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FLEXIBLE COMPLEX SYSTEM OF A DOUBLE-STRING UNDER EXTREME MOVING LOADS

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Abstract. *In this paper the dynamic response of a complex double-string system traversed by extreme moving load is considered. The paper includes the study of a dynamic behavior of a finite, simply supported double-string flexible complex system subject to moving forces with a constant velocity on the top string. The strings are identical, parallel one upon the other and continuously coupled by a linear Winkler elastic element. The moving loads are around an extreme position of the shear wave velocity of the strings. The classical solution of the response of complex systems subjected to forces moving with a constant velocity has a form of an infinite series. But also it is possible to show that in the considered case part of the solution can be presented in a closed, analytical form instead of an infinite series. The presented method to search for a solution in a closed, analytical form is based on the observation that the solution of the system of partial differential equations in the form of an infinite series is also a solution of an appropriate system of ordinary differential equations. The closed solutions take different forms depending if the velocity of a moving force is smaller, equal or larger than the shear wave velocity of the strings. This follows from the fact that in string wave phenomena may occur. The solution for the dynamic response of the composite strings under moving force is important because it can be used also in order to find the solution for other types of moving loads. The double string connected in parallel by linear elastic elements can be studied as a theoretical model of composite system or prestressed structure in which coupling effects and transverse wave effects are taken into account.*

1 INTRODUCTION

Flexible complex systems have a wide range of applications to civil, military, mechanical, transport, naval, aeronautical and aircraft as structural members with high strength to weight ratios. One of the most important issues in the dynamics of structures is moving load problems witch have been studied by many authors for many years [2-22]. Modeling of movement is very difficult in its complications and generates many mathematical problems [1,3,7,21]. Even simply models give very complex and unpredicted solutions of structural in dynamical and stability meaning. Thusly different types of structures and girders like beams, plates, shells, and frames also membranes, strings and cables have been considered. As well different models of moving loads have been assumed [3]. The paper includes the study of a dynamic behavior of a finite, simply supported double-string flexible complex system subject to moving force with a constant velocity on the top string. The strings are identical, parallel one upon the other and continuously coupled by a linear Winkler elastic element. The most critical situation is when moving loads are around an extreme position of the shear wave velocity of the strings. Responses of structures to extreme speed of load are often complex and difficult to understand, especially because of the complex nature of vibration repeatedly complicated systems. For instance, it is possible to observe some anomalies while the aircraft reaching speed of sound. The Prandtl–Glauert singularity [Fig.1] is the prediction by the Prandtl–Glauert transformation that infinite pressures would be experienced by an aircraft as it approaches the speed of sound. A certain similarity can be expected in the solid body, which can be proved by a closed-form solution of a string with a load moving at the speed of the transverse wave propagating in the string.

Figure 1: A F/A-18 Hornet during transonic flight, a Prandtl–Glauert singularity [NASA].

A string as a simple model of a one-dimensional continuous system resistant to tension but not to bending is often used in analysis of numerous engineering structures and has been a subject of great scientific interest for a considerable time. This follows from the fact that the vibrations of a string are described by the wave differential equation. This allows one to see the wave effect in a string, contrary to many more complex systems for example structural elements where it might be either not present or not clearly visible. The analogies between a string and the beams have been considered in papers [5,6,11]. Various aspects of the dynamics response of a string under a moving load have been considered, among others, in the papers [2,4,8,10,12-15,18-20,22]. The classical solution of the response of complex systems subjected to forces moving with a constant velocity has a form of an infinite series. But also it is possible to show that in the considered case part of the solution can be presented in a closed, analytical form instead of an infinite series. Using the method, of superposed deflections Kączkowski [9] has shown for a simply supported Euler-Bernoulli beam that, in the case of undamped vibration, the aperiodic part of the solution can be presented in a closed-form. Next, Reipert obtained a closed form solutions for a beam with arbitrary boundary conditions [16] and for a frame [17]. In this paper, we use a different method to obtain the solutions in a closed form. The presented method to search for a solution in a closed, analytical form is based on the observation that the solution of the system of partial differential equations in the form of an infinite series is also a solution of an appropriate system of ordinary differential equations. The closed solutions take different forms depending if the velocity of a moving force is smaller, equal or larger than the shear wave velocity of the strings. This follows from the fact that in string wave phenomena may occur. The presented solutions can be also used in axial and torsional vibration of the rod. Using this method, the closed solutions for undamped vibration of string and beam due to moving force have been obtained in the papers [18-21]. The solution for the dynamic response of the composite strings under moving force is important because it can be used also in order to find the solution for other types of moving loads. The double string connected in parallel by linear elastic elements can be studied as a theoretical model of composite system or prestressed structure in which coupling effects and transverse wave effects are taken into account.

