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INFLUENCE OF STRAIN DEFINITIONS ON TRUSSES CRITICAL LOADS

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Abstract. *In this paper, the influence of several elastic constitutive laws upon the stability analyses of plane trusses is discussed. The chosen laws are connected to the engineering strain, Green's strain and logarithmic strain concepts. A geometrically exact nonlinear formulation for a truss bar element is developed for each of them and the correspondent tangent stiffness matrices are obtained.*

The stability analyses of some sample plane trusses are performed by finding the conditions when the determinant of their respective global tangent stiffness matrices is found to be zero. The first example is a symmetric two-bar structure and both limit loads and buckling loads are plotted for the full range of possible initial inclination angles of the bars, for the three chosen constitutive laws. A similar truss with an extra vertical bar is studied to discuss possible material instability if Green's strain concept is adopted.

1 INTRODUCTION

The presented direct formulation of the geometrically exact nonlinear theory of elastic trusses is of great simplicity and elegance and has already been obtained, for plane trusses and adopting the engineering strain concept, in a pioneer paper by Turner et al (1960) [5].

The central idea of this approach can be abstracted as follows, following Levy and Spillers (1995) [2]. Let the equation of equilibrium of a system be written as

$$\mathbf{P} = \mathbf{C}\mathbf{N} \tag{1}$$

where **P** represents the applied loads, **N** the internal forcers or stresses and **C** an appropriate equilibrium operator. Under a load perturbation *d***P**, the system responds as

$$d\mathbf{P} = \mathbf{C}\, d\mathbf{N} + d\mathbf{C}\,\mathbf{N} \tag{2}$$

For discrete systems such as trussed structures it turns out to be relatively simple matter to convert this equation into the usual

$$d\mathbf{P} = \mathbf{K}_{\mathbf{T}}\, d\mathbf{q} \tag{3}$$

where $d\mathbf{q}$ represents the system's incremental generalized displacements and $\mathbf{K}_{\mathbf{T}}$ is the tangent stiffness matrix containing the usual elastic and geometric stiffness matrices. With this approach, which is followed throughout this paper, nonlinear structural analysis becomes simply an application of Newton-Raphson's method of nonlinear equations solution.

2 GENERAL THEORY

In the geometrically exact theory for trusses, no restrictions are made to the amplitude of angular deflections. Let us consider the plane truss bar of Fig. 1. Its initial length is *L*. The relationships between the vector of nodal displacements **q** of a plane truss bar and the values of θ and L' (the angle between the displaced position of the bar and its original position and the deformed length of the bar, respectively) are

$$L' - L = \mathbf{c}\,\mathbf{q} \tag{4}$$

where

$$\mathbf{c} = \begin{bmatrix} -\cos\theta & -\sin\theta & \cos\theta & \sin\theta \end{bmatrix} \tag{5}$$

$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^t \tag{6}$$

To compute the strain ε_i (with *i*= 1 for engineering strain, *i* = 2 for Green's strain and *i* = 3 for logarithmic or natural strain) the following expressions are used, according to Argyris *et al* (1960) [1],

$$\varepsilon_1 = \lambda - 1 \tag{7}$$

$$\varepsilon_2 = \frac{\lambda^2 - 1}{2} \tag{8}$$

$$\varepsilon_1 = \ln\lambda \tag{9}$$

where $\lambda = L'/L$.

