrm{T}} \boldsymbol{d} + \boldsymbol{g}_h^{\mathrm{T}} \boldsymbol{h} \\ + \frac{1}{2}\boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{\varDelta} \boldsymbol{d} + \frac{1}{2}\boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{DCu} - \frac{1}{2}\boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{s} \end{bmatrix} \mathrm{d}\tau , \tag{37}$$

in which $\boldsymbol{u}$ and $\boldsymbol{d}$ are the horizontal displacement vector and the water depth vector, respectively, and $\boldsymbol{\beta}$ is the pressure vector.

In the integrand of Eq. (37), the summation of the first three terms represents the kinetic energy, the summation of the fourth and fifth terms represents the potential energy and the summation of the last three terms represents the constrained term. Calculating the first variation of S gives

$$\begin{cases} \boldsymbol{M}_u \ddot{\boldsymbol{u}} - \boldsymbol{M}_{du}^{\mathrm{T}} \ddot{\boldsymbol{d}} - \boldsymbol{h}_x \boldsymbol{g}_h - 0.5 \boldsymbol{C}^T \boldsymbol{Bd} = 0 \\ \boldsymbol{M}_d \ddot{\boldsymbol{d}} - \boldsymbol{M}_{du} \ddot{\boldsymbol{u}} + \boldsymbol{g}_d - 0.5 \boldsymbol{\varDelta\beta} - 0.5 \boldsymbol{BCu} = 0 \\ \boldsymbol{\varDelta d} + \boldsymbol{DCu} - \boldsymbol{s} = 0 \end{cases} \tag{38}$$

which constitutes a system of differential algebraic equations (DAES).

3.2 The Zu-class method

The nonlinear DAES (38) is obtained from the first variation of Eq. (37), which corresponds to a constrained Hamilton system. For the Hamilton system, the symplecticity is a characteristic property. As the symplectic method can preserve the property of symplecticity and the approximate energy necessary for a long time computation, it is often used to simulate the Hamilton dynamical system. However, for the constrained Hamilton system, it is required for a time integration method to preserve not only the symplectic structure of the Hamilton system but also all the constraints. In Ref [8], a method preserving all the constraints was developed by Zhong et al. The numerical experiment about the double pendulum presented in [8] shows this method can preserve energy well. In Ref [9] it was named as the Zu-type method. In Ref [10], the Zu-type method was proved to be symplectic. In the Zu-type method, the constraint conditions are satisfied strictly at the integration points, and the orbit is treated as the geodesic in the state space and is, therefore, approximated by using the time finite element method. In this sub-section, the Zu-class method will be employed to solve the DAES(38). Let the time domain be discretized as

$$t = t_0, t_1, \cdots, t_k, \cdots, \qquad t_k = k \times \Delta t , \tag{39}$$

where Δt is the time step. Let the velocities in $[t_k, t_{k+1}]$ be approximated as

$$\dot{\boldsymbol{u}}_k(t) = \frac{\boldsymbol{u}_{k+1} - \boldsymbol{u}_k}{\Delta t}, \quad \dot{\boldsymbol{d}}_k(t) = \frac{\boldsymbol{d}_{k+1} - \boldsymbol{d}_k}{\Delta t}, \quad t \in [t_k, t_{k+1}], \tag{40}$$

where $\#_k = \#(t_k)$, and the displacements be approximated as

$$\boldsymbol{u}(t) = \frac{\boldsymbol{u}_{k+1} + \boldsymbol{u}_k}{2}, \quad \boldsymbol{d}(t) = \frac{\boldsymbol{d}_{k+1} + \boldsymbol{d}_k}{2}, \quad t \in [t_k, t_{k+1}]. \tag{41}$$

The pressure is approximated to be a constant, i.e.,

$$\boldsymbol{\beta}(t) = \boldsymbol{\beta}_k, \ t \in [t_k, t_{k+1}]. \tag{42}$$