2 MATHEMATICAL MODEL AND GOVERNING EQUATION

Consider the problem of a dynamic behavior of flexible complex system consist of a finite, simply supported double-string. The strings are identical, parallel one upon the other and continuously interfaced by a linear Winkler elastic element with *k* coefficient. The strings are under axial compression *N* and the system are excited by a load *p(x,t)* moving with a constant velocity *v* on a top string as on Fig.2.

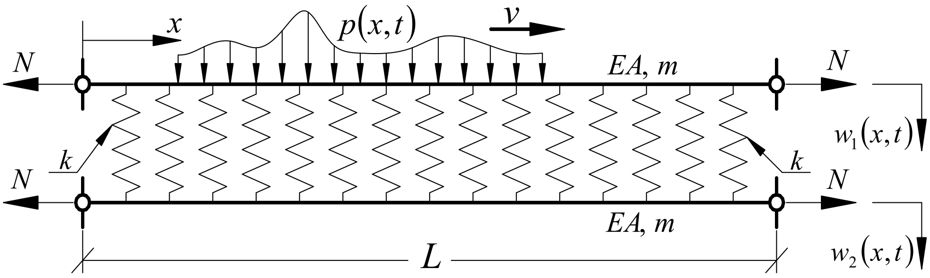

Figure 2: Double-string system under a moving force.

Vibrations describe functions $w_1(x,t)$ and $w_2(x,t)$ which solution in the classical-forms and closed-form is investigated. Hence an equation of motion of double-string system is governed by two conjugate partial differential equations:

$$-N\frac{\partial^2 w_1(x,t)}{\partial x^2}+m\frac{\partial^2 w_1(x,t)}{\partial t^2}+k\left[w_1(x,t)-w_2(x,t)\right]=p(x,t), \tag{1}$$

$$-N\frac{\partial^2 w_2(x,t)}{\partial x^2}+m\frac{\partial^2 w_2(x,t)}{\partial t^2}+k\left[w_2(x,t)-w_1(x,t)\right]=0, \tag{2}$$

where *m* is the mass per unit length ρA of each string, *k* denotes the stiffness modulus of a springs system, and furthermore, *EA* is the axial rigidity of the strings, *E* denotes Young's modulus of elasticity and A is the area of the cross-section of the strings. The load function in expression (1) and from figure 2 has a form:

$$p(x,t) = P\,\delta(x - vt), \tag{3}$$

where P is the intensity value of the load and $\delta(.)$ denotes Dirac delta. After introducing the dimensionless variables:

$$\xi = x/L, \quad T = vt/L, \quad \xi \in [0,1], \quad T \in [0,1], \tag{4}$$

the differential equations of motion of the string-string system have the form:

$$-\frac{\partial^2 w_1(\xi,T)}{\partial \xi^2} + \eta^2 \frac{\partial^2 w_1(\xi,T)}{\partial T^2} + k_o\left[w_1(\xi,T) - w_2(\xi,T)\right] = P_o\,\delta(\xi - T), \tag{5}$$

$$-\frac{\partial^2 w_2(\xi,T)}{\partial \xi^2} + \eta^2 \frac{\partial^2 w_2(\xi,T)}{\partial T^2} + k_o\left[w_2(\xi,T) - w_1(\xi,T)\right] = 0. \tag{6}$$

The parameters from equations (5) and (6) have the following designations:

$$v_s = \sqrt{N/m}, \qquad \eta = v/v_s, \qquad k_o = k\,L^2/N, \qquad P_o = P\,L/N. \tag{7}$$