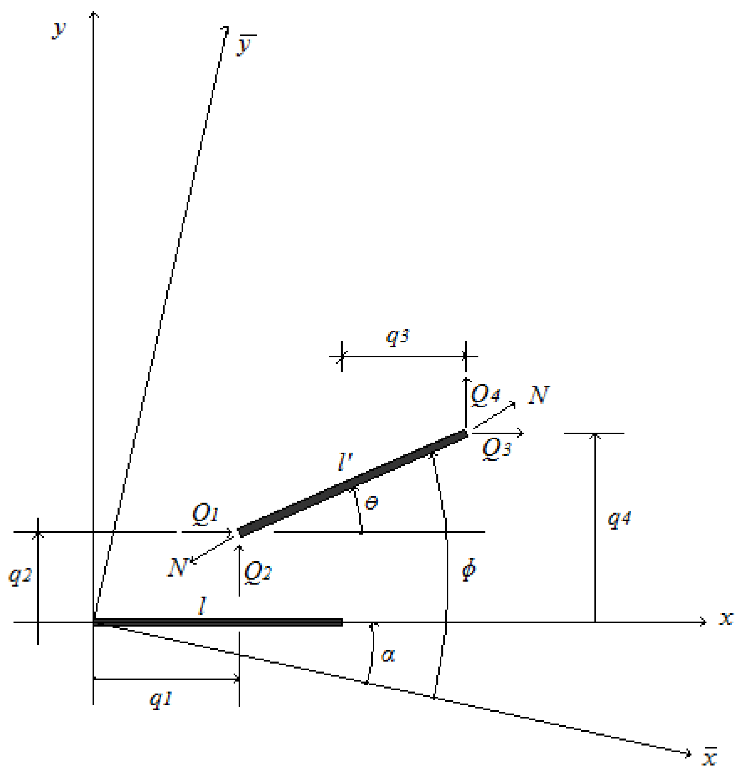

Figure 1: Reference and actual configuration of a truss element

The exact values of the components of nodal forces, as depicted in Fig. 1, are

$$\mathbf{Q} = N\,\mathbf{c}^t \tag{10}$$

where

$$\mathbf{Q} = \begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 \end{bmatrix}^t \tag{11}$$

and N is the normal force in the bar. The constitutive laws render the actual normal stress σ_i by the following expressions

$$\sigma_1 = E_1 \varepsilon_1 \tag{12}$$

$$\sigma_2 = \lambda E_2 \varepsilon_2 \tag{13}$$

$$\sigma_3 = E_3 \frac{\varepsilon_3}{\lambda} \tag{14}$$

where E_i is the elastic modulus correspondent to the chosen strain concept.

Thus, the normal force is given, for each constitutive law, by

$$N_i = \sigma_i\, A \tag{15}$$

where A is the area of the cross section of the bar.

To obtain the expression of the tangent stiffness matrix $\mathbf{k_T}$, one must differentiate Eq. 10 to get

$$d\mathbf{Q} = \mathbf{c^t}\, dN + N\,\mathbf{s^t}\, d\theta \tag{16}$$

where

$$\mathbf{s} = [\sin\theta \quad -\cos\theta \quad -\sin\theta \quad \cos\theta] \tag{17}$$

Next, one must establish relationships between the differentials $d\theta$ and dN and the displacement increments $d\mathbf{q}$. To that end, Eqs. 4 and 15 are used, to obtain

$$d\theta = \frac{1}{L}\mathbf{s}\,d\mathbf{q} \tag{18}$$

$$dN = \frac{1}{L}D_i\,A\mathbf{c}\,d\mathbf{q} \tag{19}$$

where

$$D_1 = E_1 \tag{20}$$

$$D_2 = \frac{E_2}{2}\left(3\lambda^2 - 1\right) \tag{21}$$

$$D_3 = \frac{E_3}{\lambda^2}\left(1 - \ln\lambda\right) \tag{22}$$

It is now necessary to introduce into Eq. 16 the values of $d\theta$ and dN given by Eqs. 18 and 19, to obtain

$$d\mathbf{Q} = \left(\frac{D_i A}{L}\mathbf{c}^{\mathbf{t}}\mathbf{c} + \frac{N_i}{L'}\mathbf{s}^{\mathbf{t}}\mathbf{s}\right)d\mathbf{q} \tag{23}$$

the equation that renders the tangent stiffness matrix of the exact theory for each of the elastic constitutive laws adopted:

$$\mathbf{k_T} = \frac{D_i A}{L}\mathbf{c}^{\mathbf{t}}\mathbf{c} + \frac{N_i}{L'}\mathbf{s}^{\mathbf{t}}\mathbf{s} \tag{24}$$

As in the usual matrix analysis of structures, once the tangent stiffness matrix of the bar in the local reference frame is known, its counterpart in the global one is easily obtained by the rotation formula, given, by example, by Meek (1991)[3],

$$\bar{\mathbf{k}}_{\mathbf{T}} = \mathbf{T}^{\mathbf{t}}\mathbf{k_T}\mathbf{T} \tag{25}$$

where

$$\mathbf{T} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \tag{26}$$

and α is the angle between the original direction of the bar and the $\bar{x}$ axis of the global reference frame, as shown in Fig. 1.