By substituting Eqs. (40)-(42) into Eq. (37), the kinetic energy, potential energy and constrained term can be approximated as

$$\begin{aligned}
T &= \frac{1}{2}\left(\boldsymbol{u}_{k+1}^{\mathrm{T}} - \boldsymbol{u}_k^{\mathrm{T}}\right)\boldsymbol{M}_u\left(\frac{\boldsymbol{u}_{k+1} - \boldsymbol{u}_k}{\Delta t}\right) + \frac{1}{2}\left(\boldsymbol{d}_{k+1}^{\mathrm{T}} - \boldsymbol{d}_k^{\mathrm{T}}\right)\boldsymbol{M}_d\,\frac{\boldsymbol{d}_{k+1} - \boldsymbol{d}_k}{\Delta t} \\
&\quad - \left(\boldsymbol{d}_{k+1}^{\mathrm{T}} - \boldsymbol{d}_k^{\mathrm{T}}\right)\boldsymbol{M}_{du}\left(\frac{\boldsymbol{u}_{k+1} - \boldsymbol{u}_k}{\Delta t}\right) \\
U &= \frac{\Delta t}{2}\boldsymbol{g}_d^{\mathrm{T}}\left(\boldsymbol{d}_k + \boldsymbol{d}_{k+1}\right) - \frac{\Delta t}{2}\boldsymbol{g}_h^{\mathrm{T}}\left[\boldsymbol{h}(\boldsymbol{u}_k) + \boldsymbol{h}(\boldsymbol{u}_{k+1})\right] \\
R &= \frac{1}{2}\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\Lambda}\frac{\Delta t}{2}\left(\boldsymbol{d}_k + \boldsymbol{d}_{k+1}\right) + \frac{1}{2}\boldsymbol{\beta}^{\mathrm{T}}\frac{\Delta t}{2}\left(\boldsymbol{D}_k\boldsymbol{C}\boldsymbol{u}_k + \boldsymbol{D}_{k+1}\boldsymbol{C}\boldsymbol{u}_{k+1}\right) - \frac{1}{2}\Delta t\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{s}
\end{aligned} \tag{43}$$

Substituting Eq. (43) into the action integral and taking the first variation gives

$$\begin{cases}
\dfrac{\boldsymbol{M}_u}{\Delta t}\left(\boldsymbol{u}_{k+1} - \boldsymbol{u}_k\right) - \boldsymbol{M}_{du}^{\mathrm{T}}\left(\dfrac{\boldsymbol{d}_{k+1} - \boldsymbol{d}_k}{\Delta t}\right) + \dfrac{\Delta t}{2}\boldsymbol{h}_{\boldsymbol{u}_{k+1}}\boldsymbol{g}_h + \dfrac{\Delta t}{4}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{D}_{k+1}\boldsymbol{\beta} - \boldsymbol{p}_{u,k+1} = \boldsymbol{0} \\
-\dfrac{\boldsymbol{M}_u}{\Delta t}\left(\boldsymbol{u}_{k+1} - \boldsymbol{u}_k\right) + \boldsymbol{M}_{du}^{\mathrm{T}}\left(\dfrac{\boldsymbol{d}_{k+1} - \boldsymbol{d}_k}{\Delta t}\right) + \dfrac{\Delta t}{2}\boldsymbol{h}_{\boldsymbol{u}_k}\boldsymbol{g}_h + \dfrac{\Delta t}{4}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{D}_k\boldsymbol{\beta} + \boldsymbol{p}_{u,k} = \boldsymbol{0} \\
\boldsymbol{M}_d\,\dfrac{\boldsymbol{d}_{k+1} - \boldsymbol{d}_k}{\Delta t} - \boldsymbol{M}_{du}\left(\dfrac{\boldsymbol{u}_{k+1} - \boldsymbol{u}_k}{\Delta t}\right) - \dfrac{\Delta t}{2}\boldsymbol{g}_d + \dfrac{\Delta t}{4}\boldsymbol{\Lambda}\boldsymbol{\beta} + \dfrac{\Delta t}{4}\boldsymbol{B}\boldsymbol{C}\boldsymbol{u}_{k+1} - \boldsymbol{p}_{d,k+1} = \boldsymbol{0} \\
-\boldsymbol{M}_d\,\dfrac{\boldsymbol{d}_{k+1} - \boldsymbol{d}_k}{\Delta t} + \boldsymbol{M}_{du}\left(\dfrac{\boldsymbol{u}_{k+1} - \boldsymbol{u}_k}{\Delta t}\right) - \dfrac{\Delta t}{2}\boldsymbol{g}_d + \dfrac{\Delta t}{4}\boldsymbol{\Lambda}\boldsymbol{\beta} + \dfrac{\Delta t}{4}\boldsymbol{B}\boldsymbol{C}\boldsymbol{u}_k + \boldsymbol{p}_{d,k} = \boldsymbol{0} \\
\boldsymbol{\Lambda}\boldsymbol{d}_{k+1} + \boldsymbol{D}_{k+1}\boldsymbol{C}\boldsymbol{u}_{k+1} - \boldsymbol{s} = \boldsymbol{0}
\end{cases}, \tag{44}$$

