ECCOMAS Congress 2016
VII European Congress on Computational Methods in Applied Sciences and Engineering
M. Papadrakakis, V. Papadopoulos, G. Stefanou, V. Plevris (eds.)
Crete Island, Greece, 5–10 June 2016

TOPOLOGY OPTIMIZATION OF CONTACT PROBLEMS BASED ON ALLEN-CAHN APPROACH

Andrzej M. Myśliński[1,2]

[1]Systems Research Institute
ul. Newelska 6, 01-447 Warsaw, Poland
e-mail: myslinsk@ibspan.waw.pl

[2] Warsaw University of Technology, Faculty of Manufacturing Engineering
ul. Narbutta 85, 02-524 Warsaw, Poland
e-mail: a.myslinski@wip.pw.edu.pl

Keywords: Topology optimization, Contact problems, Phase field regularization, Gradient flow.

Abstract. *The paper deals with a phase field model for formulation and solution of the topology optimization problems of bodies in unilateral contact consisting in the normal contact stress minimization. The contact problem with Tresca friction is governed by the system of elasticity equations with inequality type boundary conditions. The structural optimization problem consists in finding such material distribution within design domain to minimize the normal contact stress along the boundary of the body. The original structural optimization problem is reformulated in terms of material density function. Moreover the original cost functional is regularized using also surface and bulk energy terms. These terms allow to control global perimeter constraint and the occurence of the intermediate solution values. Using Lagrange multiplier approach the derivative of the regularized cost functional with respect to the control variable is calculated. The neccessary optimality condition is formulated in the form of Allen-Cahn gradient flow equation. The optimal topology is obtained as the steady state of the phase transition governed by this equation. This equation is discretized using finite difference and finite element methods. Numerical examples are provided.*

1 INTRODUCTION

The paper deals with the topology optimization for an elastic body in unilateral contact with a rigid foundation. This optimization problem consists in finding such topology of the domain occupied by the body and/or the shape of its boundary that the normal contact stress along the boundary of the body is minimized. Many successful methods have been proposed to analyze and to solve numerically topology optimization problems, including Simple Isotropic Material Penalization method and Evolutionary Structural Optimization approach, topology derivative method or different level set methods [1, 6, 7, 11, 19, 15].

In this paper phase field approach [4, 5, 6, 8, 17, 18] is proposed to regularize two phase topology optimization problem for unilateral elastic contact system and to solve it numerically. Material density function is a variable subject to optimization. This approach consists in using Ginzburg-Landau free energy term [8, 14, 17, 18, 19] as the regularization term rather than the perimeter constraint term. Although the proposed regularization for topology optimization of contact problems is more complicated than the perimeter one it has advantages comparing to the standard one. The derivative formula of the cost functional with respect to the material density function is calculated and is employed to formulate a necessary optimality condition for the topology optimization problem. This necessary optimality condition takes the form of the generalized Allen–Cahn equation rather than Cahn–Hilliard equation as in authors previous paper [10]. The derivative of the cost functional appears in the right hand side of these equation. Moreover the cost functional derivative is employed to calculate a descent direction in the numerical algorithm. Finite difference and finite element methods are used as the approximation methods. Implementation details are introduced. Numerical examples are provided and discussed.

2 PROBLEM FORMULATION

Consider deformations of an elastic body occupying two – dimensional domain Ω with the smooth boundary Γ (see Fig. 1). Assume $\Omega \subset D$ where D is a bounded smooth hold – all subset of R^2. The body is subject to body forces $f(x) = (f_1(x),\, f_2(x))$, $x \in \Omega$. Moreover, surface tractions $p(x) = (p_1(x), p_2(x))$, $x \in \Gamma$, are applied to a portion Γ_1 of the boundary Γ. We assume, that the body is clamped along the portion Γ_0 of the boundary Γ, and that the contact conditions are prescribed on the portion Γ_2, where $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j,\ i, j = 0, 1, 2$, $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$.