As a more practical alternative, one can introduce the new matrices

$$\bar{\mathbf{c}} = [-\cos\phi \quad -\sin\phi \quad \cos\phi \quad \sin\phi] \tag{27}$$

$$\bar{\mathbf{s}} = [\sin\phi \quad -\cos\phi \quad -\sin\phi \quad \cos\phi] \tag{28}$$

where $\phi = \theta + \alpha$, which are related to Eqs. 5 and 17 by

$$\bar{\mathbf{c}} = \mathbf{cT} \tag{29}$$

$$\bar{\mathbf{s}} = \mathbf{sT} \tag{30}$$

Thus, a new form of the tangent stiffness matrix of the truss member in the global reference frame is

$$\bar{\mathbf{k}}_{\mathbf{T}} = \frac{D_i A}{L} \bar{\mathbf{c}}^{\mathbf{t}} \bar{\mathbf{c}} + \frac{N_i}{L'} \bar{\mathbf{s}}^{\mathbf{t}} \bar{\mathbf{s}} \tag{31}$$

which is formally identical to Eq. 24.

To obtain the tangent stiffness matrix of the structure as a whole, the following summation process is extended to all members of the system

$$\mathbf{K}_{\mathbf{T}} = \sum \mathbf{A}^{\mathbf{t}} \bar{\mathbf{k}}_{\mathbf{T}} \mathbf{A} \tag{32}$$

where **A** is the usual incidence matrix of each bar.

3 THE FIRST EXAMPLE

The first chosen example is the classic two bars truss whose stability was studied by Ratzersdorfer (1936) [4], also known as von Mises truss, depicted in Fig. 2.

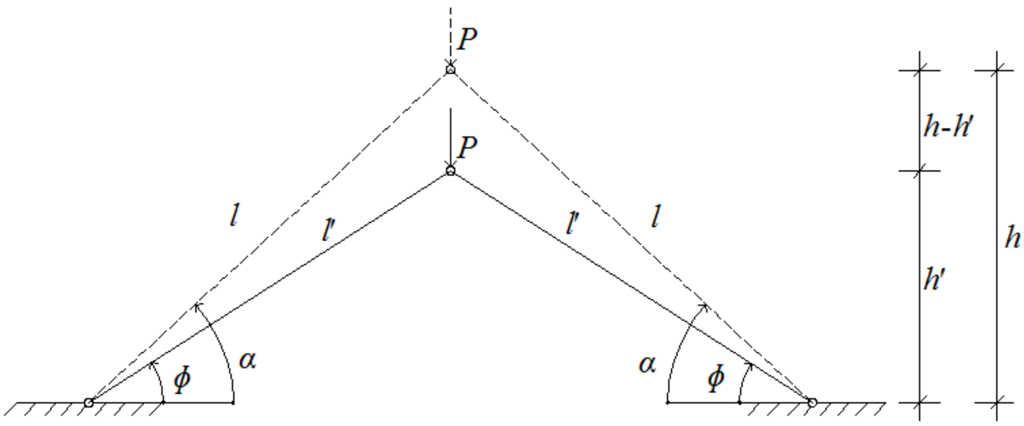

Figure 2. The first example

As it is well known, depending to the initial geometry, either bifurcation or limit point instabilities may happen. The original length of the bars is L and their original inclination α. After a vertical load P is applied to the central node the length of the bars becomes L' and their inclination ϕ.

If the exact theory is applied, the tangent stiffness matrix associated to the two degrees of freedom of the central node is

$$\mathbf{K}_{\mathbf{T}} = \frac{2D_i A}{L} \begin{bmatrix} \cos^2\phi & 0 \\ 0 & \sin^2\phi \end{bmatrix} + \frac{2N_i}{L'} \begin{bmatrix} \sin^2\phi & 0 \\ 0 & \cos^2\phi \end{bmatrix} \tag{33}$$

where the constant D_i and the normal force N_i depend on the particular constitutive law chosen.

The instability criterion is that the determinant of this matrix must be zero.