where $\boldsymbol{p}_u$ and $\boldsymbol{p}_d$ are the momentum vectors

$$\boldsymbol{p}_u = \boldsymbol{M}_u\dot{\boldsymbol{u}} - \boldsymbol{M}_{du}^{\mathrm{T}}\dot{\boldsymbol{d}}, \quad \boldsymbol{p}_d = \boldsymbol{M}_d\dot{\boldsymbol{d}} - \boldsymbol{M}_{du}\dot{\boldsymbol{u}}. \tag{45}$$

Eq. (44) is a system of nonlinear algebraic equations that can be solved by using the Newton iteration method. If $\boldsymbol{M}_d$ and $\boldsymbol{M}_{du}$ in Eq. (44) are replaced with $\boldsymbol{M}_d\varepsilon$ and $\boldsymbol{M}_{du}\varepsilon$ ($\varepsilon \ll 1$), Eq. (44) can be used to analyze the shallow water problem ignoring the effect of the vertical acceleration. In this case, the results are in agreement with the SVE solution.

4 NUMERICAL EXAMPLES

In this section, three classical numerical examples, which have been used frequently by many researchers to test different numerical methods[11-13], are used to verify the correction of the proposed SWE-DP and the effectiveness of the proposed numerical method.

4.1 Evolution of shorelines over a parabolic topography

Analytical solutions of the nonlinear shallow water equations were derived by Sampson et al.[11] for a perturbed flow in 1-dimensional container with a parabolic bed profile. This provides a perfect test for the present method in dealing with bed sloping source term and wetting/drying condition. The initial water domain is shown by Fig. 3.

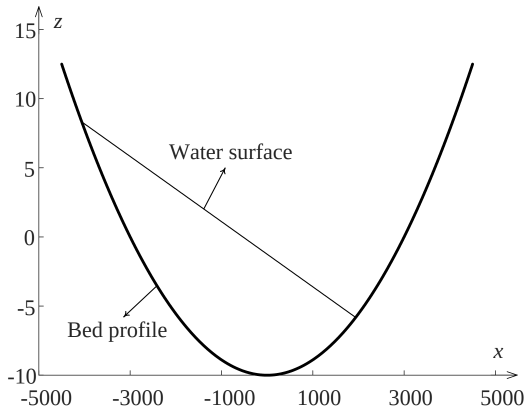

Fig.3 The water profile at $t = 0$ s

The Bed profile is defined by

$$h(x) = h_0 - h_0 \left(\frac{x}{a} \right)^2 \tag{46}$$

where $h_0 = 10$ m, $a = 3000$ m. The analytical horizontal displacement and water surface are shown as

$$u(x,t) = \frac{a^2}{2gh_0} B\omega (1 - \cos \omega t) \tag{47}$$

and

$$\eta(x,t) = -\frac{1}{g} B\omega \cos \omega t \left(x + u(x,t) \right) - \frac{a^2 B^2}{8g^2 h_0} \omega^2 \cos 2\omega t - \frac{B^2}{4g}, \quad \omega = \frac{\sqrt{2gh_0}}{a}, \tag{48}$$

respectively, in which $B = 5\ \mathrm{m/s}$, $g = 10\ \mathrm{m/s^2}$, $t \in [0\ \mathrm{s}, 6000\ \mathrm{s}]$.