Let $\rho = \rho(x) : \Omega \to R$ denote the material density function at any generic point x in a design domain Ω. It is a phase field variable taking value close to 1 in the presence of material, while $\rho = 0$ corresponds to regions of Ω where the material is absent, i.e. there is a void. In the phase field approach the interface between material and void is described by a diffusive interfacial layer of a thickness proportional to a small lenght scale parameter $\epsilon > 0$ and at the interface the phase field ρ rapidly but smoothly changes its value [6]. We require that $0 \leq \rho \leq 1$. The ρ values outside this range do not seem to correspond to admissible material distributions. The elastic tensor $\mathcal{A}$ of the material body is assumed to be a function depending on density function ρ:

$$\mathcal{A} = g(\rho)\mathcal{A}_0, \ \ \mathcal{A}_0 = \{a_{ijkl}\}_{i,j,k,l=1}^{2} \tag{1}$$

and $g(\rho)$ is a suitable chosen function [2, 4, 6, 15]. We denote by $u = (u_1, u_2)$, $u = u(x)$, $x \in \Omega$, the displacement of the body and by $\sigma(x) = \{\sigma_{ij}(u(x))\}$, $i, j = 1, 2$, the stress field in the body. Consider elastic bodies obeying Hooke's law, i.e., for $x \in \Omega$ and $i, j, k, l = 1, 2$

$$\sigma_{ij}(u(x)) = g(\rho)a_{ijkl}(x)e_{kl}(u(x)). \tag{2}$$

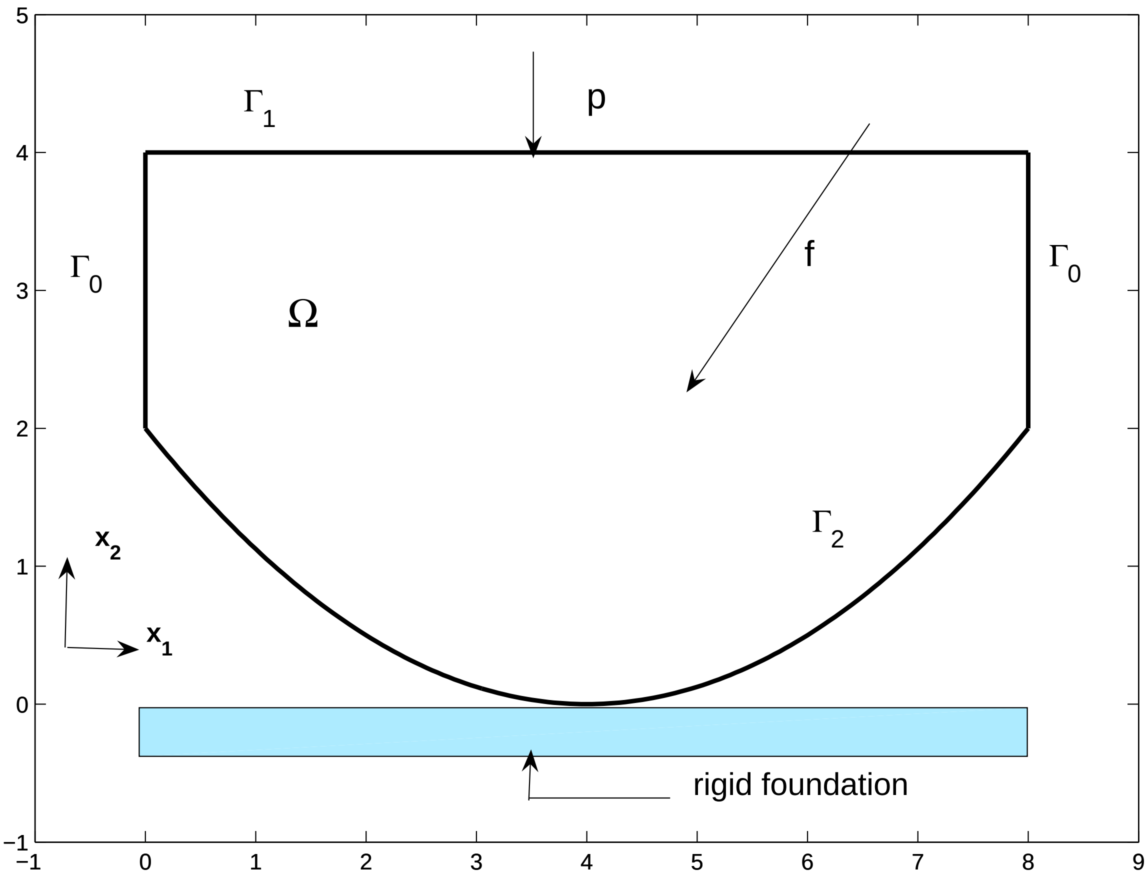

Figure 1: Initial domain Ω.

