ECCOMAS Congress 2016
VII European Congress on Computational Methods in Applied Sciences and Engineering
M. Papadrakakis, V. Papadopoulos, G. Stefanou, V. Plevris (eds.)
Crete Island, Greece, 5–10 June 2016

Singularly Perturbed Problems in Mechanics (some fundamental aspects of designing and computing)

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Abstract. It is important study, that is concerned with the development of the concepts and methods of classical stability theory in reference to the problems of designing and computing for singularly perturbed class systems , generated by engineering practice. The various aspects of complex systems dynamics are considered. Methods of the modelling and analysis on the generalized methodology base, coupling the stability theory ideas and asymptotic theory manners, are elaborated. Non-traditional, extended approach, formed on A.M.Lyapunov's theory methods, on brilliant ideas of N.G.Chetayev, P.A.Kuzmin, V.V. Rumyantsev, K.P.Persidskiy, is worked out. It gives universal tool that makes it possible to come near to the solving of fundamental problems in general modelling theory, in designing/computing. The effective algorithm of engineering level is constructed, which is perspective for multidisciplinary systems. Besides all investigated objects are interpreted from unified positions as singular ones; effectual non-traditional technology of modelling, that uses *principally non-linear approach*, is established; the simple schemes of *decomposition of original systems (models) and of dynamic properties* are worked out; *the generalization of the reduction principle,* well-known in stability theory, is got for general qualitative analysis. This manner is permitting to construct the hierarchical sequence of simplified systems (and models) as comparison ones; to determine the conditions of the qualitative equivalence between original and shortened systems; to find the areas of their acceptability for analytical or computer- analytical analysis in designing of engineering objects.

Keywords: Complex Systems, Stability Theory, Singularity, Modelling.

Author is grateful to Russian Foundation of Fundamental Investigations for support of this research (grant 15-08-00393).

1 INTRODUCTION

The first fundamental rigorous results in this direction were obtained by H.Poincare and by A.M.Lyapunov. In classical works of A.M.Lyapunov the comparison method (general method of qualitative analysis) was developed with the strong justification for the solving of stability problems. This method led to the reduction principle, well-known one in stability theory (A.M.Lyapunov, K.P.Persidskiy,…), and to the comparison principle (R.Bellman-V.Matrosov).

There is the direct methodological connection between stability theory and singular perturbations theory (I.S.Gradstein, N.G.Chetayev); between modelling problems and parametric stability theory (N.G.Chetayev, P.A.Kuzmin). With reference to Mechanics problems, formulated here, it leads to the singularly perturbed problems, with various singularities types, with specific critical cases. This research is formed on accepted here basic proposition about global, in-depth fundamental, connection between the singularly perturbed problems (and modelling problems in Mechanics) and the stability methods of Lyapunov's theory. Such tenet is ascending to well-known stability postulate (N.G.Chetayev), and singularity postulate (L.K.Kuzmina), with the extending statement about stability with parametric perturbations on singular case. All original objects may be treated from unified view point of systems of singularly perturbed class. The object state may be described by the equations with small (or big) parameters. The original mathematical model (as example, in Lagrange's form) may be represented in standard form, as singular model, with small parameters in different powers. For this it is necessary to construct the corresponding non-linear, non-singular, evenly-regular transformation of variables. It is postulated (L.K.Kuzmina), that such suitable transformation exists always, and it may be constructed by special, non-formal manners. Besides the original dynamic problems are solved as singular ones; shortened (approximate) systems are introduced as subsystems of *s*-level (*s*-systems); *reduced models are obtained as asymptotic s-models*. Here these *s*-systems are also singularly perturbed ones. It is non-traditional approach, combining the methods of stability theory and perturbations theory that allows to come near to the solving fundamental problem of modelling ,designing, computing in Mechanics via understanding it as problem of singularly perturbed class.