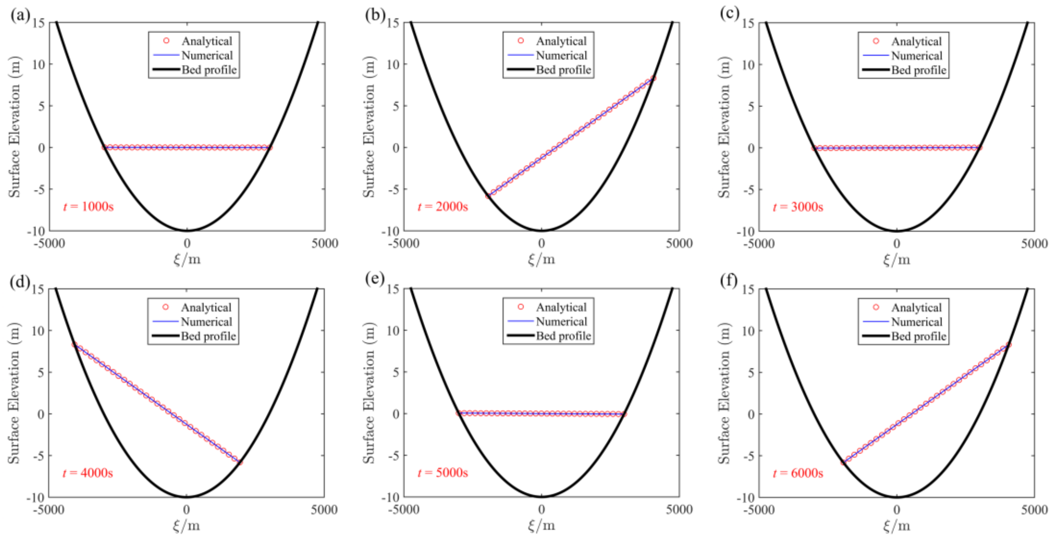

Fig.4 Water surface elevation at different times: (a) $t = 1000$ s; (b) $t = 2000$ s; (c) $t = 3000$ s; (d) $t = 4000$ s; (e) $t = 5000$ s; (f) $t = 6000$ s;

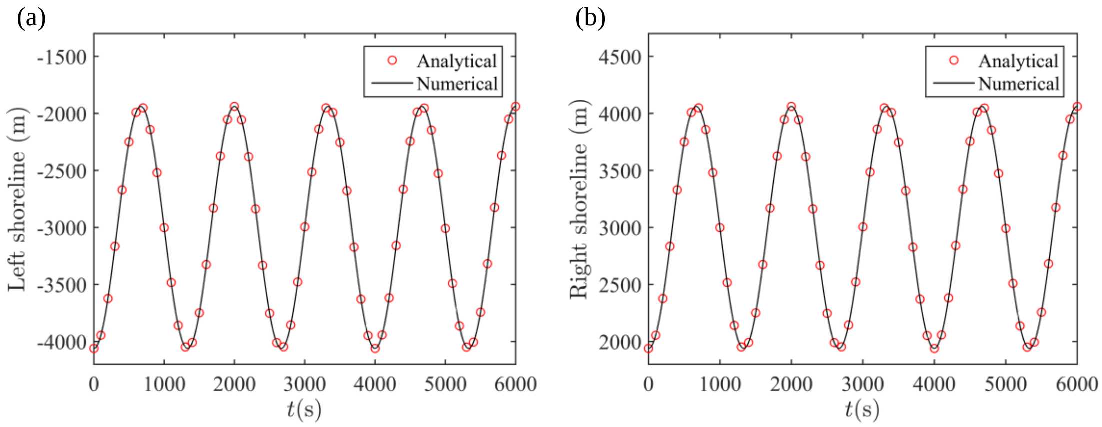

Fig.5 Location of shorelines: (a) Left shoreline; (b) Right shoreline.

Numerical simulation is performed with a time step $\Delta t = 1$ s and a uniform grid with $N_e = 300$ elements. Fig. 4 shows the predicted water surface elevation at different times. Fig. 5 shows the predicted locations of left and right shorelines. Excellent agreements are observed between the numerical prediction and analytical solutions from Figs. 4-5, which validates that the proposed method is able to deal with the shallow water with complex bed topographies and wet-dry interfaces.

4.2 Spreading of a drop of shallow water

Consider the spreading of a parabola-shaped two-dimensional drop of shallow water on a horizontal plane. The drop is initially confined to $|x| < 1$ according to

$$z = \eta_0(x) = \begin{cases} 1 - x^2 & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}, \tag{49}$$

and at rest. Upon releasing the drop, it spreads under the effect of gravity. The gravitational acceleration is $g = 1\,\mathrm{m/s^2}$. The temporal evolution of this system has been analytically investigated by Frei[14]. He noted that the parabolic shape is always retained and that the velocity across the drop is a linear function, i.e.,

$$\eta(x,t)=\lambda^{-1}\left[1-\left(\frac{\xi}{\lambda}\right)^2\right],\ \dot{u}(x)=\xi\left(\frac{\lambda_t}{\lambda}\right) \tag{50}$$

