ECCOMAS Congress 2016
VII European Congress on Computational Methods in Applied Sciences and Engineering
M. Papadrakakis, V. Papadopoulos, G. Stefanou, V. Plevris (eds.)
Crete Island, Greece, 5–10 June 2016

SQUEEZE FLOW OF VISCOPLASTIC BINGHAM MATERIAL

Larisa V. Muravleva[1]

[1] Lomonosov Moscow State University
Leninskie gory, 1, Main Building 119991 Moscow Russia
e-mail: lvmurav@gmail.com

Keywords: Viscoplastic Fluid, Squeeze Flow, Augmented Lagrangian Method.

Abstract. *We consider both planar and axisymmetric squeeze flows of a viscoplastic medium. Firstly we deal with no-slip boundary conditions. The asymptotic and the numerical solutions are developed. Previous theoretical analysis of this problem, using the standard lubrication approximation, has led to conflicting results, whereby the material around the plane of symmetry must both behaves as unyielded solid and translates in the main direction with a nonuniform velocity. This variation of the velocity implies that the plug region cannot be truly unyielded. Our solutions show that this region is a pseudo-plug region in which the leading order equation predicts a plug, but really it is weakly yielded at higher order. We follow the asymptotic technique suggested earlier by Balmforth, Craster (1999) and Frigaad, Ryan (2004). The obtained analytical expressions and numerical results are in a very good agreement with the earlier works. For numerical simulations we apply Augmented Lagrangian method (ALM) that yields superior results regarding the location of the yieldsurface. Finite-difference method on staggered grids is used as a discretization technique.*

Secondly we consider squeeze flow of a Bingham fluid subject to wall slip. If the wall shear-stress is smaller than the threshold value (the slip yield stress), the fluid adherence to the boundary is imposed and we have no-slip condition. When the wall shear stress reaches the slip yield stress, the fluid slips along the boundary. We solve numerically this problem also by ALM. The different flow regimes can be observed depending on the relative values of the yield stress and the slip yield stress. More precisely, for the case when the yield stress is smaller than the slip yield stress, there exists a particular value T_{crit} such that when the slip yield stress is greater or equal to T_{crit},the material fully sticks at the wall. When the slip yield stress is less than T_{crit} and bigger than the yield stress, the fluid sticks in the central region of the wall and slips close to the outer edges of discsorplates. For the case when the yield stress is bigger than the slip yield stress the material slips on the whole boundary.

1 INTRODUCTION

Squeeze flow is of considerable interest for fluids with a yield stress. The squeeze flow problem is well-studied, detailed review can be found in [1]. The analysis of viscoplastic materials started with the work of Scott [2] and Peek [3] who used the lubrication approximation, respectively assumed with different rigid zones in the flow region. Later, Covey and Stanmore [4] gave the approximate solutions for Bingham and HerschelBulkley fluids using lubrication theory.The lubrication solutions were developed using a no-slip wall boundary condition. An interesting controversy was initiated by Lipscomb and Denn [5] who showed that the unyielded (rigid) zone obtained by using the lubrication approximation for squeeze flow of Bingham materials includes the central plane of the material which however cannot move as a rigid plug for simple kinematic reasons. Gartling and Phan-Thien [6], ODonovan and Tanner [7], and Wilson [8] used instead of the original Bingham model the biviscosity fluid model both for analytical [8] and numerical [6],[7] studies. ODonovan and Tanner [7] solved the squeeze flow problem numerically and showed that unyielded material arises only adjacent to the center of the two disks.

The excellent work by Smyrnaios and Tsamopoulos [9] provided the locations of rigid zones either qualitatively or quantitatively. They employed both the original Bingham constitutive equation and the the Papanastasiou model and showed that unyielded material could only exist around the two stagnation points of flow extending in this way the work by ODonovan and Tanner [7]. Matsoukas and Mitsoulis [10], [11] also solved numerically the squeeze flow of viscoplastic materials, for both planar and axisymmetric flow, confirming the earlier results by Smyrnaios and Tsamopoulos.

Roussel at al.[12] studied squeeze flow of a Bingham fluid by means of a variational approach [13] They divided the zone of deformation into two kinematically admissible velocity fields,central region of pure extensional flow and sheared regions adjacent to the plates. Minimizing the energy dissipation functional with respect to a parameter defining the horizontal boundary of the two zones, they obtained lower-bound solutions for Bingham materials. The work of Adams et al. [14] provided experimental evidence that the yield surfaces predicted by the lubrication solution for viscoplastic fluids are erroneous. Besides, Meeten [15] carried out both theoretical and experimental investigations of the squeeze flow of Herschel Bulkley fluid.

It has been well established that the application of the lubrication solution incorporating the Bingham and HerschelBulkley relationships leads to a profound kinematic inconsistency in the calculated velocity fields.

Balmforth and Craster [16] and Frigaard and Ryan [17] suggested the asymptotic technique that resolves the lubrication paradox and builds the consistent solution for thin layer problems. In a recent paper by Muravleva [18] the planar squeeze flow of a Bingham fluid is studied exploiting the asymptotic technique introduced in [16],[17],[19]. Besides, the results of computations in [18] show presence of unyielded regions near the two stagnation points of flow close to centers of plates (that was found earlier by many researchers) and new additional unyielded regions at the outer edge of the material (both for short and long plates). More recently Fusi at all [20] investigated these unyielded regions, adjacent to the outer edge exploiting the integral formulation of the linear momentum balance. In many instances, the boundary condition on the wall is not perfectly no-slip as has been assumed above. It is frequently observed that some viscoplastic fluids slip along boundaries under some conditions which depend on the solid-fluid interaction [21], [22]. A number of slip conditions with variable degrees of complexity have been employed in the past for modeling squeeze flows of viscoplastic materials. For viscoplastic materials,the shear stress at the plates are usually assumed to be a constant fraction of the yield stress of the material and was consequently independent of position. Sherwood and Durban analyzed squeeze flow of a Bingham [23] and Herschel-Bulkley [24] materials under this slip wall boundary conditions by using an asymptotic expansion of the velocities in h/r, where r is a radial coordinate, and h the instantaneous distance between the plates. Adams et al. [14] presented the results of squeeze-flow experiments of a HerschelBulkley material between two rigid plates, investigated both experimentally and computationally. Stick and slip boundary conditions were simulated and, in the case of the later, a Coulombic boundary condition was adopted. Lawal and Kalyon [] considered Herschel-Bulkley materials with Navier slip boundary conditions following the lubrication approximation,and obtained numerical solutions. Karapetsas and Tsamopoulos [25] examined the transient squeeze flow of a viscoplastic material between two parallel coaxial disks. The Papanastasiou model was used. On the surface of the disk, the sip conditions are imposed. This slip model divides the wall boundary into a slip region and a no-slip region. To this end, the slip coefficient is an exponential function of the radial distance from the contact point and in this way it achieves a continuous transition between the slip and the no-slip region. Yang and Zhu [26] and later Ayadi [27] analyzed the squeeze flow of biviscous fluids with the Navier slip condition, based on lubrication analysis.

The experimental data show that slip often occurs only when a critical value of the shear-stress is reached [28].If the flow field is such that the wall shear-stress is smaller than the critical value, the fluid adherence to the boundary and we have no-slip condition. When the wall shear stress reaches the threshold value, the fluid slips along the boundary with a given friction coefficient. The boundary condition can be written as follows :

$$\begin{aligned} \tau_w = \tau_c + C_f u_w, &\quad \text{if } \tau_w > \tau_c, \\ u_w = 0, &\quad \text{if } \tau_w \leq \tau_c \end{aligned} \tag{1}$$

where u_w denote the tangential velocity, τ_w is the wall shear stress and C_f is the friction coefficient, τ_c is the critical value of the shear stress under which no slip will occur,known as the slip yield stress. If τ_c is zero, we have the usual Navier slip condition:

$$\tau_w = -C_f u_w \tag{2}$$

The no-slip and the perfect-slip cases are obtained for C_f tends to ∞ and $C_f = 0$, respectively.

The numerical implementation of slip boundary condition 1 Fortin et al. [29] applied the augmented Lagrangian method to impose a slip boundary condition on walls 1 for the round Poiseuille flow of a Bingham plastic. Huilgol [31] analyzed the variational principle of a yield-stress fluid together with a slip yield condition. Roquet and Saramito [30] investigated in detail the Poiseuille flow of a Bingham fluid in a square duct with slip yield boundary condition at the wall.They used the augmented Lagrangian algorithm. Damianou et al. [32] solved numerically the cessation of axisymmetric Poiseuille flow of a HerschelBulkley fluid under the assumption that slip occurs along the wall with slip yield stress. Damianou and Georgiou used regularized versions of both the constitutive and the slip equations along with finite elements in order to solve the steady-state flow of a Herschel–Bulkley fluid in a rectangular duct with wall slip and non-zero slip yield stress [33].