We use here and throughout the paper the summation convention over repeated indices [9]. The strain $e_{kl}(u(x))$, $k, l = 1, 2$, is defined by:

$$e_{kl}(u(x)) = \frac{1}{2}(u_{k,l}(x) + u_{l,k}(x)), \tag{3}$$

where $u_{k,l}(x) = \frac{\partial u_k(x)}{\partial x_l}$. The stress field σ satisfies the system of equations in the domain Ω [9]

$$-\sigma_{ij}(x)_{,j} = f_i(x) \quad x \in \Omega, i, j = 1, 2, \tag{4}$$

where $\sigma_{ij}(x)_{,j} = \frac{\partial \sigma_{ij}(x)}{\partial x_j}$, $i, j = 1, 2$. The following boundary conditions are imposed on the boundary $\partial\Omega$

$$u_i(x) = 0 \quad \text{on} \quad \Gamma_0, \quad i = 1, 2, \tag{5}$$

$$\sigma_{ij}(x) n_j = p_i \quad \text{on} \quad \Gamma_1, \quad i, j = 1, 2, \tag{6}$$

$$u_N \leq 0, \quad \sigma_N \leq 0, \quad u_N \sigma_N = 0 \quad \text{on } \Gamma_2, \tag{7}$$

$$|\ \sigma_T\ | \leq 1, \quad u_T \sigma_T + |\ u_T\ | = 0 \quad \text{on } \Gamma_2, \tag{8}$$

where $n = (n_1, n_2)$ is the unit outward versor to the boundary Γ. Here $u_N = u_i n_i$ and $\sigma_N = \sigma_{ij} n_i n_j$, $i, j = 1, 2$, represent the normal components of displacement u and stress σ, respectively. The tangential components of displacement u and stress σ are given by $(u_T)_i = u_i - u_N n_i$ and $(\sigma_T)_i = \sigma_{ij} n_j - \sigma_N n_i$, $i, j = 1, 2$, respectively. $|\ u_T\ |$ denotes the Euclidean norm in R^2 of the tangent vector u_T.

2.1 Structural Optimization Problem

Before formulating a structural optimization problem for (4)-(8) let us introduce the set U_{ad} of admissible domains. This set has the form

$$U_{ad} = \{\Omega \ : \ E \subset \Omega \subset D \subset R^2 \ : \ \Omega \text{ is Lipschitz continuous}, \quad Vol(\Omega) - Vol^{giv} \leq 0, \ \ Vol(\Omega) = \int_\Omega \rho(x)dx.\}, \tag{9}$$

where $E \subset R^2$ is a given domain such that Ω as well as all perturbations of it satisfy $E \subset \Omega$. The constant $const_1 > 0$ is assumed to exist. The set U_{ad} is assumed to be nonempty. The constant $Vol^{giv} = const_0 > 0$ is given. For the shape optimization prpblem for system (4)-(8) the optimized domain Ω is assumed to satisfy equality volume condition, i.e., (9) is assumed to be satisfied as equality. In a case of topology optimization Vol^{giv} is assumed to be the initial domain volume and (9) is satisfied in the form $Vol(\Omega) = r_{fr}Vol^{giv}$ with $r_{fr} \in (0, 1)$ [15]. Recall from [11, 12] the cost functional approximating the normal contact stress on the contact boundary

$$J_\eta(u(\Omega)) = \int_{\Gamma_2} \sigma_N(u)\eta_N(x)ds, \tag{10}$$

depending on the auxiliary given bounded function $\eta(x) \in M^{st}$. The auxiliary set $M^{st} = \{\eta = (\eta_1, \eta_2) \in [H^1(D)]^2 : \eta_i \leq 0 \ \text{ on } D, \ i = 1, 2, \quad \| \eta \|_{[H^1(D)]^2} \leq \ 1\}$. Functions σ_N and η_N are the normal components of the stress field σ corresponding to a solution u satisfying system (4) - (8) and the function η, respectively. The cost functional 10) approximates the normal contact stress and is associated with the elastic energy functional [9]. Consider the following structural optimization problem: *for a given function $\eta \in M^{st}$, find a domain $\Omega^\star \in U_{ad}$ such that*

$$J_\eta(u(\Omega^\star)) = \min_{\Omega \in U_{ad}} J_\eta(u(\Omega)). \tag{11}$$

Adding to (9) a perimeter constraint $P_D(\Omega) \leq const_1$, where $P_D(\Omega) = \int_\Gamma dx$ is a perimeter of a domain Ω in D [5, 11, 16] and $const_1 > 0$ is a given constant the existence of an optimal domain $\Omega^\star \in U_{ad}$ to the problem (11) is ensured (see [4, 5, 16]).