The quantity v_s represents velocity of the transverse wave in the system of double-string. On the other hand the boundary conditions for both Eq. (5) and Eq. (6) take the form:

$$w_j(0,T) = 0, \qquad w_j(1,T) = 0, \quad j = \{1,\ 2\} \tag{8}$$

whereas the initial conditions are the following:

$$w_j(\xi,0) = 0, \qquad \left.\frac{\partial w_j(\xi,T)}{\partial T}\right|_{T=0} = 0, \quad j = \{1,\ 2\}. \tag{9}$$

It is easy to see that if you add together the equations (5) and (6) and introduce a new function as $w_I(\xi,T)$ we get:

$$-\frac{\partial^2 w_I(\xi,T)}{\partial^2 \xi} + \eta^2 \frac{\partial^2 w_I(\xi,T)}{\partial^2 T} = P_o\,\delta(\xi - T), \tag{10}$$

which describes vibrations of a single string. But then again, when we take the difference of these equations and differences of deflection functions take as a new function $w_{II}(\xi,T)$, that can be saved:

$$-\frac{\partial^2 w_{II}(\xi,T)}{\partial^2 \xi} + \eta^2 \frac{\partial^2 w_{II}(\xi,T)}{\partial^2 T} + 2k_o\,w_{II}(\xi,T) = P_o\,\delta(\xi - T). \tag{11}$$

In turn, this one describes vibrations of a single string resting on an elastic Winkler support with double parameter k ($2k_o$). So the solution (10) and (11) is also a solution of expressions (5) and (6), after appropriate transformation function $w_I(\xi,T)$ and $w_{II}(\xi,T)$ into $w_1(\xi,T)$ and $w_2(\xi,T)$. In addition, the solutions of equations (5), (6), (10) and (11) for boundary conditions (8) and initial conditions (9) are assumed to be in the form of sine series:

$$w_k(\xi,T) = \sum_{n=1}^{\infty} y_{kn} \sin n\pi\xi, \quad k = \{1,\ 2, I, II\}. \tag{12}$$

By substituting expression (10) into equations (5), (6), (10) or (11) and using the orthogonalization method one obtains set of uncoupled ordinary differential equations. Eventually, the solution of the above differential equations are sums of the particular integrals $w_k^A(\xi,T)$ and general integrals $w_k^S(\xi,T)$. We know that the angle of inclination of the tangent to the deflection function $w_k(\xi,T)$ can be presented:

$$\varphi_k(\xi,T)=\frac{\partial w_k(x,t)}{\partial x}=\frac{1}{L}\frac{\partial w_k(\xi,T)}{\partial \xi}, \tag{13}$$

additionally the dynamic component of the tension in the strings is given by:

$$N_k(\xi,T)=EA\frac{\partial w_k(x,t)}{\partial x}=\frac{EA}{L}\frac{\partial w_k(\xi,T)}{\partial \xi}. \tag{14}$$

3 THE CLASSICAL SOLUTIONS

The classical solution has a form of an infinite series, so classical part of the particular and general solution of the first string presents itself as:

$$\begin{aligned} w_1(\xi,T)=w_1^A(\xi,T)+w_1^S(\xi,T)=\frac{P_o}{1-\eta^2}\Bigg\{&\sum_{n=1}^{\infty}\frac{\left[2\,(n\pi)^2\,(1-\eta^2)+k_o\right]\,\sin n\pi T}{(n\pi)^4\,(1-\eta^2)\,+(n\pi)^2k_o}\sin n\pi\xi+\\ &-\sum_{n=1}^{\infty}\frac{\left[\frac{(n\pi)^3}{(1-\eta^2)^{-1}}\sin\left[\frac{\sqrt{(n\pi)^2+k_o}}{\eta}T\right]+\sqrt{(n\pi)^2+k_o}\left[\frac{(n\pi)^2}{(1-\eta^2)^{-1}}+k_o\right]\sin\frac{n\pi}{\eta}T\right]}{\eta^{-1}\,(n\pi)^2\sqrt{(n\pi)^2+k_o}\left[(n\pi)^2(1-\eta^2)+k_o\right]}\sin n\pi\xi\Bigg\}, \end{aligned} \tag{15}$$