Here we shall apply the asymptotic technique introduced in [16], [17] to obtain a consistent thin-layer solution for the squeeze problem of viscoplastic material. Further we compare our analytical solution with the numerical one. The main difficulty in the numerical simulation of viscoplastic fluid flow is related to the non-differentiable form of constitutive law and inability to evaluate the stresses in regions where the material has not yielded. There are two principal approaches that have been proposed in the literature to overcome the mathematical problem of viscoplastic fluid flow. The first one, known as regularization method, consists in approximating the constitutive equation by a smoother one. Regularization replaces the rigid zones by very viscous fluid. The yield surface is computed a posteriori and the determination of the unyielded regions relies on the Von Mises stress criterion. The criterion based on the strain rate tensor cannot be used since in the unyielded regions is never truly zero, but only close to zero. The second approach is based on the theory of variational inequalities [34]. In the latter case, the problem reduces to the minimization of a functional and a further solution of the equivalent saddle-point problem which is solved iteratively (see for a detailed review [35]). Augmented Lagrangian method (ALM) introduced by Fortin and Glowinski [36]has become the most popular in this family of methods. The main advantage of these methods is that they involve the true constitutive relation and the algorithm gives truly unyielded regions with exactly zero strain rate. ALM is generally slower than codes with regularization methods. Regularization methods produce smoother solutions for the velocity field. This smoothness allows the application of more sophisticated flow solvers and leads to faster computations. Another important computation issue is the accurate tracking of the yield surfaces. Compared to the regularization method, the augmented Lagrangian method yields superior results regarding the location of the yield surface and explorations of the plastic limit, for example, in simulation of the flow cessation or no-flow limit. According to [37], pragmatically, the choice between augmented Lagrangian and regularization is related to whether one needs to determine the position of the yield surface, or whether a reasonable approximation to the velocity field is sufficient. Since the question of the existence and the position of the unyielded regions is of particular interest in the squeeze flow, we choose ALM for validation of our asymptotic solution.

2 PROBLEM STATEMENT

The goal of this study is to obtain a consistent solution for a squeeze flow of an incompressible Bingham viscoplastic fluid for axisymmetric geometry. and $2H$ and R are the gap and radius of the plates, respectively.

The flow is governed by the dimensionless conservation equations of momentum and mass:

$$\begin{aligned} Re\Big(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\Big) &= -\frac{\partial p}{\partial r} + \varepsilon^2\frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \varepsilon^2\frac{\tau_{rr}-\tau_{\theta\theta}}{r}, \\ \varepsilon^2 Re\Big(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\Big) &= -\frac{\partial p}{\partial z} + \varepsilon^2\frac{\partial \tau_{rz}}{\partial r} - \varepsilon^2\frac{\partial \tau_{rr}}{\partial z} - \varepsilon^2\frac{\partial \tau_{\theta\theta}}{\partial z} + \varepsilon^2\frac{\tau_{rz}}{r}, \\ \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} &= 0 \end{aligned} \tag{3}$$

The z-coordinate is aligned with the layer axis, p is the pressure, τ_{ij} is the deviatoric stress tensor. The components of velocity in the(r, θ, z) directions are $(u, 0, w)$. We have scaled lengths in the r and z directions differently, with the disk radius $\hat{R}$ as the horizontal length-scale and with the half the distance between the disks $\hat{H}$ as a characteristic thickness in the z-direction, respectively, $\hat{W}$ is taken as the charasteristic velocity in the transverse direction, and the velocity component in the r-direction is scaled with $\hat{U}$ ($\hat{U}\hat{H}/\hat{R} = \hat{W}$). The pressure is scaled with $\hat{\mu}\hat{U}\hat{L}/\hat{H}^2$, and time with $\hat{L}/\hat{U}$. The shear-stress components are scaled with $\hat{\mu}\hat{U}/\hat{H}$. The extensional stresses are scaled with $\hat{\mu}\hat{U}/\hat{R}$. We denote dimensional variable with a hat symbol. In the above equations, the small aspect ratio ε and Reynolds number Re are defined by

$$\varepsilon = \frac{\hat{H}}{\hat{R}}, \quad Re = \frac{\hat{\rho}\hat{U}\hat{H}}{\hat{\mu}} \tag{4}$$

where $\hat{\rho}$ and $\hat{\mu}$ are the density and plastic viscosity(both dimensional).

The constitutive laws for a Bingham fluid are

where $\dot{\boldsymbol{\gamma}}$ is strain rate tensor

$$\dot{\gamma}_{rr} = 2\frac{\partial u}{\partial r}, \quad \dot{\gamma}_{rz} = \Big(\frac{\partial u}{\partial z} + \varepsilon^2\frac{\partial w}{\partial r}\Big), \quad \dot{\gamma}_{zz} = 2\frac{\partial w}{\partial z}, \quad \dot{\gamma}_{\theta\theta} = 2\frac{u}{r} \tag{5}$$

and τ, $\dot{\gamma}$ are the second invariants of $\boldsymbol{\tau}$ and $\dot{\boldsymbol{\gamma}}$, i.e.

$$\text{with } \tau = \sqrt{\tau_{rz}^2 + \varepsilon^2(\tau_{rr}^2 + \tau_{\theta\theta}^2 + \tau_{rr}\tau_{\theta\theta})} \tag{6}$$

$$\text{with } |\dot{\gamma}| = \sqrt{(\frac{\partial u}{\partial z})^2 + 4\varepsilon^2\Big((\frac{\partial u}{\partial r})^2 + (\frac{u}{r})^2 + \frac{\partial u}{\partial r}\frac{u}{r}\Big) + 2\varepsilon^2\frac{\partial u}{\partial z}\frac{\partial w}{\partial r} + \varepsilon^4(\frac{\partial w}{\partial z})^4} \tag{7}$$

The dimensionless number appearing above is the Bingham number, B:

$$B = \frac{\hat{\tau}_0\hat{H}}{\hat{\mu}\hat{U}} \tag{8}$$

where $\hat{\tau}_0$ is yield stress. The Bingham number represents the ratio of yield stress to viscous stress.

We consider $Re \ll 1$. Neglecting the inertial terms, 3 are replaced by:

$$\begin{aligned} -\frac{\partial p}{\partial r} + \varepsilon^2\frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \varepsilon^2\frac{\tau_{rr} - \tau_{\theta\theta}}{r} &= 0, \\ -\frac{\partial p}{\partial z} + \varepsilon^2\frac{\partial \tau_{rz}}{\partial r} - \varepsilon^2\frac{\partial \tau_{rr}}{\partial z} - \varepsilon^2\frac{\partial \tau_{\theta\theta}}{\partial z} + \varepsilon^2\frac{\tau_{rz}}{r} &= 0, \\ \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} &= 0 \end{aligned} \tag{9}$$

The above equations are solved together with appropriate boundary conditions.

The studied domain now becomes a the upper-right quarter of the space occupied by the material and the following on the plane conditions are imposed on the plane of a symmetry Exploiting the double symmetry of the axisymmetric squeeze flow we consider the boundary condition for part of the material contained between the r and z axes and the upper plate. On the plane of symmetry $z = 0$ (central plane) :

$$\tau_{rz} = 0, \quad w = 0 \tag{10}$$

no slip condition at the contact with the upper plate ($z = 1$):

$$u = 0, \quad w = -1 \tag{11}$$

on the axis of symmetry $r = 0$:

$$u = 0, \quad \tau_{rz} = 0 \tag{12}$$

zero tangential and normal stress condition on the free surface $r = 1$:

$$\sigma_{rr} = -p + \varepsilon^2\tau_{rr} = 0, \quad \tau_{rz} = 0 \tag{13}$$

3 ASYMPTOTIC EXPANSIONS

We now solve the equations by introducing an asymptotic expansion. First we consider shear flow near the plate for which we may find a solution through a straightforward expansion of the equations. We consider regular expansions in ε of form:

$$\begin{aligned} u &= u^0 + \varepsilon u^1 + \varepsilon^2 u^2 \ldots, \\ w &= w^0 + \varepsilon w^1 + \varepsilon^2 w^2 \ldots, \\ p &= p^0 + \varepsilon p^1 + \varepsilon^2 p^2 \ldots, \\ \tau_{ij} &= \tau_{ij}^0 + \varepsilon \tau_{ij}^1 + \varepsilon^2 \tau_{ij}^2 \ldots \end{aligned} \tag{14}$$

We substitute these expansions into the governing equations 9, and collect together terms of the same order.

The lubrication equations for the first two orders are:

$$\mathcal{O}(1) \qquad \begin{aligned} & -\frac{\partial p^0}{\partial r} + \frac{\partial \tau_{rz}^0}{\partial z} = 0, \\ & -\frac{\partial p^0}{\partial z} = 0, \\ & \frac{\partial u^0}{\partial r} + \frac{u^0}{r} + \frac{\partial w^0}{\partial z} = 0 \end{aligned} \tag{15}$$

$$\mathcal{O}(\varepsilon) \qquad \begin{aligned} & -\frac{\partial p^1}{\partial r} + \frac{\partial \tau_{rz}^1}{\partial z} = 0, \\ & -\frac{\partial p^1}{\partial z} = 0, \\ & \frac{u^1}{r} + \frac{\partial u^1}{\partial r} + \frac{\partial w^1}{\partial z} = 0 \end{aligned} \tag{16}$$

3.1 Shear region

After the solution of the two first equations of 15 we have

$$p^0 = p_0(r), \quad \tau_{rz}^0 = z p_0'(r) \tag{17}$$

where p_0 is a function only of r and the prime $()'$ represents derivative $\frac{d}{dr}$. The leading order second invariants of strain rate and stress are given by

$$\tau^0 = |\tau_{rz}^0|, \quad \dot{\gamma}^0 = \left|\frac{\partial u^0}{\partial z}\right| \tag{18}$$

Provided the yield stress is exceeded, the stress tensor components become

$$\tau_{rz}^0 = \frac{\partial u^0}{\partial z} + Bsgn(\frac{\partial u^0}{\partial z}), \quad \tau_{rz}^1 = \frac{\partial u^1}{\partial z}, \tau_{rr}^0 = 2\left(\frac{\partial u^0}{\partial r} + \frac{B\frac{\partial u^0}{\partial r}}{\left|\frac{\partial u^0}{\partial z}\right|}\right) \tag{19}$$

Since we are looking for a solution with $\frac{\partial u^0}{\partial r} < 0$ in the domain $r > 0$, $s > 0$, we get

$$\tau_{rz}^0 = \frac{\partial u^0}{\partial z} - B \tag{20}$$

Integrating the last expression and using no-slip boundary conditions 11 we have

$$u^0 = \frac{p_0'(r)}{2}(z^2-1) + B(z-1) \tag{21}$$

We utilize the continuity equation and obtain

$$w^0 = -\frac{z^3-3z+2}{6}\Big(p_0''(r) + \frac{p_0'(r)}{r}\Big) - \frac{B}{2r}(z-1)^2 - 1 \tag{22}$$

We now focus on the first-order approximation. Integrating two first equations of 16 and 19, using $u^1 \mid_{z=1} = 0$, we receive:

$$p^1 = p_1(r), \quad \tau_{rz}^1 = z \cdot p_1'(r) + g(r), \quad u^1 = \frac{p_1'(r)}{2}(z^2-1) + g(r)(z-1) \tag{23}$$

where p_1 is a function only of r and $g(r)$ is an unknown function of integration.