3 PHASE FIELD BASED TOPOLOGY OPTIMIZATION PROBLEM

Let us introduce the regularized cost functional $J(\rho, u)$ in the form:

$$J(\rho, u) = J_\eta(u) + E(\rho), \tag{12}$$

where the functional $J_\eta(u)$ is given by (10). The Ginzburg-Landau free energy functional $E(\rho)$ is expressed as

$$E(\rho) = \int_\Omega \psi(\rho)d\Omega, \ \ \psi(\rho) = \frac{\gamma\epsilon}{2} \mid \nabla\rho \mid^2 + \frac{\gamma}{\epsilon}\psi_B(\rho), \tag{13}$$

where $\epsilon > 0$ is a constant, $\gamma > 0$ is a parameter related to the interfacial energy density. Function $\psi_B(\rho) = \rho^2(1 - \rho^2)$ is a double-well potential [10] which characterizes the two phases [2, 6]. The structural optimization problem (11) takes the form: *find $\rho^\star \in U^\rho_{ad}$ such that*

$$J(\rho^\star, u^\star) = \min_{\rho \in U^\rho_{ad}} J(\rho, u), \tag{14}$$

where $u^\star = u(\rho^\star)$ denotes a solution to the state system (4)-(8) depending on $\rho^\star$ and $U^\rho_{ad} = \{\rho : Vol(\Omega) \leq Vol^{giv}\}$ denotes the set of admissible material density functions.

In order to compute the first variation of the cost functional (12) we apply a formal Lagrangian approach combined with Allen–Cahn approach [2]. Let us introduce the Lagrangian $L(\rho) = L(\rho, u, \lambda, p^a, q^a, \mu)$:

$$L(\rho, u, \lambda, p^a, q^a, \mu) = J_\eta(u) + E(\rho) + \int_\Omega g(\rho) a_{ijkl} e_{ij}(u) e_{kl}(p^a) dx - \tag{15}$$
$$\int_\Omega f_i p_i^a dx - \int_{\Gamma_1} p_i p_i^a ds + \int_{\Gamma_2} \lambda p_T^a ds + \int_{\Gamma_2} q^a u_T ds + \mu(\int_\Omega \rho(x) dx - Vol^{giv}),$$

where $(p^a, q^a) \in K_1 \times \Lambda_1$ denotes an adjoint state defined as follows:

$$\int_\Omega g(\rho) a_{ijkl} e_{ij}(\eta + p^a) e_{kl}(\varphi) dx + \int_{\Gamma_2} q^a \varphi_T ds = 0 \quad \forall \varphi \in K_1, \tag{16}$$

and

$$\int_{\Gamma_2} \zeta(p_T^a + \eta_T) ds = 0 \ \ \forall \zeta \in \Lambda_1. \tag{17}$$

The sets K_1 and Λ_1 are given by

$$K_1 = \{\xi \in V_{sp} \ : \ \xi_N = 0 \ \text{ on } A^{st}\ \}, \tag{18}$$

$$\Lambda_1 = \{\zeta \in \Lambda \ : \ \zeta(x) = 0 \text{ on } B_1 \cup B_2 \cup B_1^+ \cup B_2^+\}, \tag{19}$$

while the coincidence set $A^{st} = \{x \in \Gamma_2 \ : \ u_N + v = 0\}$. Moreover $B_1 = \{x \in \Gamma_2 : \lambda(x) = -1\}$, $B_2 = \{x \in \Gamma_2 : \lambda(x) = +1\}$, $\tilde{B}_i = \{x \in B_i : u_N(x) + v = 0\}$, $i = 1, 2$, $B_i^+ = B_i \setminus \tilde{B}_i$, $i = 1, 2$. The derivative of the Lagrangian L with rescpect to ρ has the form:

$$\int_\Omega \frac{\partial J}{\partial \rho}(\rho, u)\zeta dx = \int_\Omega \frac{\partial L}{\partial \rho}(\rho, u, \lambda, p^a, q^a, \mu)\zeta dx = \int_\Omega [\gamma\epsilon \nabla\rho \cdot \nabla\zeta + \frac{\gamma}{\epsilon}\psi_B'(\rho)\zeta + \mu\zeta] dx +$$
$$\int_\Omega [g'(\rho) a_{ijkl} e_{ij}(u_\epsilon) e_{kl}(p^a + \eta) - f(p^a + \eta)]\zeta dx, \ \ \forall \zeta \in H^1(\Omega) \tag{20}$$

Using (20) we formulate a modified Allen-Cahn equation with constant mobility function as a gradient flow dynamic problem in an artificial time variable. It leads to a pseudo time stepping approach. This problem is as follows: *find sufficiently regular* $(\rho, u, \lambda, p^a, q^a, \mu)$ *satisfying (4)-(8), (16)-(17) as well as*

$$\frac{\partial \rho}{\partial t} = \varphi_E(\rho) \qquad \text{in } \ \Omega, \ \forall t \in [0, T), \tag{21}$$
$$\nabla\rho \cdot n = 0 \qquad \text{on } \ \partial\Omega, \ \forall t \in [0, T), \tag{22}$$
$$\rho(0, x) = \rho_0(x) \qquad \text{in } \ \Omega, \ t = 0. \tag{23}$$

where the potential function φ_E is given by

$$\varphi_E = -\gamma\epsilon \triangle \rho + \frac{\gamma}{\epsilon}\psi_B'(\rho) + \mu -$$
$$g'(\rho) a_{ijkl} e_{ij}(u_\epsilon) e_{kl}(p^a + \eta) - f(p^a + \eta), \quad \text{a.e. in } \ \Omega. \tag{24}$$

The necessary optimality condition to optimization problem (14) has the form: if $(\rho^\star, u^\star, \lambda^\star, p^{a\star}, q^{a\star}, \mu^\star)$ is an optimal solution to structural optimization problem (14) than it satisfies (4)-(8), (16)-(17) and (21)-(24).

4 NUMERICAL RESULTS

The discretized structural optimization problem (14) is solved numerically. Time derivatives are approximated by the forward finite difference. Piecewise constant and piecewise linear finite element method is used as disretization method in space variables. The derivative of the double well potential is linearized with respect to ρ. Primal-dual active set method has been used to solve state and adjoint systems (4)-(8) and (16)-(17). Biconjugate gradient method has been used to solve (21)-(23). The algorithms are programmed in Matlab enviroment. As an example a body occupying 2D domain

$$\Omega = \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \ \wedge \ 0 < v(x_1) \leq x_2 \leq 4\}, \tag{25}$$

is considered. The boundary Γ of the domain Ω is divided into three pieces

$$\begin{aligned} \Gamma_0 = \{(x_1, x_2) \in R^2 : x_1 = 0, 8 \ \wedge \ 0 < v(x_1) \leq x_2 \leq 4\}, \\ \Gamma_1 = \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \ \wedge \ x_2 = 4\}, \\ \Gamma_2 = \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \ \wedge \ v(x_1) = x_2\}. \end{aligned} \tag{26}$$

The domain Ω and the boundary Γ_2 depend on the function v. The initial position of the boundary Γ_2 is given as in Fig. 1. The computations are carried out for the elastic body characterized by the Poisson's ratio $\nu = 0.29$, the Young modulus $E = 2.1 \cdot 10^{11} N/m^2$. The body is loaded by boundary traction $p_1 = 0$, $p_2 = -5.6 \cdot 10^6 N$ along Γ_1, body forces $f_i = 0$, $i = 1, 2$. Auxiliary function η is selected as piecewise constant (or linear) on D and is aproximated by a piecewise constant (or bilinear) functions. The computational domain $D = [0, 8] \times [0, 4]$ is selected. Domain D is discretized with a fixed rectangular mesh of 80 $\times$ 40. Other parameters are: $\epsilon = 0.02$, $\gamma = 1$, $T = 0.200$. Following [17] $g(\rho) = \frac{\rho}{1+\exp(-40\rho)} + g_\epsilon$, $g_\epsilon = 0.02$, $\psi_B(\rho) = \rho^2(1-\rho)^2(\frac{1}{10}\exp(15(1/2-\rho)^2)+1)$.