in which, ξ is defined by Eq. (1), λ describes the half-width of the drop, and $\lambda_t=\partial\lambda/\partial t$ is the velocity of the leading edge. Following Refs. [14, 15], the $\lambda(t)$ is obtained numerically as the root of the follow equation

$$t=\frac{1}{2}\left[\sqrt{\lambda(\lambda-1)}+\ln\left(\sqrt{\lambda-1}+\sqrt{\lambda}\right)\right], \tag{51}$$

and

$$\lambda_t=2\sqrt{1-\lambda^{-1}}\,. \tag{52}$$

The proposed SWE-DP is used to model this problem, and the numerical simulation is performed with $N_e=100$ elements. The time step is selected as $\Delta t=0.01\,\mathrm{s}$, and the numerical simulation lasts for $2\,\mathrm{s}$. To ignore the effect of the vertical acceleration, $\boldsymbol{M}_d$ and $\boldsymbol{M}_{du}$ in Eq. (44) are replaced with $0.001\boldsymbol{M}_d$ and $0.001\boldsymbol{M}_{du}$, respectively. The analytical solutions and the numerical results computed using the proposed method are compared in Figs. 6-7. The example used here was also discussed in Ref. [16] where the SVE was used to model the drop and the second order high resolution algorithm was applied to solve the SVE. The numerical velocities given in Ref. [16] is also displayed in Fig. 7.

It can be observed from Figs. 6-7 that the numerical results computed using the proposed method are in excellent agreement with the analytical solution. However the numerical velocities given in Ref. [16] are different from the analytical solutions at the left and right moving wet-dry interfaces. Hence it can be concluded that the proposed model can correctly handle the problem of wet-dry interfaces, and perform better than the SVE.

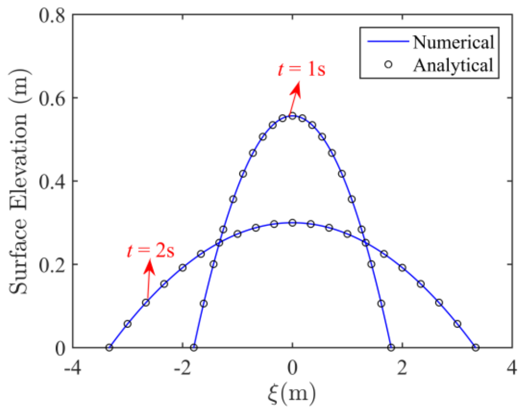

Fig.6 Water free surface

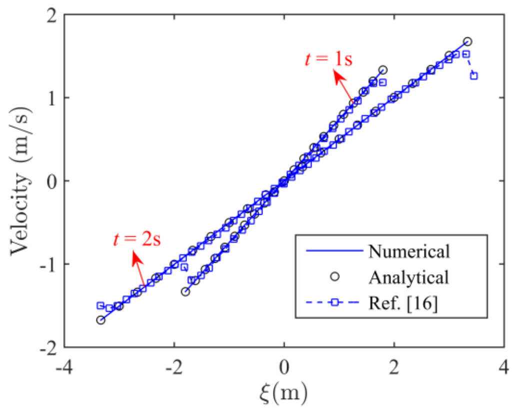

Fig.7 Comparison of velocity

4.3 Dam break on dry bed

Consider a reservoir with length 150 m and water depth 10 m; see Fig. 8. Suppose that the right dam is broken suddenly and the water pour out of the reservoir. The right side of the dam is the dry bed. Initially, the water surface is defined as

$$z = \eta_0(x) = \begin{cases} h_0, & \text{if} \quad x \le x_0 \\ 0, & \text{if} \quad x > x_0 \end{cases} \tag{53}$$

where $x_0 = 150\,\mathrm{m}$ and $h_0 = 10\,\mathrm{m}$. The initial velocity is $\dot{u}(x,0) = 0\,\mathrm{m/s}$. The gravitational acceleration is $g = 9.81\,\mathrm{m/s}$.