3.1 Engineering Stain

For the engineering strain constitutive law, the instability criterion is

$$\left(\sin^2\phi\cos\phi-\cos\alpha\right)\left(\cos^3\phi-\cos\alpha\right)=0 \tag{34}$$

The condition for lateral buckling of the structure is given by

$$\sin^2\phi\cos\phi=\cos\alpha \tag{35}$$

and the corresponding buckling load is

$$P_b=2E_1A\cos^2\phi\sin\phi \tag{36}$$

Limit point instability (snap-through) may be reached according to the condition

$$\cos^3\phi=\cos\alpha \tag{37}$$

and the corresponding buckling load is

$$P_l=2E_1A\sin^3\phi \tag{38}$$

In Fig. 3, both bucking and limit loads are plotted for the full range of possible initial inclination angles of the bars. One should note that for α > 68.898^o bifurcation instability (lateral buckling) will prevail with bucking loads given by the lower branch of the respective curve. For α bellow that value the structure may undergo snap through instability. It is important to see that this value, denoted as point A in the figure, is slightly different from that of the tangential point between the limit point and buckling curves, where ϕ = 45^o and α = 69.295^o.

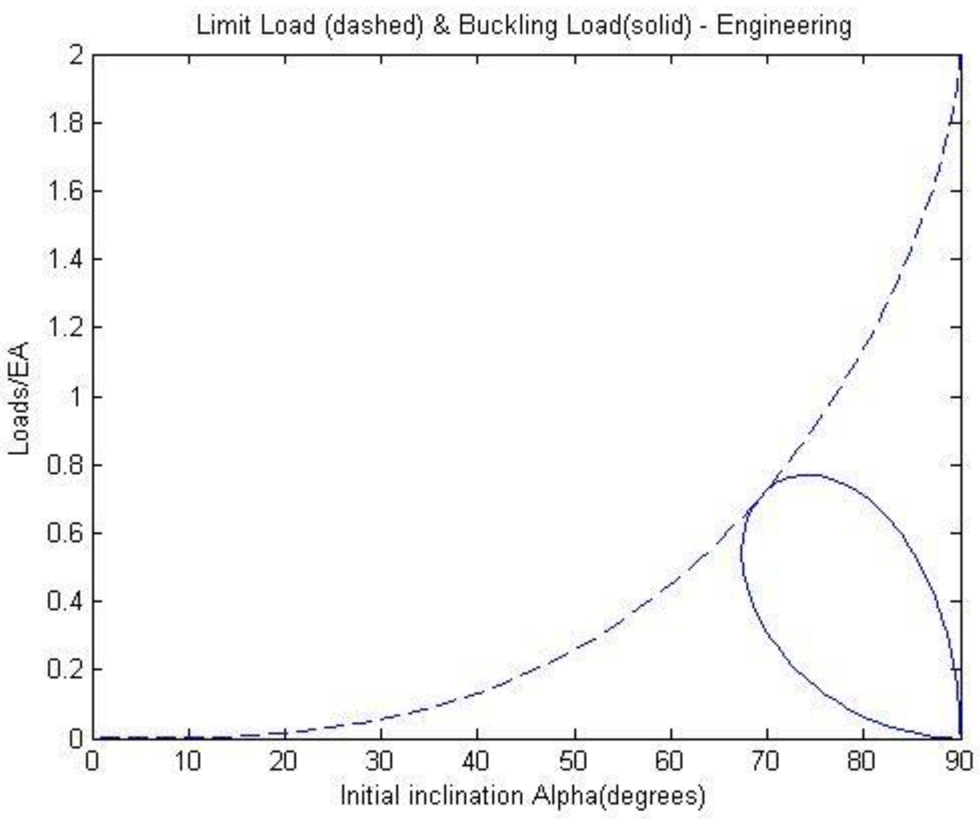

Figure 3: Instability loads, First Example, Engineering Strain

3.2 Green's Strain

For this strain definition, the condition for lateral buckling of the structure is given by

$$\tan^2\phi = \tan^2\alpha - 2 \tag{39}$$

and the corresponding buckling loads are

$$P_b = 2E_2 A\cos^3\alpha\sqrt{\tan^2\alpha - 2} \tag{40}$$

For Green's strain, limit point instability (snap-through) may be reached according to the condition

$$\tan\phi = \frac{\sqrt{3}}{3}\tan\alpha \tag{41}$$

and the limit load is

$$P_l = \frac{2\sqrt{3}}{9}E_2 A\sin^3\alpha \tag{42}$$

In Fig. 4, both bucking and limit loads are plotted for the full range of possible initial inclination angles of the bars. One should note that for $\alpha > 60^o$ bifurcation instability (lateral buckling) will prevail with bucking loads given by the right branch of the respective curve. For α bellow that value the structure may undergo snap through instability. It is important to see that this value is exactly that of the tangential point between the limit point and buckling curves.