Fig. 2 presents the optimal domain obtained by solving structural optimization problem (14) in the computational domain D using the optimality condition (21)-(24). The areas with low values of density function appear in the central part of the body and near the fixed edges. The obtained normal contact stress is almost constant along the optimal shape boundary and has been significantly reduced comparing to the initial one (see Fig. 3).

5 CONCLUSIONS

The structural optimization problem for elastic contact problem with the prescribed friction is solved numerically in the paper. Obtained numerical results indicate that the proposed numerical algorithm allows for significant improvements of the structure from one iteration to the next. Phase field approach based on the Allen-Cahn equation is flexible and can be easily combined with material density field. In this sense this approach follows SIMP method. On the other hand this approach can be also coupled with other physical fields allowing to consider different topology optimization problems.

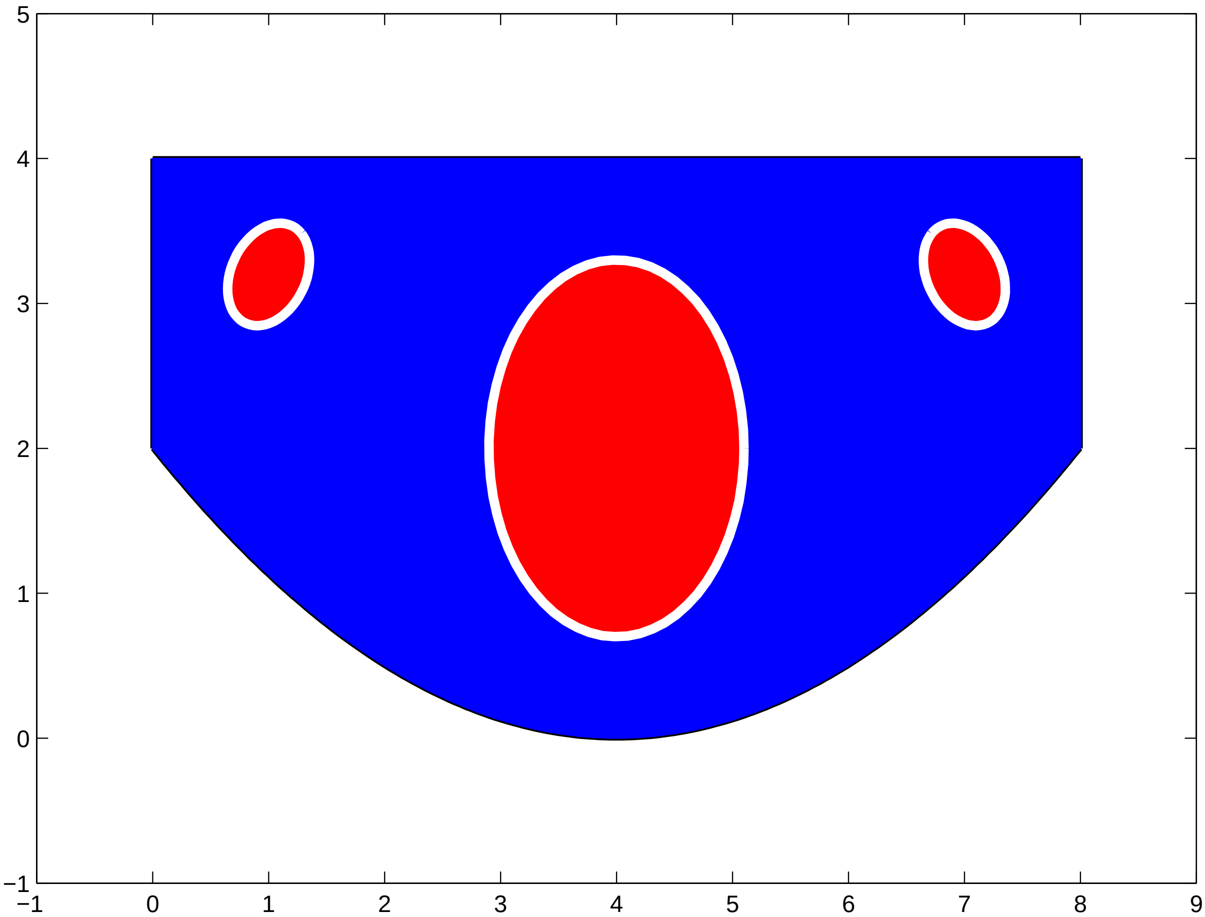

Figure 2: Optimal material density distribution in domain $\Omega^{\star}$.