3.2 Plastic region.

At leading order, at each $r \in [0,1]$ we have $\tau_{rz}^0 = zp_0'(r)$. The $|\tau_{rz}^0|$ exceeds its maximum value at point $z=1$ and vanish at point $z=0$. Hence, there exists the point $z=z_0$ at which $|\tau_{rz}^0| = B$, that means $\frac{\partial u^0}{\partial z} = 0$. But according to 18 $\tau^0 = B$ and $\dot{\gamma}^0 = 0$. Hence, at leading order, the yield condition holds at this point and $z_0(r) = \frac{B}{|p_0'(r)|}$ is the position of pseudo-yield surface. For $z \in [0,\, z_0]$ we have $\tau^0 < B$ and $\dot{\gamma}^0 = 0$.

The expression for the velocity can be written as follows:

$$u^0(r,z) = \begin{cases} \frac{B}{2z_0}(1-z_0)^2, & z \in [0,\, z_0], \\ \frac{B}{2z_0}\Big((1-z_0)^2 - (z-z_0)^2\Big), & z \in (z_0,\, 1] \end{cases} \tag{24}$$

We denote the pseudo-plug velocity by $u_0(r)$:

$$u_0(r) = u^0(z_0(r)) = \frac{B}{2z_0}(1-z_0)^2 \tag{25}$$

To determine pseudo-yield surface $z = z_0(r)$, we use the flow rate for the channel.

$$\int_0^1 u^0(r,z)\, dz = Q(r) \equiv \frac{r}{2} \tag{26}$$

Substituting 24 into the last equation leads to the cubic equation (rewritten the Buckingham equation), where the unknown is the pseudo-yield surface $z_0(r)$

$$z_0^3 - z_0\Big(3 + \frac{3r}{B}\Big) + 2 = 0 \tag{27}$$

This equation can be solved by the method proposed in [38]. There is single root of this equation for which $0 < z_0 \le 1$ and can be written as

$$z_0(r) = -2\sqrt{1+\frac{r}{B}}\cos\Big(\frac{1}{3}\arccos\frac{1}{\sqrt{(1+\frac{r}{B})^3}} - \frac{2\pi}{3}\Big) \tag{28}$$

It can be seen that this zeroth order solution has the characteristic Bingham-Poiseuille profile of velocity in sheared layer. But it's evident that the plug region isn't a true plug region as the leading order velocity varies in the r-direction. This is the essence of the lubrication paradox for yield stress fluids.

The source of the problem is revealed by the diagonal components of the stress. From the above expression for u_0 we can easily see that $\frac{\partial u^0}{\partial r}$ will not in general vanish while $\frac{\partial u^0}{\partial z}$ does. In the domain $|z| \le z_0$ we should consider the higher-order equations and pay attention to the diagonal stress components.

We assume that the domain occupied by the medium can be separated into two subdomains:

1. The external regions are situated near the plates where the yield criterion is overcome.There the shear stress is dominant. We name them "shear regions".

2. The inner region includes the centerplane, where yield criterion is not reached, the shear stress is smaller and in the centerplane equal to zero. The flow is close to extensional flow. We call this region "pseudoplug" or "plastic region".

These regions are separated by an interface represented by the smooth surface pseudo-plug $z = z_0(r)$.

Let us consider the domain near the centerplane of thickness $0 < z < z_0$. Below the fake yield surface, $z = z_0(r)$, the asymptotic expansion described above breaks down. To find a solution appropriate to this region we look for a slightly different set of asymptotic sequences. The principal difference is in the expansion of the horizontal velocity component:

$$
\begin{aligned}
u &= u^{p,0}(r) + \varepsilon u^{p,1}(r,z) + \varepsilon^{p,2} u^2(r,z) + \ldots, \\
w &= w^{p,0}(r,z) + \varepsilon w^{p,1}(r,z) + \varepsilon^2 w^{p,2}(r,z) + \ldots, \\
p &= p^{p,0}(r,z) + \varepsilon p^{p,1}(r,z) + \varepsilon^2 p^{p,2}(r,z) + \ldots
\end{aligned} \tag{29}
$$

where the property that $\frac{\partial u^{p,0}}{\partial z} = 0$ at z_0 is explicitly built in. We use the same symbol for the pseudo-plug solution. When we match the two solutions, we add a subscript p to explicitly distinguish the pseudo-plug solution.

Thus, by introducing an asymptotic expansion, we find

$$
\begin{aligned}
\tau_{rr}^{p,-1} &= \frac{2B}{\dot\gamma^{p,0}} \frac{\partial u^{p,0}}{\partial r}, \\
\tau_{rz}^{p,0} &= \frac{B}{\dot\gamma^{p,0}} \frac{\partial u^{p,1}}{\partial z}, \\
\tau_{\theta\theta}^{p,-1} &= \frac{2B}{\dot\gamma^{p,0}} \frac{u^{p,0}}{r}
\end{aligned} \tag{30}
$$

where the leading order term in the rate of strain tensor is

$$
\dot\gamma^{p,0} = \sqrt{\Big(\frac{\partial u^{p,1}}{\partial z}\Big)^2 + 4\Big(\Big(\frac{\partial u^{p,0}}{\partial r}\Big)^2 + \Big(\frac{u^{p,0}}{r}\Big)^2 + \frac{\partial u^{p,0}}{\partial r}\frac{u^{p,0}}{r}\Big)} \tag{31}
$$

The lubrication equations for the first two orders are:

$$
\begin{aligned}
\mathcal{O}(1) \qquad & -\frac{\partial p^{p,0}}{\partial r} + \frac{\partial \tau_{rz}^{p,0}}{\partial z} = 0, \\
& -\frac{\partial p^{p,0}}{\partial z} = 0,
\end{aligned} \tag{32}
$$

$$\mathcal{O}(\varepsilon) \qquad \begin{aligned} &\frac{\partial u^{p,0}}{\partial r} + \frac{u^{p,0}}{r} + \frac{\partial w^{p,0}}{\partial z} = 0 \\ &-\frac{\partial p^{p,1}}{\partial r} + \frac{\partial \tau_{rr}^{p,-1}}{\partial r} + \frac{\tau_{rr}^{p,-1} - \tau_{\theta\theta}^{p,-1}}{r} + \frac{\partial \tau_{rz}^{p,1}}{\partial z} = 0, \\ &-\frac{\partial p^{p,1}}{\partial z} - \frac{\partial(\tau_{rr}^{p,-1} + \tau_{\theta\theta}^{p,-1})}{\partial z} = 0, \\ &\frac{\partial u^{p,1}}{\partial r} + \frac{u^{p,1}}{r} + \frac{\partial w^{p,1}}{\partial z} = 0 \end{aligned} \tag{33}$$

Using symmetry about the centerplane, we have $w^p \mid_{z=0} = 0$, $\tau_{rz}^p \mid_{z=0} = 0$, $\frac{\partial u^p}{\partial z} \mid_{z=0} = 0$. Integrating the first two equations of 32 and enforcing continuity p^0, τ_{rz}^0, u^0 at $z = z_0(r)$ leads to

$$p^{p,0}(r) = p_0(r), \quad p_0' = -\frac{B}{z_0}, \quad \tau_{rz}^{p,0} = -\frac{Bz}{z_0}, \quad u^{p,0} = u_0(r) = \frac{B}{2z_0}(1 - z_0)^2 \tag{34}$$

From the last equation of 32 we obtain

$$w^{p,0} = -z\Big(u_0' + \frac{u_0}{r}\Big) \tag{35}$$

Differentiation of the expressions 27,34 after minor manipulations gives

$$z_0' = \frac{3z_0^2}{2B(z_0^3 - 1)} \quad u_0' = \frac{3(1 + z_0)}{4(1 + z_0 + z_0^2)} \tag{36}$$

The continuity condition of $w^{p,0}$ and $w^{s,0}$22 at pseudo yield surface is automatically satisfied by substituting the expression for z_0', u_0', r 27,36.