Following to ideas of N.G.Chetayev's, in accordance with Lyapunov's methods, the singular problems may be solved (stability, proximity, optimality, quickness,…) both for non-critical and for critical cases; with simple and multiple roots;… Also for singular systems with the peculiarities (critical spectrums) the reduction conditions may be determined. In these cases the direct use of known results of singular perturbations theory (A.N.Tikhonov, A.Naifeh,…) is non-suitable: eigen-values of corresponding matrices are zero- and imaginary ones. Mechanical systems, modelling the technical objects, are "quasi-Tikhonov's systems" (N.N.Moiseev). *Therefore special, novel, manners are necessary*. Methods, based on Lyapunov's methodology, Chetayev's ideas, elaborated here, give powerful tool; bring new interesting results, perspective both for perturbations theory (singular problems in specific, critical cases are solvable) and for applications to Mechanics, for general modelling theory.

2 INITIAL GENERAL PRINCIPLES OF METHODOLOGY

It is well known [1-16], many applied investigations lead to the mathematical problems, having important general features: the reduction of the order of the differential equations, the loss of continuity, boundary condition. We shall call the singular systems such ones, when the transition to the reduced system is accompanied by the lowering of the model order, with the structural change. For these systems the initial mathematical model may be presented in standard form of system with the singular perturbations. It enables to consider the original objects on typical scheme, to construct by regular manners the "idealized" mechanical-mathematical models, that are interesting for engineering applications, to get strict conditions of their acceptability in dynamics.

For singularly perturbed objects a motion consists of components of various multi-scale classes, from fast to slow, and their differential equations (mathematical models) can be led to the form of equations with small parameters before higher derivatives. Therefore in practice for analysis of such systems the reduced models of lower order are being used as working models. Let the differential equations of the perturbed motion of considered system can be led to a form (let consider the systems with the steady states set)

$$M(m)\frac{dy}{dt} = Y(t, m, y) \qquad (1)$$

where $y = \|z, x\|^T$, *z*, *x* are *m*-, *n*-dimensional vectors; $m > 0$ is a small dimensionless parameter; $M(m) = \|M_{i,j}(m)\|$; $Y(t,m,y) = \|Z(t,m,y); P(m)x + X(t,m,y)\|^T$; $P(m) = \|P_{i,j}(m)\|$;

$M_{i,j}$, $P_{i,j}$ are submatrices of the appropriate sizes; $Z(t, m, y)$, $X(t, m, y)$ are non-linear vector-functions, holomorphic (in appropriate domain) on the totality of variables z, x, in which coefficients are continuous, limited functions of t, m; $Z(t, m, z, 0)= 0$, $X(t, m, z, 0)= 0$; $M_{i,j}(m) = m^{a_i} I$, $0 \le a_i \le r$; I are identity matrices. We shall be able to consider the critical cases of system (1), where z are critical variables [1]. Taking into consideration in (1) only members containing m in power not more than s, $s < r$, we shall receive the shortened approximate system of type

$$M_s(m)\frac{dy}{dt} = Y_s(t, m, y) \tag{2}$$

We shall call (2) the shortened system of s-level, s-system (s-approximation on m). For singularly perturbed systems that are considered here, the order of system (2) is lower than order of full system (1). In applications to mechanics this shortened system leads to the reduced model as the asymptotic model of s-level. From the point of view of mechanics a transition to the reduced model is accompanied by a decrease of freedom degrees number. The system (2) is system of differential-algebraic equations.

We shall be able to obtain the sequence of asymptotic models (as designing-basic models) in mechanics, corresponding the sequence of shortened s-systems (s=0,1,2,…r-1). The following problem is important both for the theory and applications: in which cases and under what conditions it is possible to reduce the system (1)analysis to shortened s-system? The similar problem for equations with small parameter under higher derivatives was considered by many authors [2-8]. We shall show the solving of some concrete problems of mechanics on examples of systems with big and small papameters, that lead to particular cases of the system (1) of special class, that are not embraced by already known fundamental results. This methodology allows to realize the parallel computing in engineering for original very complex system.