and the second string:

$$\begin{aligned} w_2(\xi,T)=w_2^A(\xi,T)+w_2^S(\xi,T)=\frac{P_o}{1-\eta^2}\Bigg\{&\sum_{n=1}^{\infty}\frac{k_o\,\sin n\pi T}{(n\pi)^4\,(1-\eta^2)\,+(n\pi)^2k_o}\sin n\pi\xi+\\ &-\sum_{n=1}^{\infty}\frac{\left[\frac{(n\pi)^3}{(1-\eta^2)^{-1}}\sin\left[\frac{\sqrt{(n\pi)^2+k_o}}{\eta}T\right]-\sqrt{(n\pi)^2+k_o}\left[\frac{(n\pi)^2}{(1-\eta^2)^{-1}}+k_o\right]\sin\frac{n\pi}{\eta}T\right]}{\eta^{-1}(n\pi)^2\sqrt{(n\pi)^2+k_o}\left[(n\pi)^2(1-\eta^2)+k_o\right]}\sin n\pi\xi\Bigg\}. \end{aligned} \tag{16}$$

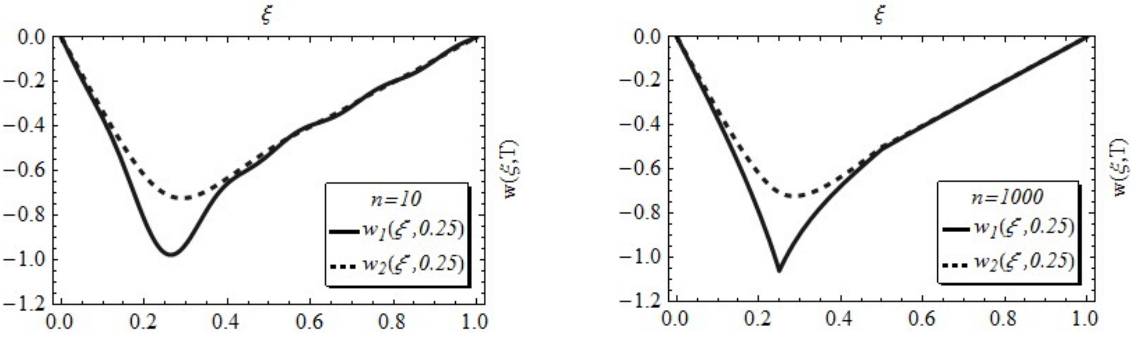

Figure 3: Deflection of the strings system depending on the number of approximation functions.

Analysis of functions (15) and (16) at the critical speed is difficult, even impossible, for example when $v = v_s$ ($\eta = 1$). On the one hand, the accuracy of the solution depends on respectively large number of approximation functions.

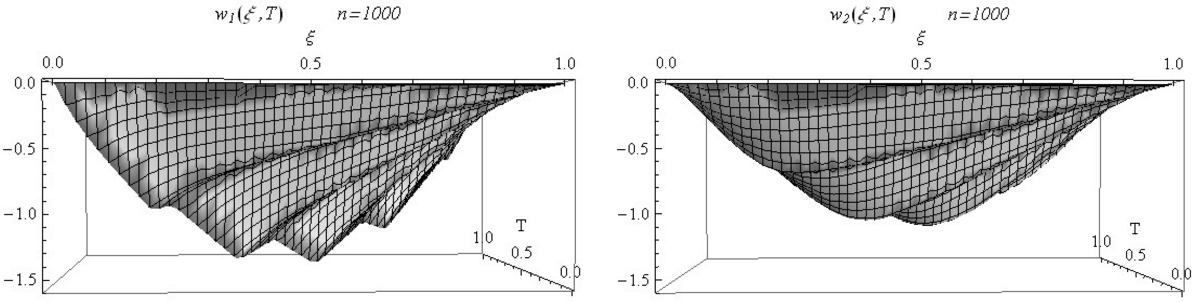

Figure 4: Overlap transverse waves in time of the double-string system under a moving force.