From 30,34 we obtain

$$\begin{aligned} (\tau^{p,-1})^2 = (\tau_{rr}^{p,-1})^2 + (\tau_{\theta\theta}^{p,-1})^2 + \tau_{rr}^{p,-1}\tau_{\theta\theta}^{p,-1} + (\tau_{rz}^{p,0})^2 = \\ \frac{4B^2}{(\dot\gamma^{p,0})^2}\Big((u_0')^2 + (\frac{u_0}{r})^2 + u_0'(r)\frac{u_0}{r}\Big) + \frac{B^2}{(\dot\gamma^{p,0})^2}(\frac{\partial u^{p,1}}{\partial z})^2 = B^2 \end{aligned} \tag{37}$$

We denote

$$\eta = \sqrt{4\Big((u_0')^2 + (\frac{u_0}{r})^2 + u_0'\frac{u^0}{r}\Big)} \tag{38}$$

Substituting $\tau_{rz}^{p,0} = -\frac{Bz}{z_0}$ 30,34 into 37 leads to the following expression

$$\frac{B^2\eta^2}{(\dot\gamma^{p,0})^2} = \frac{B^2(z_0^2 - z^2)}{z_0^2} \tag{39}$$

After minor calculation we have

$$\dot\gamma^{p,0} = \frac{\eta z_0}{\sqrt{z_0^2 - z^2}} \tag{40}$$

Substituting the expression 38 into 31 gives

$$(\frac{\partial u^{p,1}}{\partial z})^2 + \eta^2 = (\dot\gamma^{p,0})^2 \tag{41}$$

Inserting 40 into 41 we obtain the equation for $\frac{\partial u^{p,1}}{\partial z}$ Solving this equation and integrating we obtain

$$u^{p,1} = \eta\sqrt{z_0^2 - z^2} + u_1^* \tag{42}$$

where $u_1^*(r)$ is an unknown function of integration. Obviously the function $u_1^*(r)$ is the value of the first order velocity u^1 at the pseudo-yield surface $z = z_0(r)$. We now focus on the first-order approximation. Integrating the equation **??** gives

$$p^{p,1} = -\tau_{rr}^{p,-1} - \tau_{\theta\theta}^{p,-1} + \psi(r) \tag{43}$$

where $\psi(r)$ is an unknown function of integration. Enforcing continuity of the pressure at the pseudo-yield surfaces yields $\psi(r) = p_1(r)$. Substituting 43 into the first equation of 33 gives

$$\frac{\partial}{\partial r}(2\tau_{rr}^{p,-1} + \tau_{\theta\theta}^{p,-1}) - p_1' + \frac{\tau_{rr}^{p,-1} - \tau_{\theta\theta}^{p,-1}}{r} + \frac{\partial \tau_{rz}^{p,1}}{\partial z} = 0 \tag{44}$$

Inserting 34,40 into 30 we find that

$$\tau_{rr}^{p,-1} = \frac{2B}{\eta z_0} u_0' \sqrt{z_0^2 - z^2} \quad , \tau_{\theta\theta}^{p,-1} = \frac{2B}{\eta z_0}\frac{u_0}{r}\sqrt{z_0^2 - z^2} \tag{45}$$

Substituting $\tau_{rr}^{p,-1}$ and $\tau_{\theta\theta}^{p,-1}$ into 44 leads to

$$\begin{aligned} 2B\frac{d}{dr}\Big(\frac{2u_0' + \frac{u_0}{r}}{\eta z_0}\Big)\sqrt{z_0^2 - z^2} + 2B\Big(\frac{2u_0' + \frac{u_0}{r}}{\eta z_0}\Big)\frac{z_0' z_0}{\sqrt{z_0^2 - z^2}} - \\ -p_1' + \frac{2B(u_0' - \frac{u_0}{r})}{r\eta z_0}\sqrt{z_0^2 - z^2} + \frac{\partial \tau_{rz}^{p,1}}{\partial z} = 0 \end{aligned} \tag{46}$$

After integrating (in respect that $\tau_{rz}^{p,1}\mid_{z=0} = 0$)

$$\begin{aligned} \tau_{rz}^{p,1} = -B(z\sqrt{z_0^2 - z^2} + z_0^2 \arcsin\frac{z}{z_0})\Big(\frac{d}{dr}\Big(\frac{2u_0' + \frac{u_0}{r}}{\eta z_0}\Big) + \frac{(u_0' - \frac{u_0}{r})}{r\eta z_0}\Big) - \\ -2B\Big(\frac{2u_0' + \frac{u_0}{r}}{\eta z_0}\Big) z_0' z_0 \arcsin\frac{z}{z_0} + p_1' z \end{aligned} \tag{47}$$

Enforcing continuity of the first order shear stress τ_{rz} at the pseudo-yield surface leads to

$$g(r) = -\frac{\pi B}{2}\Big(\frac{d}{dr}\Big(\frac{z_0}{\eta}(2u_0' + \frac{u_0}{r})\Big) + \frac{z_0}{\eta r}(u_0' - \frac{u_0}{r})\Big) \tag{48}$$

The unknown $u_1^*(r)$ can be determined by matching the first order velocities at $z = z_0$

$$\begin{aligned} u_1^* &= \frac{p_1'}{2}(z_0^2 - 1) + g(z_0 - 1), \\ u^s(r,z) &= -\frac{B}{2z_0}(z^2 - 1) + B(z - 1) + \varepsilon(\frac{p_1'}{2}(z^2 - 1) + g(z - 1)), \\ u^p(r,z) &= \frac{B}{2z_0}(1 - z_0^2) + \varepsilon(\eta\sqrt{z_0^2 - z^2} + \frac{p_1'}{2}(z_0^2 - 1) + g(z_0 - 1)) \end{aligned} \tag{49}$$

However, function p_1 is still unknown. To find p_1, we consider the flowrate constraint:

$$\int_0^1 u(r,\,z)\,dz = \int_0^{z_0} u^p\,dz + \int_{z_0}^1 u^s\,dz =$$
$$\frac{B(z_0-1)^2(z_0+2)}{6z_0} + \varepsilon\Big(\frac{z_0^2\pi\eta}{4} + \frac{p_1'}{3}(-1+z_0^3) + \frac{g}{2}(-1+z_0^2)\Big) = \frac{r}{2} \tag{50}$$

$Q(r) = Q_0(r) + \varepsilon Q_1(r)$. We require that $Q = Q_0$, $Q_1 = 0$. Thus we obtain the equation for the border z_0:

$$z_0^3 - z_0\Big(3 + \frac{3r}{B}\Big) + 2 = 0 \tag{51}$$

which is equivalent to 27 and expression for p_1'

$$p_1' = \frac{3z_0^2\eta\pi}{4(1-z_0^3)} - \frac{3g(1+z_0)}{2(1+z_0+z_0^2)} = -\frac{\pi B\eta z_0'}{2} - 2gu_0' \tag{52}$$

As we have mentioned above the numerical modeling [10] shows the presence of the rigid zones near the stagnation points at the center of the plate. It is interesting to investigate the obtained asymptotic solution near the stagnation point. We will examine the second invariant of the stress τ at the plate at $z = 1$.

The leading order τ^0 in the shear region equals the shear stress $\tau^0(r,1) = |\tau^0_{rz}(r,1)| = \frac{B}{z_0(x)}$ and exceed the yield stress because $z_0(r) \le 1$. Thus, in accordance with the zero-order solution, the Bingham fluid in the shear region is yielded.

Then we take into account the first order approximation and consider the stress along the $z = 1$

$$\tau(r,1) = \tau^0(r,1) + \varepsilon\tau^1(r,1) + O(\varepsilon^2) = |\tau^0_{rz}(r,1) + \varepsilon\tau^1_{rz}(r,1) + O(\varepsilon^2)| = \frac{B}{z_0(r)} - \varepsilon(p_1'(r) + g(r)) + O(\varepsilon^2).$$

where $p_1'(r)$ and $g(r)$ are determined by 52, 48 To analyze the behavior of the function $\tau^0_{rz}(r,1)$ near the axis of symmetry, we use the Taylor series. We introduce the new variable $\tilde{\zeta} = \left(\frac{r}{B}\right)^{1/2}$ and expand the function $z_0(r(\tilde{\zeta}))$, using 28 in Taylor series. After that, we return to the variable r and obtain

$$z_0(r) = 1 - \sqrt{\frac{r}{B}} + \frac{r}{3B} + \frac{r}{18B}\sqrt{\frac{r}{B}} - \frac{r^2}{27B^2} - \frac{5r^2}{216B^2}\sqrt{\frac{r}{B}} + \frac{r^3}{243B^3} \tag{54}$$

$$\tau^0_{rz} = -B(1 + \sqrt{\frac{r}{B}} + \frac{2r}{3B} + \frac{5r}{18B}\sqrt{\frac{r}{B}} + \frac{r^2}{27B^2} - \frac{11r^2}{216B^2}\sqrt{\frac{r}{B}} - \frac{13r^3}{243B^3} + O(r)^{7/2}) \tag{55}$$

$$\tau^1_{rz} = \frac{\pi}{\sqrt{3}}(\frac{3\sqrt{B}}{4\sqrt{r}} - \frac{67}{192} - \frac{581\sqrt{r}}{2304\sqrt{B}} + O(r)) \tag{56}$$

The first term in the brackets on the right-hand side of 56 is positive and tends to infinity when $r \to 0$ and the other terms are finite for $0 \le r \le 1$. For $r \to 0$ we have $-\tau_{rz}(r,1) = \frac{B}{z_0(r)} - \varepsilon(p_1'(r) + g(r)) \to -\infty$. This means that near $r = 0$ there exists a point $r = r_0$ at which $-\tau_{rz} = B$ and $\tau = B$. For $r \le r_0$ we have $\tau < B$ and the material is unyielded.

Figure 1:

So, asymptotic analysis predicts a region of unyielded material near the plane. Consequently our asymptotic expansion breaks down. We give in Fig. 1(a) the magnitude of the tangential stress $-\tau_{rz}(r,1)$, acting on the disc surface for $B=1$, $\varepsilon=0.1$. The solid line corresponds to $\tau_{rz}(r,1)=\tau^0_{rz}(r,1)+\varepsilon\tau^1_{rz}(r,1)$, dotted line — tangential stress zero-order $\tau^0(r,1)$, dashed line — $z=B$. The solid line is below the dotted line and approaches it for $r\to 1$. The abscissa of the point of intersection of the solid line and a dotted horizontal line is the radius yield surface on the disk ($\tau=B$). Fig. 1(b) shows the stress intensity along the disk surface for $\varepsilon=0.1$ and four Bingham numbers, 0.1, 1,10,100. The solid line corresponds to $\tau(r,1)=\tau^0(r,1)+\varepsilon\tau^1(r,1)$, dotted line — $\tau^0(r,1)$. The uppermost curve in this figure is for the highest Bn value, they decrease gradually as Bn decreases . The stresses are increase monotonically with increasing r and decrease sharply near the axis of symmetry $r=0$. The stresses equal to the corresponding Bingham numbers are marked points on the solid lines. The yield surfaces (i.e. the surfaces on which $\tau=B$) are marked points on the solid lines. As expected, the size of the unyielded spot increases with B.(This figure qualitatively follow fig14a presented in)This figure is in qualitative agreement with fig.14a presented in [9].