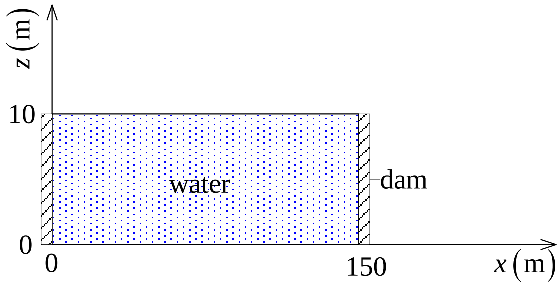

Fig. 8 The dam

The analytical solution of this problem is given as[13]

$$\eta(x,t) = \begin{cases} h_0, & \text{if } \xi \le x_0 - t\sqrt{gh_0} \\ \dfrac{1}{9g}\left(2\sqrt{gh_0} - \dfrac{\xi - x_0}{t}\right)^2, & \text{if } x_0 - t\sqrt{gh_0} < \xi \le x_0 + 2t\sqrt{gh_0} \\ 0, & \text{if } \xi > x_0 + 2t\sqrt{gh_0} \end{cases} \tag{54}$$

and

$$\dot{u}(x,t) = \begin{cases} 0, & \text{if } \xi \le x_0 - t\sqrt{gh_0} \\ \dfrac{2}{3}\left(\sqrt{gh_0} + \dfrac{\xi - x_0}{t}\right), & \text{if } x_0 - t\sqrt{gh_0} < \xi \le x_0 + 2t\sqrt{gh_0} \\ 0, & \text{if } \xi > x_0 + 2t\sqrt{gh_0} \end{cases} \tag{55}$$

where ξ is defined by Eq. (1). Numerical simulation is performed with a time step $\Delta t = 0.01\,\mathrm{s}$ and a uniform grid with $N_e = 300$ elements. The numerical simulation lasts for 6 s. To ignore the effect of the vertical acceleration, $\boldsymbol{M}_d$ and $\boldsymbol{M}_{du}$ in Eq. (44) are replaced with $0.001\boldsymbol{M}_d$ and $0.001\boldsymbol{M}_{du}$, respectively. Fig. 9 shows the predicted water surface elevation at $t = 6\,\mathrm{s}$. Fig. 10 shows the predicted discharges $\eta\dot{u}$ at $t = 6\,\mathrm{s}$. Excellent agreements are observed between the numerical prediction and analytical solutions from Figs. 9-10, which validates that the proposed method is able to deal with the shallow water with wet-dry interfaces.

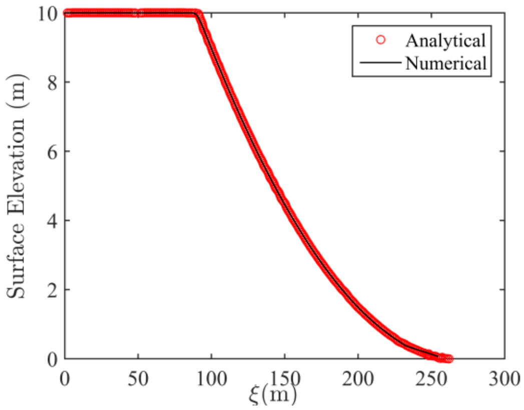

Fig. 9 Water surface at 6 s

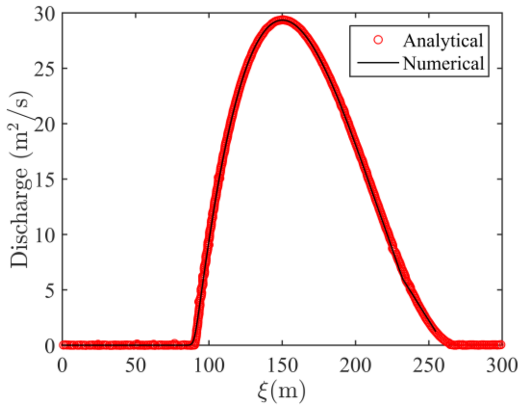

Fig. 10 Water discharge at 6 s

5 CONCLUSIONS

A constrained Hamilton variational principle for shallow water with sloping bottom is developed. Based on the constrained Hamilton variational principle, a shallow water equation based on displacement and pressure (SWE-DP) is developed. A hybrid numerical method combining the finite element method for spatial discretization and the Zu-class method for time integration is constructed for the SWE-DP. Three numerical examples are used to test correctness of the proposed SWE-DP and the effectiveness of the hybrid numerical method proposed for the SWE-DP. Numerical examples show that the proposed method performs well in the simulation of the shallow water with sloping depths and wet-dry interfaces.

In this paper, we only consider 1-dimensional shallow water. However, the basic ideas could be expanded to other hydrodynamics problems. We believe that the displacement method will play a major role in accurately and efficiently simulating hydrodynamics problems. In our next work, the proposed method will be expanded to shallow water with two dimensions.

6 ACKNOWLEDGEMENTS

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