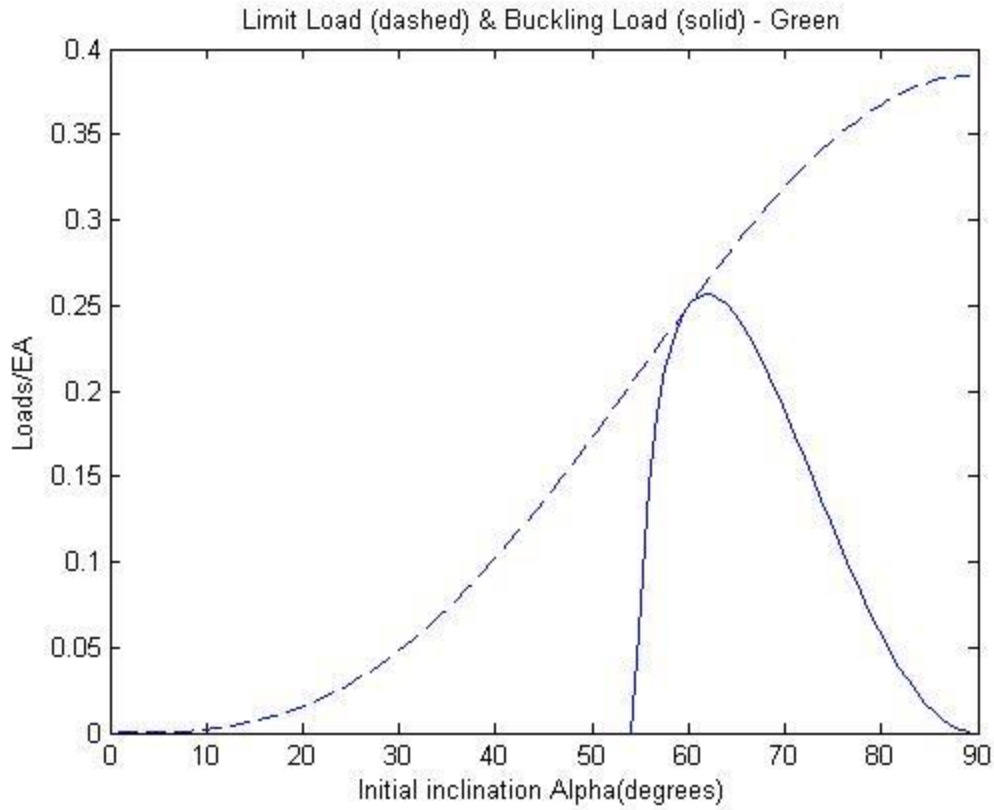

Figure 4: Instability loads, First Example, Green's Strain

3.3 Logarithmic Strain

For this strain definition, the condition for lateral buckling of the structure is given by

$$\lambda^2 \ln\lambda + (1-2\ln\lambda)\cos^2\alpha = 0 \tag{43}$$

For natural strain, limit point instability (snap-through) may be reached according to the condition

$$\lambda^2(1-\ln\lambda) - (1-2\ln\lambda)\cos^2\alpha = 0 \tag{44}$$

Instability loads for both cases are given by

$$P = -\frac{2E_3A}{\lambda^2}\ln\lambda\sqrt{\lambda^2-\cos^2\alpha}. \tag{45}$$

In Fig. 5, both bucking and limit loads are plotted for the full range of possible initial inclination angles of the bars. It is important to see that the two curves never touch each other and for α >71.278° bifurcation instability (lateral buckling) will prevail with bucking loads given by the lower branch of the respective curve. For α bellow that value the structure may undergo snap through instability.