4 Slip condition

We consider another boundary condition at the contact with the upper plate ($z=1$):

$$\hat{u}= \begin{array}{ll} -\Big(1-\frac{\hat{\tau}_c}{|\hat{\tau}_{rz}|}\Big)\frac{\hat{\tau}_{rz}}{\hat{c}_f}, & |\hat{\tau}_{rz}|>\hat{\tau}_c, \\ 0, & otherwise \end{array} \quad (57)$$

here $\hat{s}_0$ is the slip yield stress, $\hat{c}_f$ is . The slip yield dimensionless number B_w is defined as the ratio of $\hat{s}_0$ to a characteristic stress $\hat{\mu}\hat{U}/\hat{R}$: $B_w=\hat{s}_0\hat{R}/\hat{\mu}\hat{U}$, the friction dimensionless number C_f is defined by $C_f=\hat{c}_f\hat{R}/\hat{\mu}$. We introduce $\beta=1/C_f$. The three dimensionless numbers B, B_w and β characterize the squeeze problem with the stick-slip law at the wall. We can rewrite in dimensionless form:

$$u= \begin{array}{ll} -\beta\tau_{rz}\Big(1-\frac{B_w}{|\tau_{rz}|}\Big), & |\tau_{rz}|>B_w, \\ 0, & otherwise \end{array} \quad (58)$$

This slip model divides the wall boundary into the slip region and no-slip region. The stick and the slip regions are separated by the transition point r_T. β is a parameter used to adjust the level of the slip velocity in comparison to the wall shear and a B_w is a second parameter used to adjust the length of the slip region. The choice $\beta=0$ leads naturally to a no-slip boundary condition, whereas when B_w becomes zero this model reduces to the standard Navier slip model and slip occurs over the entire wall boundary with a constant slip coefficient.

In this section we will assume, that the condition $|\tau_{rz}|>B_w$ is satisfied, therefore the velocity boundary condition becomes the following:

$$u\,|_{z=1}=-\beta(\tau_{rz}-B_w\frac{\tau_{rz}}{|\tau_{rz}|}), \quad w\,|_{z=1}=-1. \quad (59)$$

4.1 Shear region

Following section 2 we look for a solution in which the main variables of the problem can be expressed as power series of ε, namely

$$u=u^0+\varepsilon u^1+\varepsilon^2u^2\ldots,$$

$$
\begin{aligned}
w &= w^0 + \varepsilon w^1 + \varepsilon^2 w^2 \ldots, \\
p &= p^0 + \varepsilon p^1 + \varepsilon^2 p^2 \ldots, \\
\tau_{ij} &= \tau_{ij}^0 + \varepsilon \tau_{ij}^1 + \varepsilon^2 \tau_{ij}^2 \ldots
\end{aligned}
\tag{60}
$$

We start from the zero-order approximation. In the shear region the equations 15 remain valid and we have

$$p^0 = p_0(r), \quad \tau_{rz}^0 = z p_0'(r). \tag{61}$$

Instead of the boundary conditions 11, we have

$$u^0 = \beta(-p_0' - B_w), \quad w = -1. \tag{62}$$

From 18,20,62 we find

$$u^0 = \frac{p_0'}{2}(z^2 - 1) + B(z - 1) + \beta(-p_0' - B_w) \tag{63}$$

We utilize the continuity equation from15 (last one), the second boundary condition 62 and obtain

$$w^0 = -\frac{z^3 - 3z + 2}{6}\Big(p_0'' + \frac{p_0'}{r}\Big) - \frac{B}{2r}(z-1)^2 + \beta\Big(p_0'' + \frac{p_0' + B_w}{r}\Big)(z - 1) - 1. \tag{64}$$

Figure 2:

Just as in Section 2, we will consider two subdomains: the shear regions (near the plates), where the yield criterion at leading order is overcome ($\tau^0 = |\tau_{rz}^0| = |z p_0'(r)| > B$), and the plastic region (surrounding the centerplane), where the yield criterion is not reached. These regions are separated by pseudo-yield surface $z_0(r) = \frac{B}{|p_0'(r)|}$. The expression for the velocity $u^0(r, z)$ can be written as follows:

$$u^0(r,z) = \begin{array}{rl} \frac{B}{2z_0}(1 - z_0)^2 + \beta(\frac{B}{z_0} - B_w), & z \in [0,\, z_0], \\ -\frac{B}{2z_0}(z^2 - 1) + B(z - 1) + \beta(\frac{B}{z_0} - B_w), & z \in (z_0,\, 1]. \end{array} \tag{65}$$

To determine pseudo-yield surface $z = z_0(r)$, we use the flow rate for the channel

$$\int_0^1 u^0\, dz = \frac{B(z_0 - 1)^2(z_0 + 2)}{6z_0} + \beta(\frac{B}{z_0} - B_w) = \frac{B(z_0^3 - 3z_0 + 2) + 6\beta(B - B_w z_0)}{6z_0} = \frac{r}{2}. \tag{66}$$

The rearrangement of 66 gives the cubic equation where the unknown is the pseudo-yield surface z_0:

$$z_0^3 - 3z_0\Big(1 + \frac{r}{B} + 2\beta\frac{B_w}{B}\Big) + 2 + 6\beta = 0. \tag{67}$$

We denote

$$f_0(z_0) := z_0^3 - 3z_0(1 + \frac{r}{B} + 2\beta\frac{B_w}{B}) + 2 + 6\beta. \tag{68}$$

As $f_0(0) = 2+6\beta > 0$, $f_0(1) = -3\frac{r}{B} - 6\beta(\frac{B_w}{B} - 1) < 0$, then for $r > 0$, $z_1 < 0 < z_2 < 1 < z_3$. Thus for all $r > 0$, $\beta > 0$,$B_w > B$ on the interval $(0,\ 1)$ equation 67 has the only root:

$$z_0(r) = -2\sqrt{1 + \frac{r}{B} + 2\beta\frac{B_w}{B}}\cos\Big(\frac{1}{3}\arccos\Big(\frac{1+3\beta}{\sqrt{(1+\frac{r}{B}+2\beta(\frac{B_w}{B}))^3}} - \frac{2\pi}{3}\Big)\Big). \tag{69}$$

Now, differentiating z_0 and u_0 with respect to r, we obtain

$$z_0' = -\frac{3z_0^2}{2B(1+3\beta - z_0^3)}, \quad u_0' = \frac{3(1+2\beta - z_0^2)}{4(1+3\beta - z_0^3)}. \tag{70}$$

Exploiting the continuity equation we find w^0 in the plastic region. Finally, we obtain

$$w^0(r,z) = -z(u_0' + \tfrac{u_0}{r}), \quad z \in [0,\ z_0], \tag{71}$$

$$(z^3 - 3z + 2 - 6\beta(z-1))\Big(-\tfrac{Bz_0'}{6z_0^2} + \tfrac{B}{6rz_0}\Big) - \tfrac{B}{2r}(z^2-1) + \beta\tfrac{B_w}{r}(z-1) - 1, \quad z \in (z_0, 1] \tag{72}$$

Using formulae 66, 70, it is easy to see that the axial velocity w^0 is continuous on the pseudo-yield surface $z_0(r)$.

The boundary conditions 62 hold for $\frac{B}{z_0} > B_w$, otherwise no-slip boundary conditions hold. The condition $|\tau^0_{rz}(r_*, 1)| = B_w$ defines $r = r_*$ as the "stick-slip" transition point of zero-order . Obviously, $z_0^* = z_0(r_*) = B/B_w$, thus, in accordance with 66,

$$r_* = \frac{1}{3}(\frac{B^3}{B_w^2} - 3B + 2B_w). \tag{73}$$

We denote $\tilde{z_0}(r)$ pseudo-yield surface, corresponding to the equation 67, notation $z_0(r)$ remains for the solution of the equation 27. We observe 36, 70 that $z_0' < 0$, $\tilde{z_0'} < 0$, so the pseudo-yield surfaces $z = z_0(r)$, $z = \tilde{z_0}(r)$ are the decreasing functions of radius. If $z_0(r_*) = B/B_w < z_0(1)$, then the curves $z_0(r)$, $\tilde{z_0}(r)$ intersect at $r_* > 1$ (horizontal line $z = B/B_w$ is below $z_0(r)$ at $0 \le r \le 1$). Thus, the material adheres to the entire surface of the disks. It is evident that $z_0(r_*) = \tilde{z_0}(r_*)$. In order to determine the relative position of the graphs $z_0(r)$ and $\tilde{z_0}(r)$, we subtract the expression 27 from 67 and obtain:

$$(z_0 - \tilde{z_0})(3 + \frac{3r}{B} - (z_0^2 + z_0\tilde{z_0} + \tilde{z_0}^2)) = \frac{6\beta B_w}{B}(\tilde{z_0} - \frac{B}{B_w}). \tag{74}$$

If $r < r_*$, then $\tilde{z_0} - \frac{B}{B_w} > 0$ and $z_0 > \tilde{z_0}$ and vice versa (the second factor on the left side is positive, since $z_0, \tilde{z_0} \in (0,\ 1]$) .