4 THE CLOSED-FORM SOLUTIONS

The classical solution of the response of complex systems subjected to forces moving with a constant velocity has a form of an infinite series, expressions (15) and (16). But also it is possible to show that in the considered case part of the solution can be presented in a closed, analytical form instead of an infinite series. The presented method to search for a solution in a closed, analytical form is based on the observation that the solution of the system of partial differential equations in the form of an infinite series is also a solution of an appropriate system of ordinary differential equations. For instance, let's take into consideration the particular solution Eq. (10) in the form of a series that is:

$$w_I^A(\xi,T) = \frac{2P_o}{1-\eta^2}\sum_{n=1}^{\infty}\frac{\sin n\pi T \sin n\pi\xi}{(n\pi)^2}. \tag{17}$$

It can be proved that the expression (17) is also integral ordinary differential equation:

$$-\frac{d^2 w_I^A(\xi,T)}{d\xi^2} = \frac{2P_o}{1-\eta^2}\delta(\xi - T), \tag{18}$$

where T is only a parameter of time. Solving formula (18), by a finite Fourier sine transform, we get:

$$(n\pi)^2\, \overline{w}_I^A(n,T) = \frac{P_o}{1-\eta^2}\sin n\pi T, \tag{19}$$

where

$$\overline{w}_I^A(n,T) = \int_0^1 w_I^A(\xi,T)\sin n\pi\xi \; d\xi, \tag{20}$$

and thus we return to the expression (17), which proves our assumption. Therefore if the function (17) satisfies the equation (18), it can be represented as closed-solution:

$$w_I^A(\xi,T) = \frac{P_o}{1-\eta^2}\left[(1-T)\xi - (\xi - T)H(\xi - T)\right], \tag{21}$$

where $H(.)$ denotes the Heaviside step function, or the unit step function.

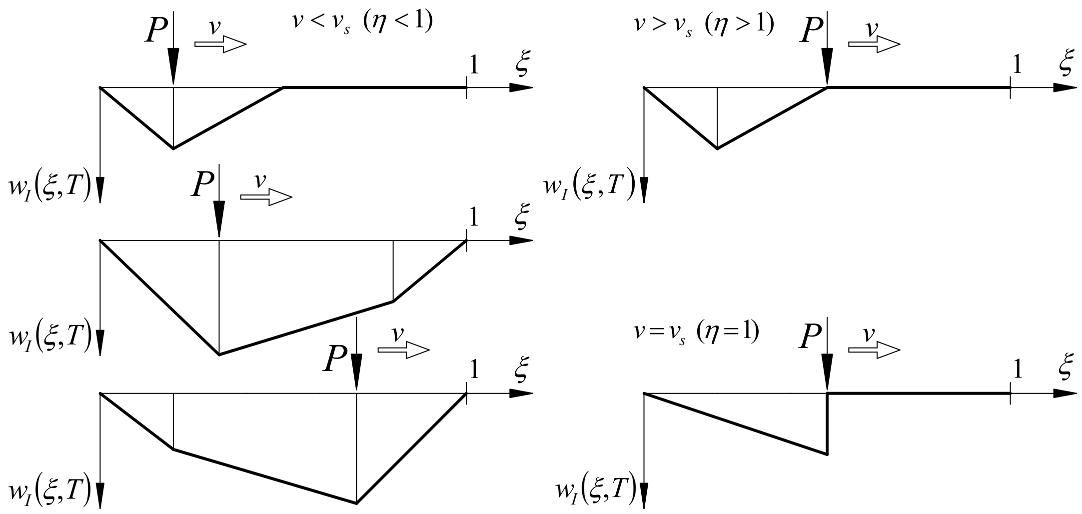

Figure 5: Deflection of string depending on velocity of moving force v and velocity of propagation transverse wave in string v_s.