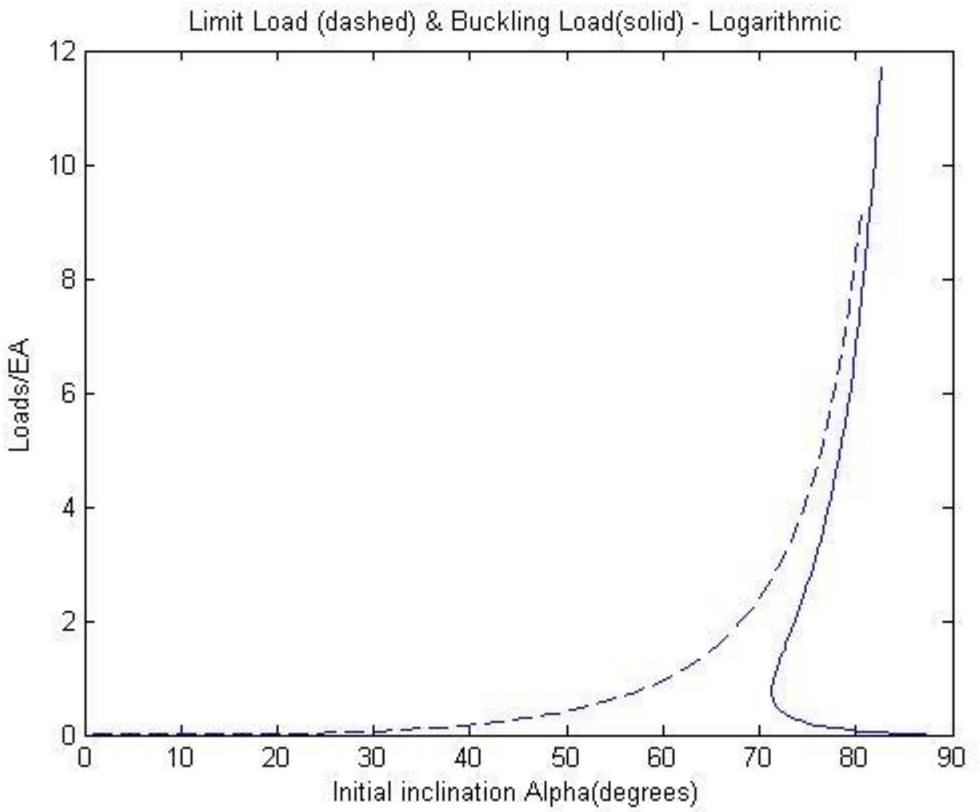

Figure 5: Instability loads, First Example, Natural or Logarithmic Strain

4 THE SECOND EXAMPLE

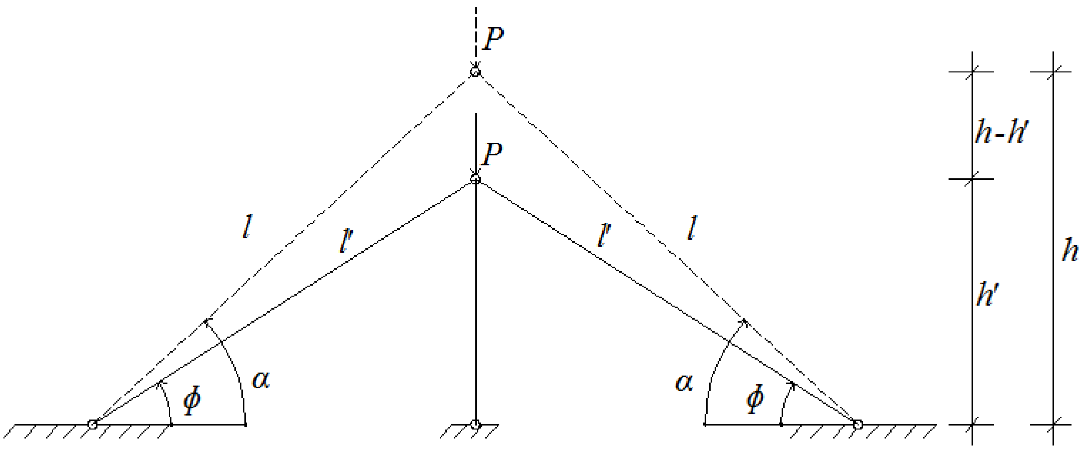

Figure 6: The Second Example

The structure of Fig. 6, similar to the first example except for an extra vertical bar under the upper node, is now analyzed using Green's strain constitutive law. The starting point is Eq. 32 which gives the contribution of the two inclined bars to the tangent stiffness matrix of the structure.

By adding to this equation the new term:

$$\mathbf{k_T} = \frac{E_2 A}{2h}\left(3\lambda_V^2 - 1\right)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \frac{V}{h'}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{46}$$

where

$$\lambda_V = \frac{h'}{h} = \frac{\tan\phi}{\tan\alpha} \tag{47}$$

is the stretching of the vertical bar, the complete $\mathbf{K_T}$ is found to be

$$\mathbf{K_T} = \frac{2D_2 A}{L}\lambda^2 \begin{bmatrix} \cos^2\phi & 0 \\ 0 & \sin^2\phi\left(1 + 1/2\ \csc^3\alpha\right) \end{bmatrix} + \left(\frac{2N_i}{L'} + \frac{V}{L'}\csc\phi\right)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{48}$$

In Eq. 47,

$$N = \frac{E_2 A \lambda(\lambda^2 - 1)}{2} \tag{49}$$

and

$$V = \frac{E_2 A \lambda_V (\lambda_V{}^2 - 1)}{2} \tag{50}$$

according to the assumed constitutive law.