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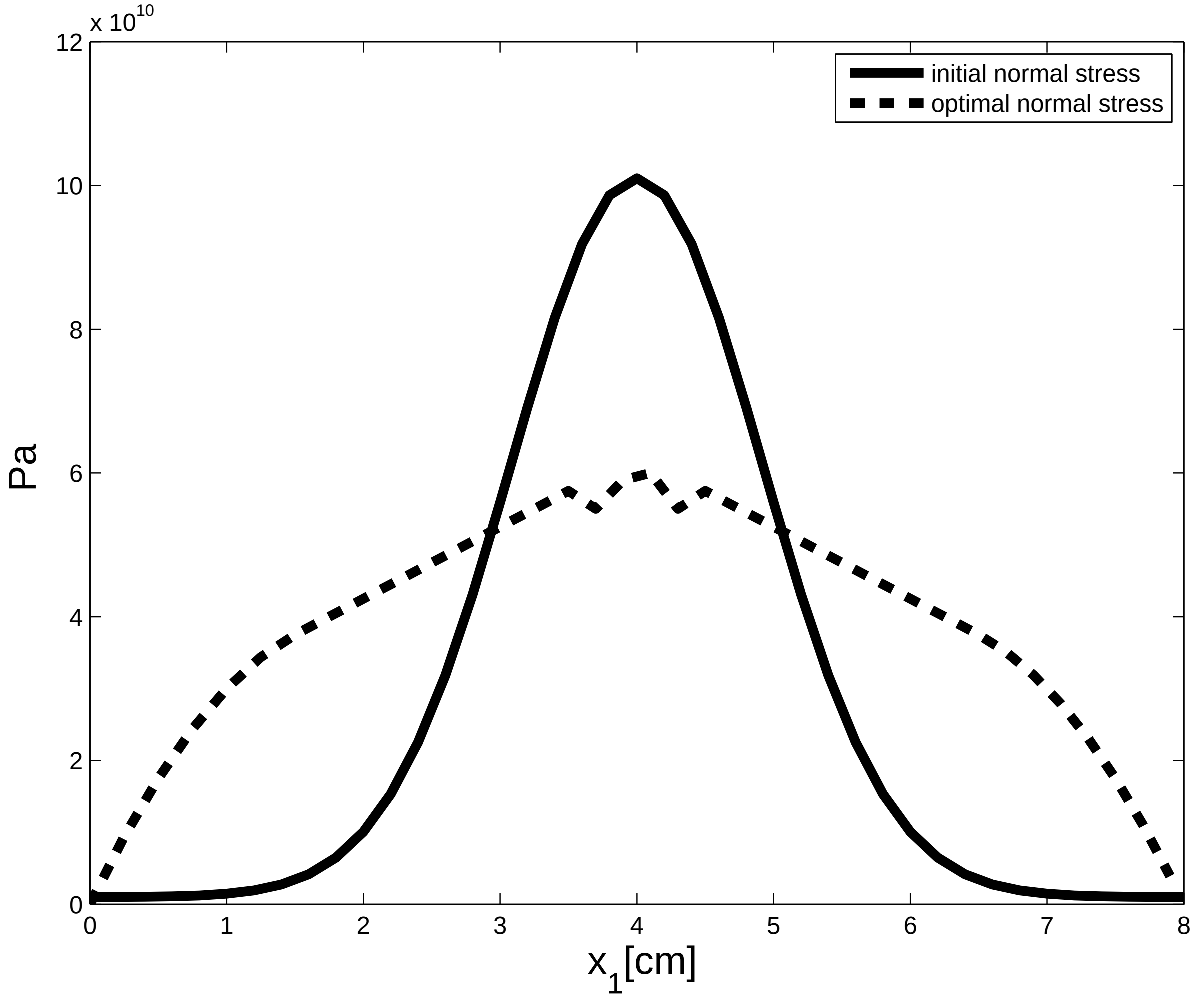

Figure 3: Initial and optimal normal contact stress.