Fig. 2 (a) shows the graphs of the pseudo-yield surfaces $z = z_0(r)$, $z = \tilde{z_0}(r)$ for three values of the Bingham number $B = 0.1, 1, 100$. The black lines correspond to the pseudo-yield surfaces $z = z_0(r)$ with no-slip condition. For the slip condition pseudo-yield surfaces $z = \tilde{z_0}(r)$ are painted by color lines: red lines correspond to $B_w/B = 1.2$, blue line - $B_w/B = 1.5$, turquoise line - $B_w/B = 2$. We can see that the size of the stick region enlarges monotonically as the ratio B_w/B increases (B is fixed); consequently, the smaller is the B_w/B, the higher is the corresponding curve $z = \tilde{z_0}(r)$. We observe that for $B = 10$ slip occurs only for $B_w = 12$. For $B_w = 15$ and $B_w = 20$ no-slip condition is performed on the entire surface of the disks. For $B = 100$ and for all three $B_w = 120, 150, 200$ the no-slip condition is valid on the whole surface of the disks.

On Fig. 2(b) we have plotted $z = z_0(r)$, $z = \tilde{z}_0(r)$ for $B = 1$, $B_w = 2$ and various slip coefficients β. As mentioned above, this parameter adjust the level of the local slip velocity in comparison to the wall shear. More specifically, when the value of β increases the value of slip velocity grows. It can be easily seen that for the small values of β the curve $\tilde{z}_0(r)$ is close to the curve $z_0(r)$. When β becomes bigger the graph of the function $\tilde{z}_0(r)$ is close to a horizontal line in accordance with 70.

Let us return to Fig. 1 (a) and remember that the absolute value of shear stress $|\tau_{rz}|$ is smaller than $|\tau^0_{rz}|$, so $|\tau^0_{rz}(r_*)| = B_w$, but $|\tau_{rz}(r_*)| < B_w$. Accounting for the first approximation for the shear stress leads to decreasing of the absolute magnitude of the shear stress, as can be clearly seen from Fig. 1. Therefore, at the point r_* shear stress is above the critical B_w and actually slippage occurs at point r_T ($r_T > r_*$). Thus the fulfillment of the slip condition for zero approximation does not guarantee the fulfillment of this condition if we take into consideration the first-order approximation. There is a transition region (r_*, r_T) where $|\tau^0_{rz}| > B_w$ and $|\tau^0_{rz} + \varepsilon\tau^1_{rz}| < B_w$. We call this case fake slip and consider later.

In this subsection we deal with the case $|\tau_{rz}| > B_w$. If the condition $|\tau^0_{rz} + \varepsilon\tau^1_{rz}| > B_w$ is fulfilled then the slip occurs and we have the following boundary condition for the first approximation:

$$u^1 \mid_{z=1} = -\beta\tau^1_{rz}. \tag{75}$$

Equation 75 can be written in the general form

$$u^1 \mid_{z=1} = u^1_b, \tag{76}$$

where

$$u^1_b = -\beta\tau^1_{rz}. \tag{77}$$

The equations 16 remain valid but instead of no-slip conditions 11 we consider 76. We solve these equations as in the subsection 2.1 and have the following expressions:

$$p^1 = p_1(r), \quad \tau^1_{rz} = z \cdot p_1'(r) + g(r), \quad u^1 = \frac{p_1'(r)}{2}(z^2 - 1) + g(r)(z-1) + u^1_b. \tag{78}$$

4.2 Plastic region.

In the plastic region we introduce the asymptotic expansion 29. The expressions 30–31 hold. For the plastic region the solution repeats the solution for the plastic region with no-slip condition. Instead of expressions 34, we obtain the following items:

$$p^0(r) = p_0(r), \quad p_0' = -\frac{B}{z_0}, \quad \tau^0_{rz} = -\frac{Bz}{z_0}, \quad u^0 = u_0(r) = \frac{B}{2z_0}(1-z_0)^2 + \beta(\frac{B}{z_0} - B_w). \tag{79}$$

The equations 37–47 are valid, but for the functions z_0, z_0', u_0, u_0' we use the expressions 69,70,79 from the previous subsection. Given the continuity of pressure and stress on the pseudo-yield surface we obtain:

$$\begin{aligned} p^s(r) =& \quad p_0(r) + \varepsilon p_1(r) + \dots, \quad p^p(r) = p_0(r) + \varepsilon(p_1(r) - \frac{2B}{\eta z_0}(u_0' + \frac{u_0}{r})\sqrt{z_0^2 - z^2}) + \dots, \\ \tau^s_{rz}(r) =& \quad -\frac{Bz}{z_0} + \varepsilon(zp_1'(r) + g(r)) + \dots, \\ \tau^p_{rz}(r,z) =& \quad -\frac{Bz}{z_0} + \varepsilon\Big(p_1'z + \frac{2g}{\pi}\arcsin\frac{z}{z_0} - Bz\sqrt{z_0^2 - z^2}\Big(\frac{d}{dr}\Big(\frac{2u_0' + \frac{u_0}{r}}{\eta z_0}\Big) + \frac{u_0' - \frac{u_0}{r}}{r\eta z_0}\Big)\Big) + \dots, \end{aligned} \tag{80}$$

where

$$g(r) = -\frac{\pi B}{2}\Big(\frac{d}{dr}\Big(\frac{z_0}{\eta}(2u_0' + \frac{u_0}{r})\Big) + \frac{z_0}{\eta r}(u_0' - \frac{u_0}{r})\Big). \tag{81}$$

From continuity of u^1 at $z = z_0$ it follows that

$$u^{1s} = \frac{p_1'}{2}(z^2-1)+g(z-1)+u_b^1, \quad u^{1p} = \eta\sqrt{z_0^2 - z^2}+u_1^* \Rightarrow u_1^* = \frac{p_1'}{2}(z_0^2-1)+g(z_0-1)+u_b^1.$$

Substituting 78 into expression 77 gives

$$u_b^1 = -\beta(p_1'(r) + g(r)). \tag{82}$$

We can express the velocity in terms of $z_0(r)$:

$$\begin{aligned} u^s(r,z) &= -\tfrac{B}{2z_0}(z^2-1) + B(z-1) + \beta(\tfrac{B}{z_0} - B_w) + \varepsilon(\tfrac{p_1'}{2}(z^2-1) + g(z-1) + u_b^1), \\ u^p(r,z) &= \tfrac{B}{2z_0}(1-z_0^2) + \beta(\tfrac{B}{z_0} - B_w) + \varepsilon(\eta\sqrt{z_0^2 - z^2} + \tfrac{p_1'}{2}(z_0^2-1) + g(z_0-1) + u_b^1). \end{aligned}$$

To find p_1 we consider the flow rate through the gap:

$$\int_0^1 u(r,\, z)\, dz = \int_0^{z_0} u^p\, dz + \int_{z_0}^1 u^s\, dz =$$

$$\frac{B(z_0^2 - 3z_0 + 2) + 6\beta(B - B_w z_0)}{6z_0} + \varepsilon\Big(\frac{z_0^2\pi\eta}{4} + \frac{p_1'}{3}(-1+z_0^3) + \frac{g}{2}(-1+z_0^2) + u_b^1\Big) = \frac{r}{2}. \tag{83}$$

Just as in the subsection 2.2 we obtain expression for p_1':

$$p_1' = \frac{3z_0^2\pi\eta}{4(1-z_0^3)} - \frac{3g(1-z_0^2)}{2(1-z_0^3)} + \frac{3u_b^1}{(1-z_0^3)}. \tag{84}$$

Using 82 we have:

$$p_1' = \frac{3z_0^2\eta\pi}{4(1+3\beta-z_0^3)} - \frac{3g(1+2\beta-z_0^2)}{2(1+3\beta-z_0^3)} = -\frac{\pi B\eta z_0'}{2} - 2gu_0'. \tag{85}$$

The coordinate of the transition point r_T can be found from the fact that at the transition point the shear stress is equal to the critical value $-B_w$: $\tau_{rz}^0(r_T) + \varepsilon\tau_{rz}^1(r_T) = -B_w$. We find the root of the next equation $z_0(r)$ numerically:

$$\frac{B}{z_0} + \varepsilon(\frac{\pi B\eta(z_0)z_0'}{2} + g(z_0)(2u_0'(z_0) - 1)) = B_w, \tag{86}$$

and then substitute it into the formula 66 and define r_T.

4.3 Fake slip

Let us return to the fake slip case. There is an interval (r_*, r_T), for which at the boundary $z = 1$ no-slip condition is satisfied, i.e., velocity at the boundary should be zero. Since the zero-order approximation of the velocity at the boundary is positive, this value should be compensated by the boundary condition for the first approximation. Therefore:

$$u_b^1 = -\frac{\beta}{\varepsilon}(\frac{B}{z_0} - B_w) \tag{87}$$

Figure 3:

In both regions, fake slip and slip, the solutions of the first-order are described by the same formulae except the expression for the function u_b^1 in the right-hand side of the boundary condition. The expression 84 after the substitution of 87 takes the following form:

$$p_1' = \frac{z_0^2 \eta \pi 3}{4(1 - z_0^3)} - \frac{3g(1 + z_0)}{2(1 + z_0 + z_0^2)} - \frac{3\beta}{\varepsilon(1 - z_0^3)}(\frac{B}{z_0} - B_w). \tag{88}$$

Figure 3 shows the graphs of the shear stress for Bingham number $B = 1$ and different wall Bingham numbers $B_1 = 1.5, 2, 2.5$. The purple line corresponds to τ_{rz}^0 - zero approximation with no-slip condition, the black line corresponds to $\tau_{rz} = \tau_{rz}^0 + \varepsilon \tau_{rz}^1$ with no-slip condition. The solutions at the fake slip and the slip regions are depicted in color: red line corresponds to $B_w = 1.5$, blue line - $B_w = 2$, green line - $B_w = 2.5$. The asterisks on r-axis mean transition from the no-slip region to the fake slip region. In other words, they are the abscissas $r = r_*$ of the intersection points of the purple line and the dotted lines $z = B_w$ ($|\tau_{rz}^0(r_*)| = B_w$). The color of the asterisk corresponds to the value of B_w. The circles on r-axis mean transition from the fake slip region to the slip region, namely, they are the abscissas $r = r_T$ of the intersection points of the dotted line $z = B_w$ and the corresponding color line. Thus, each graph is composed from three parts: for $r \leq r_*$ it is solution of the no-slip problem, for $r_* < r < r_T$ it is solution of the fake slip problem, for $r \geq r_T$ it is solution of the slip problem. One can easily see that rising of B_w entails shortening of the fake slip region. The slip condition reduces the absolute value of the shear stress compared with the stick condition. The smaller is ratio B_w/B (for fixed B), the sooner τ_{rz} exceeds the value B_w and the closer to the axis of symmetry is the fake slip region. We note that for case when the fake slip region is located near the axis of symmetry (for example, $B_w = 1.5$), the absolute value of the shear stress in this region (red line) is slightly less in comparison with solution with no-slip condition (black line). If the fake slip region is located in the middle ($B_w = 2$) or closer to the edge of the disk ($B_w = 2.5$), the shear stress in the fake slip region coincide with solution with no-slip condition. In the slip region shear stresses are nearly the linear functions and visually the slopes are almost identical.