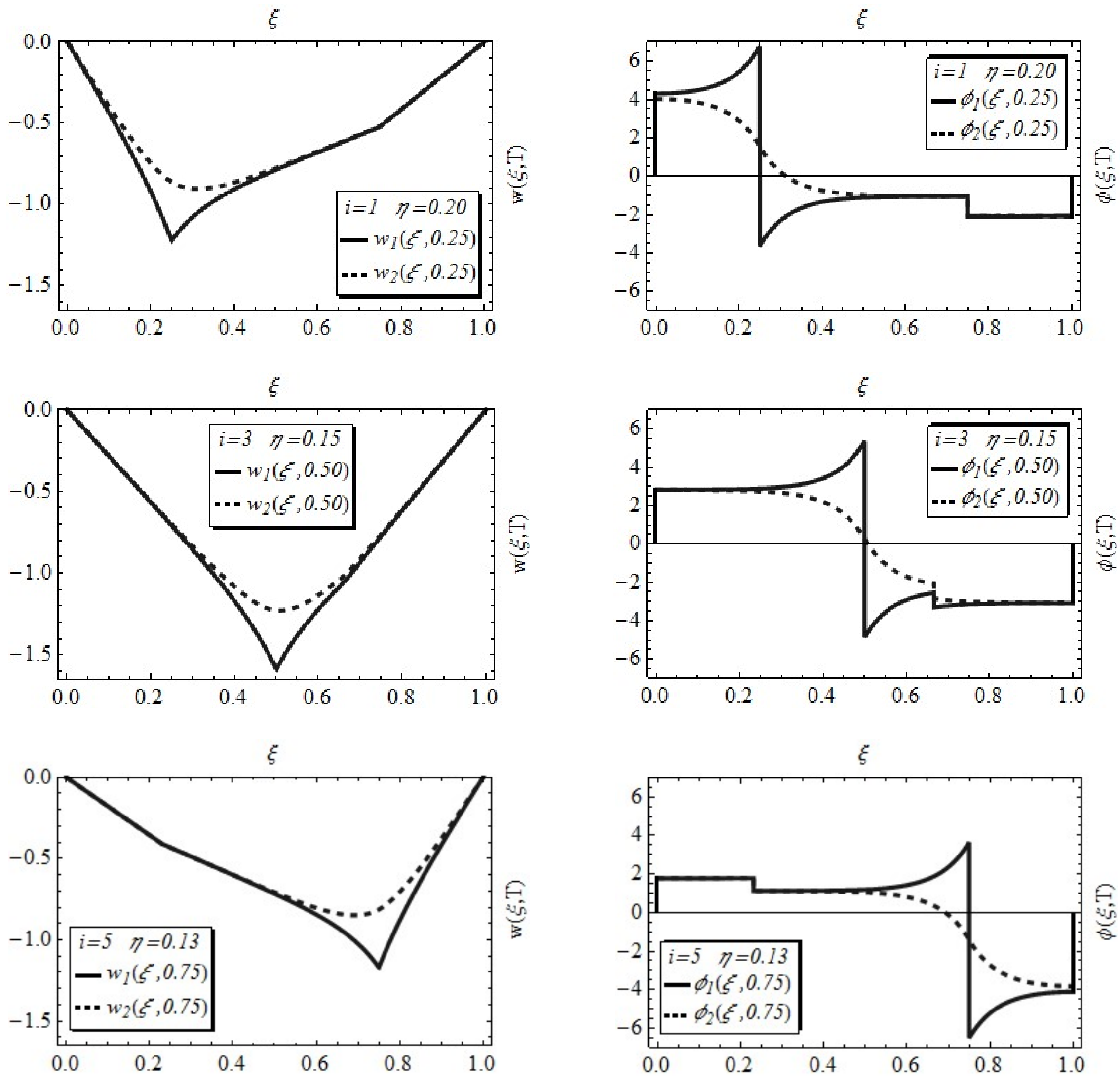

Figure 6: Deflection $w_i(\xi,T)$ and rotation $\varphi_i(\xi,T)$ of double-string system under a moving force.