The instability criterion requires the determinant of this matrix to be zero. The first condition, corresponds to the lateral buckling of the structure and leads to the buckling load:

$$P_b = 2E_2 A\cos^3\alpha\,\tan\phi \tag{51}$$

The second condition corresponds to a null value of the stiffness coefficient related to the vertical displacement of the upper node. However, this is not an indication of a genuine snap snap-through phenomenon, but is a consequence of the material instability inextricably present in the constitutive law.

This assertion can be easily proven from Eqs. 13 and 21 and:

$$\sigma_2 = \lambda E_2 \varepsilon_2 = \frac{E_2}{2}\lambda(\lambda^2 - 1) \tag{52}$$

The zero value of its derivative is

$$3\lambda^2 - 1 = 0 \tag{53}$$

which gives the limit value:

$$\lambda_l = \frac{\sqrt{3}}{3} \tag{54}$$

This is the very value that is obtained combining the definition of λ_V given by Eq. 46 with the condition of Eq. 53:

$$\lambda_V = \frac{\tan\phi}{\tan\alpha} = \frac{\sqrt{3}}{3} \tag{55}$$

In Fig. 7, both bucking and limit loads are plotted for the full range of possible initial inclination angles of the bars. It is important to see that limit point and buckling curves touch each other in two points. One should note that for $\alpha < 10^o$ bifurcation instability (lateral buckling) will prevail with bucking loads given by the left branch of the respective curve. For $10^o<\alpha<50^o$the structure may undergo snap through instability. For larger values of α the structure may experience bifurcation (lateral buckling) instability, with buckling loads given by the right branch of the respective curve.

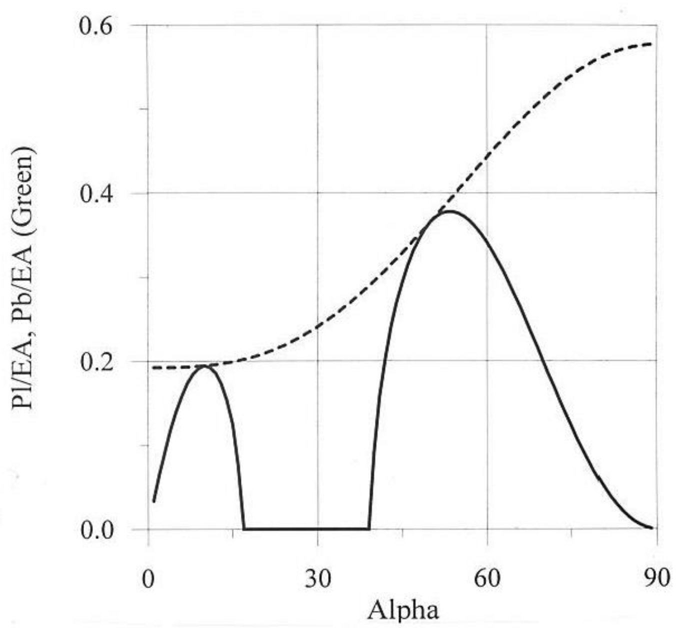

Figure 7: Instability loads, Second Example, Green's Strain

5 CONCLUSIONS

In this paper, the influence of several elastic constitutive laws upon the stability analyses of plane trusses is discussed. The chosen laws are connected to the engineering strain, Green's strain and logarithmic strain concepts. A geometrically exact nonlinear formulation for a truss bar element is developed for each of them and the correspondent tangent stiffness matrices are obtained.

The stability analyses of some sample plane trusses are performed by finding the conditions when the determinant of their respective global tangent stiffness matrices is found to be zero. The first example is a symmetric two-bar structure and both limit loads and buckling loads are plotted for the full range of possible initial inclination angles of the bars, for the three chosen constitutive laws. A similar truss with an extra vertical bar is studied to discuss possible material instability if Green's strain concept is adopted.

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