In Figure 3(b) we examine the influence of the parameter β on the value of the shear stress with fixed $B = 1$. We choose the color of the lines as well as in the previous figure, while the solid line corresponds to $\beta = 1$, dashed line corresponds $\beta = 0.5$, dotted line $-\beta = 0.1$. The fluid velocity at the disks surfaces has positive value, which increases as β increases. Since the mass flow rate is kept constant, as we vary the slip coefficient β, the increase in the fluid velocity in the disk surfaces region will result in decreased velocity gradients there, and in the center region the fluid will be correspondingly slowed down as β is increased, otherwise the mass conservation requirement will be violated.A reduction in velocity gradient will necessarily lead to a decrease in shear stress, as indicated in 3(b) As expected, for the small values of β ($\beta = 0.1$) the graph is close to the line, corresponding to the no-slip solution.

The fluid velocity at the disks surfaces has positive value, which increases as β increases. Since the mass flow rate is kept constant, as we vary the slip coefficient β, the increase in the fluid velocity in the disk surfaces region will result in decreased velocity gradients there, and in the center region the fluid will be correspondingly slowed down as β is increased, otherwise the mass conservation requirement will be violated. A reduction in velocity gradient will necessarily lead to a decrease in shear stress , as indicated in Fig. 6b, the pressure gradient sharply decreases with increased p at any radial coordinate.

We see that the location of the transition point r_T is independent of β when $B_w = 2$ and $B_w = 2.5$. For $B_w = 1.5$ magnitude of the shear stress in the fake region increases with decreasing of β, so the critical value B_w is reached at smaller radius, therefore, the transition points r_T are moved to the left. But the difference between them is negligibly small and these points are almost indistinguishable.

Figure 4: Velocity profiles .

Figure 4 shows the radial velocity profiles: red lines correspond to the no-slip solution and black lines correspond to the slip solution, solid lines are used for composite solution, dotted lines - for zero-order solution. Fig. 4 (a) corresponds to $B = 10$, $B_w = 13$, $\varepsilon = 0.1$, radial location is $r = 0.9$. This location belongs to the slip region. We see that the stick solution(corresponding to the no-slip condition) has zero velocity at the disk while the slip solution has positive velocity at the disk. Wherein the velocity magnitude for zero-order approximation is bigger than the composite solution. In consequence of the flow rate conservation, slip solution velocity at the plane of symmetry ($z = 0$) is reduced compared with stick solution. The axial distributions of the radial velocity profiles at selected radial coordinates are shown in Fig. 4(b),(c). Fig. 4 (b) corresponds to $B = 10$, $B_w = 13$, $\varepsilon = 0.05$, radial locations are (from left to right) $r = 0.6, 0.7, 0.8, 0.9, 1$. The first profile is located in the no-slip region, thus solutions coincide and we see the only red line. Next position $r = 0.7$ is located in the fake slip region. The red and black contours are almost identical, excluding small deviation near the pseudo-yield surface. The locations $r = 0.8, 0.9, 1$ are in the slip region. It is easy to see that under the slip condition the velocity profiles are compressed in the radial direction compared with no-slip case. At any radial coordinate r, in the region close to the disks surfaces the velocity of the fluid which corresponds to the slip condition, is more than the velocity corresponding to the no-slip condition. Since the same mass flow rate is imposed, in order to satisfy the mass-conservation constraint, the increase in fluid velocity in the wall regions will be compensated for by the decrease in fluid velocity as the center region.

Fig. 4 (c) corresponds to $B = 1$, $B_w = 2$, $\varepsilon = 0.05$, radial locations are (from left to right) $r = 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$. The first profile belongs to the no-slip region, in all other positions the slip condition is performed.

5 NUMERICAL SOLUTION

In order to check our asymptotic solution were performed the simulation for the squeeze flow of viscoplastic Bingham media. This section is dedicated to the solution of the system 3 using an Augmented Lagrangian method.This method to solve the problem of Bingham fluid flow is detailed in a previous papers.3–8 We denote $\Omega = [-L, L] \times [-H, H]$, $\Gamma = \partial\Omega$,

$$\Gamma_u = \{(x, y) \,|\, -L \le x \le L, y = \pm H\}; \Gamma_s = \Gamma \setminus \Gamma_u; \tag{89}$$

$$\boldsymbol{u}_V = \quad (0, -V) \quad \text{if } y = H, \tag{90}$$

$$(0, V) \quad \text{if } y = -H. \tag{91}$$

We introduce following convex set

$$\boldsymbol{U}_V = \{\boldsymbol{u} \in (H^1(\Omega))^2 \,|\, \boldsymbol{u} = \boldsymbol{u}_V \text{ on } \Gamma_u, \nabla \cdot \boldsymbol{u} = 0\} \tag{92}$$

and the functional $J : \boldsymbol{U}_V \to R$ such that

$$J(\boldsymbol{u}) = \frac{\mu}{2}\int_\Omega |\gamma(\boldsymbol{u})|^2 d\boldsymbol{x} + \tau_0 \int_\Omega |\gamma(\boldsymbol{u})| d\boldsymbol{x}. \tag{93}$$

The set $\boldsymbol{U}_V$ consists of function in $(H^1(\Omega))^2$ that are divergence free and for which the essential boundary conditions are satisfied. The solution of problem 3 – 8 expressed as a minimization point of J on $\boldsymbol{U}_V$:

$$J(\boldsymbol{u}) = \min_{\boldsymbol{v}\in\boldsymbol{U}_V} J(\boldsymbol{v}). \tag{94}$$

The difficulty in solving problem 93, 94 is related to non-differentiability of the right-hand size second term of 93

We introduce the following functional space $\boldsymbol{Q} = \{\boldsymbol{q}|\ \boldsymbol{q} \in (L^2(\Omega))^4; \boldsymbol{q}^T = \boldsymbol{q}\}$, the additional variable

$$\boldsymbol{q} = \boldsymbol{\gamma}(\boldsymbol{v}) \in \boldsymbol{Q}. \tag{95}$$

The constraint 95is relaxed thanks to the introduction of Lagrange multiplier that coincides with the stress $\boldsymbol{\tau} \in \boldsymbol{Q}$. The augmented Lagrangian $L_r : \boldsymbol{U} \times \boldsymbol{Q} \times \boldsymbol{Q} \to R$ is given by

$$\mathcal{L}_r(\boldsymbol{u};\boldsymbol{q};\boldsymbol{\tau}) = \frac{\mu}{2}\int_\Omega |\boldsymbol{q}|^2\, d\boldsymbol{x} + \tau_0\int_\Omega |\boldsymbol{q}|\, d\boldsymbol{x} + \frac{1}{2}\int_\Omega (\gamma(\boldsymbol{u}) - \boldsymbol{q}) : \boldsymbol{\tau}\, d\boldsymbol{x} + \frac{r}{2}\int_\Omega |\gamma(\boldsymbol{u}) - \boldsymbol{q}|^2\, d\boldsymbol{x}, \quad r \geq 0. \tag{96}$$

The minimization problem 93, 94 is replaced by saddle point problem of $\mathcal{L}_r$ which we can solve by the following Uzawa-type algorithm proposed in [**?**]

$$\{\boldsymbol{q}^0, \boldsymbol{\tau}^1\} \text{ are given } \in \boldsymbol{Q}\times\boldsymbol{Q}; \tag{97}$$

for $n = 0, 1, 2\ldots$, assume $\boldsymbol{q}^n$ and $\boldsymbol{\tau}^n$ are being known, perform the next steps:

Step 1: find $\boldsymbol{u}^{n+1}$ e p^{n+1} such that:

$$\begin{aligned}
-r\triangle\boldsymbol{u}^{n+1} + \nabla p^{n+1} &= \nabla\cdot(\boldsymbol{\tau}^n - r\boldsymbol{q}^n) \text{ in } \Omega,\\
\nabla\cdot\boldsymbol{u}^{n+1} &= 0 \text{ in } \Omega,\\
r\boldsymbol{\gamma}(\boldsymbol{u}^{n+1})\cdot\boldsymbol{n} &= (r\boldsymbol{q}^n - (-p^{n+1}\boldsymbol{I} + \boldsymbol{\tau}^n))\cdot\boldsymbol{n} \text{ on } \Gamma_s,\\
\boldsymbol{u}^{n+1} &= \boldsymbol{u}_V \text{ on } \Gamma_u.
\end{aligned}$$

Step 2: compute explicitly $\boldsymbol{q}^{n+1}$ as

$$\boldsymbol{q}^{n+1} := \begin{cases} 0, & \text{if } |\boldsymbol{\tau}^n + r\boldsymbol{\gamma}(\boldsymbol{u}^{n+1})| < \tau_0,\\ (1 - \frac{\tau_0}{|\boldsymbol{\tau}^n + r\boldsymbol{\gamma}(\boldsymbol{u}^{n+1})|})\frac{\boldsymbol{\tau}^n + r\boldsymbol{\gamma}(\boldsymbol{u}^{n+1})}{(r+\mu)}, & \text{otherwise.}\end{cases} \tag{98}$$

Step 3: explicit update of the Lagrange multiplier $\boldsymbol{\tau}^{n+1}$

$$\boldsymbol{\tau}^{n+1} := \boldsymbol{\tau}^n + r(\boldsymbol{\gamma}(\boldsymbol{u}^n) - \boldsymbol{q}^n). \tag{99}$$

If $|\boldsymbol{\tau}^{n+1} - \boldsymbol{\tau}^n| > \varepsilon$ for $\varepsilon > 0$, go to step 1.