The general integral $w_I^S(\xi,T)$ can be represented as closed-form solution, too. In a period of time $i\,\eta \le T \le (i+1)\,\eta \le 1$, where $i = 0, 2, 4,\ldots, 2n$, it satisfies the differential relationship:

$$\frac{d^2 w_I^S(\xi,T)}{d\xi^2} = \frac{P_o\,\eta}{1-\eta^2}\,\delta\left[\xi - \left(\frac{T}{\eta} - i\right)\right], \tag{22}$$

with the closed-form solution:

$$w_I^S(\xi,T) = \begin{cases} -\dfrac{P_o}{1-\eta^2}\,\xi\,\left[(i+1)\eta - T\right] & \text{for} \quad \xi \le \dfrac{T}{\eta} - i, \\ -\dfrac{P_o}{1-\eta^2}(1-\xi)(i\,\eta - T) & \text{for} \quad \xi \ge \dfrac{T}{\eta} - i. \end{cases} \tag{23}$$

For example, assuming $i = 0$ and consider the cases when the velocity of a moving force is smaller, equal or larger than the shear wave velocity of the strings. In a situation where $\eta < 1$ ($v < v_s$) and $\eta > 1$ ($v > v_s$) we get the formulas:

$$\eta < 1 \quad w_I^S(\xi,T) = \begin{cases} \dfrac{P_o}{1+\eta}\,\xi & \text{for } \xi \le T, \\ \dfrac{P_o}{1-\eta^2}(T - \eta\,\xi) & \text{for } T \le \xi \le \dfrac{T}{\eta}, \\ 0 & \text{for } \dfrac{T}{\eta} \le \xi \le 1, \end{cases} \qquad \eta > 1 \quad w_I^S(\xi,T) = \begin{cases} \dfrac{P_o}{1+\eta}\,\xi & \text{for } \xi \le \dfrac{T}{\eta}, \\ \dfrac{P_o}{1-\eta^2}(\xi - T) & \text{for } \dfrac{T}{\eta} \le \xi \le T, \\ 0 & \text{for } \xi \ge T. \end{cases} \tag{24}$$

Whereas when $\eta = 1$ ($v = v_s$):

$$w_I^S(\xi,T) = \begin{cases} \dfrac{P_o}{2}\,\xi & \text{for} \quad \xi < T, \\ 0 & \text{for} \quad \xi \ge T. \end{cases} \tag{25}$$

The closed-form solutions take different forms depending on the velocity of a moving force in relation to shear wave velocity of the strings. The graphic interpretation of this relationship is shown in Figure 5. And Figure 6 shows graphs in a period of time $i\,\eta \le T \le (i+1)\,\eta \le 1$, where $i = 1, 3, 5,\ldots, 2n\text{–}1$. More examples of closed-form solutions can be found [8,18-21].

5 SOME NUMERICAL RESULTS

In Figures 3-6, we presented deflections $w_k(\xi,T)$ and rotations $\varphi_k(\xi,T)$ of double-string complex system under the concentrated moving force. And the following dimensionless values of the parameters are used in the numerical calculations: $n = \{10, 1000\}$, $P_o = 10$, $k_o = 200$, $i = \{1, 3, 5\}$, $\eta = \{0.10, 0.13, 0.15, 0.20, 1.00\}$ and $T = \{0.25, 0.50, 0.75\}$. The results for different location of the moving force are presented in graphical form in Figs. 3-6. The continuous line represents the functions of the loaded string. The dashed line shows the functions of the second string for which the load is transferred with the coupling.

6 CONCLUSIONS

- The solution of the response of string-string complex systems subjected to forces moving with a constant velocity has a form of an infinite series, but also it is possible to show that in the considered case part of the solution can be presented in a closed, analytical form.

- A system of partial differential equations with solution in the form of an infinite series can be also a solution of an appropriate system of ordinary differential equations where a variable of time becomes only a parameter of time.
- The closed solutions take different forms depending if the velocity of a moving force is smaller, equal or larger than the shear wave velocity of the strings.
- The wave phenomena may occur in a complex system of double-string under moving forces.
- It is predicted a nonlinear solution in solid under load moving at a speed of transverse wave propagating in a solid.
- It is easier to observe the wave effect in a string, contrary to many more complex systems where it might be either not present or not clearly visible.
- The accuracy of solution gives the locate position of a transverse wave's front and a soliton which is a self-reinforcing solitary wave that maintains its shape while it propagates at a constant velocity.
- The double string connected in parallel by linear elastic elements can be studied as a theoretical model of composite system or prestressed structure in which coupling effects and transverse wave effects are taken into account.

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