The advantage of this algorithm is that it transforms the non-differentiable problem 93, 94 into a family of a modified Stokes problem with an additional right hand side 98 and local explicit calculations.

The system of equations is discretized in space by finite differences on a uniform staggered grid with meshsize $h = h_1 = h_2$, where h_1 and h_2 are the step discretization in each direction. The discrete values of the pressure are located at the center of each cell and the velocity components u and v are located at the middle of the cell faces. The components γ_{11}, γ_{22}, q_{11}, q_{22}, τ_{11}, τ_{22} of strain-rate and Lagrange multipliers tensors are located at the cell centers whereas the non-diagonal components γ_{12}, q_{12}, τ_{12} ($\gamma_{12} = \gamma_{21}$) are located at the grid nodes. This grid, called Marker And Cell (MAC), allows to write first derivatives with a second-order centered scheme [**?**, **?**].

6 CONCLUSIONS

We have obtained the consistent asymptotic solution which is free from lubrication paradox. We provide expressions for velocity, stress,strain. The numerical solution that we have computed using the augmented Lagrangian method has been compared favorably with the asymptotic solutions, providing validation of the approach.

REFERENCES

[1] J. Engmann, C. Servais, A. Burbidge, Squeeze flow theory and applications to rheometry: A review. *Journal of Non-Newtonian Fluid Mechanics*, **132**, 1–27, 2005.

[2] J.R. Scott, Theory and application of the parallel-plate plastometer. *Trans. Inst. Rubber Ind.*, **7**, 169–186, 1931.

[3] R.L. Peek, Parallel-plate plastometry. *Journal of Rheology*, **3 (3)**, 345–372, 1932.

[4] G.H. Covey, B.R. Stanmore, Use of the parallel-plate plastometer for the characterisation of viscous fluids with a yield stress. *Journal of Non-Newtonian Fluid Mechanics*, **8**, 249–260, 1981.

[5] G.G. Lipscomb, M.M. Denn, Flow of Bingham fluids in complex geometries. *Journal of Non-Newtonian Fluid Mechanics*, **14**, 337–346, 1984.

[6] D.K. Gartling and N. Phan-Thien, A numerical simulation of a plastic fluid in a parallel-plate viscometer. *Journal of Non-Newtonian Fluid Mechanics*, **14**, 347–360, 1984.

[7] E.J. O'Donovan, R.I. Tanner, Numerical study of the Bingham squeeze film problem. *Journal of Non-Newtonian Fluid Mechanics*, **15**, 75–83, 1984.

[8] S.D.R. Wilson, Squeezing flow of a Bingham material. *Journal of Non-Newtonian Fluid Mechanics*, **47**, 211–219, 1993.

[9] D.N. Smyrnaios, J.A. Tsamopoulos, Squeeze flow of Bingham plastics. *Journal of Non-Newtonian Fluid Mechanics*, **100**, 165–190, 2001.

[10] A. Matsoukas, E. Mitsoulis, Geometry effects in squeeze flow of Bingham plastics. *Journal of Non-Newtonian Fluid Mechanics*, **109**, 231–240, 2003.

[11] E. Mitsoulis, A. Matsoukas, Free surface effects in squeeze flow of Bingham plastics. *Journal of Non-Newtonian Fluid Mechanics*, **129**, 182–187, 2005.

[12] N. Roussel, C. Lanos, Z. Toutou, Identification of Bingham fluid flow parameters using a complex squeeze test. *Journal of Non-Newtonian Fluid Mechanics*, **135**, 1–7, 2006.

[13] K.J. Zwick, P.S. Ayyaswamy, I.M.Cohen, Variational analysis of the squeezing flow of a yield stress fluid. *Journal of Non-Newtonian Fluid Mechanics*, **63**, 179–199, 1996.

[14] M.J. Adams, I. Aydin, B.J. Briscoe, S.K. Sinha, A finite element analysis of the squeeze flow of an elasto-viscoplastic paste material. *Journal of Non-Newtonian Fluid Mechanics*, **71**, 41–57, 1997.

[15] G.H. Meeten, Effect of plate roughness in squeeze flow rheometry. *Journal of Non-Newtonian Fluid Mechanics*, **124**, 51–60, 2004.

[16] N.J. Balmforth, R.V. Craster, A consistent thin-layer theory for Bingham fluids. *Journal of Non-Newtonian Fluid Mechanics*, **84**, 65–81, 1999.

[17] I.A. Frigaard, D.P. Ryan, Flow of a visco-plastic fluid in a channel of slowly varying width, *Journal of Non-Newtonian Fluid Mechanics*, **123**, 67–83, 2004.

[18] L. Muravleva, Squeeze plane flow of viscoplastic Bingham material. *J. Non-Newtonian. Fluid Mech.*, **220**, 148–161, 2015.

[19] A. Putz, I.A. Frigaard, D.M. Martinez, On the lubrication paradox and the use of regularisation methods for lubrication flows. J. Non-Newtonian Fluid. Mech. 63 (2009) 62-77.

[20] L.Fusi, A.Farina, F.Rosso, Planar squeeze flow of a bingham fluid. *Journal of Non-Newtonian Fluid Mechanics*, **225**, 1–9, 2015.

[21] H.A. Barnes, A review of the slip (wall depletion) of polymer solutions, emulsions and particle slip in viscometers - its cause, character and cure. *Journal of Non-Newtonian Fluid Mechanics*, **56**, 221–251, 1995.

[22] M.M. Denn, Extrusion instabilities and wall slip, Ann. Rev. Fluid Mech. 33, 265-287 (2001).

[23] J.D. Sherwood, D. Durban, Squeeze flow of a power-law viscoplastic solid. *Journal of Non-Newtonian Fluid Mechanics*, **62**, 35–54, 1996.

[24] J.D. Sherwood, D. Durban, Squeeze flow of a Herschel-Bulkley fluid. *Journal of Non-Newtonian Fluid Mechanics*, **77**, 115–121, 1998.

[25] G. Karapetsas, J. Tsamopoulos, Transient squeeze flow of viscoplastic materials. *Journal of Non-Newtonian Fluid Mechanics*, **133**, 35–56, 2006.

[26] S.P. Yang, K.Q. Zhu, Analytical solutions for squeeze flow of Bingham fluid with Navier slip condition. *Journal of Non-Newtonian Fluid Mechanics*, **138**, 173–180, 2006.

[27] A. Ayadi, Exact analytic solutions of the lubrication equations for squeeze-flow of a biviscous fluid between two parallel disks, *Journal of Non-Newtonian Fluid Mechanics*, **166**, 1253–1261 (2011).

[28] J.R.A. Pearson, C.J.S. Petrie, Proceedings of the Fourth International Congress on Rheology, Part 3, Wiley, New-York, 1965, pp. 265–282.

[29] A. Fortin, D. Cote, P.A. Tanguy, On the imposition of friction boundary conditions for the numerical simulation of Bingham fluid flows, Comp. Methods Appl. Mech. Eng. 88 (1991) 97-109.

[30] P. Saramito, N. Roquet, An adaptive finite element method for viscoplastic fluid flows in pipes, Comput. Methods Appl. Mech. Eng. 190 (2001) 53915412.

[31] R.R. Huilgol, Variational principle and variational inequality for a yieldstress fluid in the presence of slip, *Journal of Non-Newtonian Fluid Mechanics*, **75 (2-3)**, 231–251, 1998.

[32] Y.Damianou, M.Philippou, G. Kaoullas, G.C. Georgiou, Cessation of viscoplastic Poiseuille flow with wall slip, *Journal of Non-Newtonian Fluid Mechanics*, **203**, 24–37, 2014.

[33] Y.Damianou, G.C. Georgiou, Viscoplastic Poiseuille flow in a rectangular duct with wall slip, *Journal of Non-Newtonian Fluid Mechanics*,

[34] R. Glowinski, J.L. Lions, R. Tremolieres, Numerical Analysis of Variational Inequalities, North-Holland, Amsterdam, 1981.

[35] E.J. Dean, R. Glowinski, Operator-splitting methods for the simulation of Bingham viscoplastic flow, Chin. Ann. Math. 23 (2002) 187204.

[36] M. Fortin, R. Glowinski, Augmented Lagrangian Methods, North-Holland, 1983.

[37] N.J. Balmforth, I.A. Frigaard, Yielding to stress: recent developments in viscoplastic fluid mechanics, Annu. Rev. Fluid Mech. 46 (2014) 121146.

[38] V.I. Lebedev, On formulae for roots of cubic equation, *Soviet Journal of Numerical Analysis and Mathematical Modelling*, **4**, 315–324, 1991.

[39] L. Muravleva, Uzawa-like methods for numerical modeling of unsteady viscoplastic Bingham medium flows, Appl. Numer. Math