3 MECHANICAL SYSTEMS WITH NONRIGID ELEMENTS

As an example of such technical/mechanical system we shall consider the systems of gyrostabilization, modelling ones as mechanical (or electromechanical) systems with controlling gyroscopic elements. Here there is a critical case of zero roots. We shall solve a stability problem of the steady motion for such system, supposing that the elements of the system are not absolutely rigid (we neglect the mass of elastic elements). Differential equations of perturbed motion we shall accept in a form of Lagrange's equations (as in [5, 15])

$$\frac{d}{dt} a\dot{q}_M + (b+g)\dot{q}_M + cq_M = Q''_M, \quad \frac{dq_M}{dt} = \dot{q}_M \tag{3}$$

Here $q_M = \|q_1, q_2, q_3, q_4\|^T$ is n - dimensional vector of mechanical generalized coordinates, where q_1 is l-dimensional vector of the gyroscopes precessions angles; q_2 is (m-l)-dimensional vector of angles deviations of own rotations of gyroscopes from their values in steady motion; q_3 is (s-m)-dimensional vector of stabilization angles, s=m+l; q_4 is (n-s)-dimensional vector of elastic elements deformations; a, b, g are square $n\times n$ - matrices of forms of the system kinetic energy, dissipative function of friction forces, gyroscopic coefficients accordingly; $c = \|c_{i,j}\|$, $b = \|b_{i,j}\|$ (i, j=1,...,4), c_{ij} and b_{ij} are submatrices of an appropriate sizes; b_{44} is square (n-s)×(n-s) -matrix of dissipative function of internal friction forces in material of elastic bodies; c_{44} is square $(n\text{-}s)\times(n\text{-}s)$ -matrix, corresponding to potential energy of elasticity forces.

We assume that all functions in (3) are holomorphic (on the totality of their variables) in certain area.

For solving of this problem (and choosing of a reduced model) we shall lead equations (3) to a form (1) with singular perturbations. For this, first, we must introduce in equations (3) a small parameter, using physical considerations. We suppose, that the elements of the considered systems are of a sufficiently high rigidity and according to that $c_{44} = c_{44}^* / m^2$, $b_{44} = b_{44}^* / m$, where $m>0$ is a small parameter. Now, using the constructed transformation of variables

$z = \left\| a_1, a_2 \right\|^T \dot{q}_M + \left\| b_1^0 + g_1^0,\, b_2^0 + g_2^0 \right\|^T q_M$, $k_1 = \left\| a_1, a_2, a_3 \right\|^T \dot{q}_M$, $k_2 = a_4 \dot{q}_M$, $q_j=q_j$, $(j = 1, 4)$

where a_i, b_i, g_i ($i = 1,\ldots,4$) are submatrices of matrices a, b, g correspondingly, we shall lead equations (3) to the singularly perturbed form. This transformation is the non-linear, non-singular under condition that $\left| b_{i,j}^0 + g_{i,j}^0 \right|_{i=1,2}^{j=2,3} \neq 0$, evenly regular [6], not changing the statement of the stability problem. System (3) in new variables has a form (1)

$$\frac{dz}{dt} = Z(t,m,z,x), \qquad M(m)\frac{dx}{dt} = P(m) + X(t,m,z,x) \tag{4}$$

where $x = \left\| x_1, x_2, x_3 \right\|^T$, $x_1 = \left\| k_1, q_1 \right\|^T$, $x_2=k_2$, $x_3=q_4$; $a_1=0$, $a_2=2$, $a_3=0$; $P_{2i}(m) = mP_{2i}'(m)$ $(i=1,2)$.

The characteristic equation has m zero-roots. Other roots can be found from the equation $d(l, m)=0$. We assume the shortened system of 0-level (degenerated system) as an approximate one for a system (4), marking it (4′) without writing. In old variables it is the system

$$\frac{d}{dt} a^* \dot{q} + (b^* + g^*)\dot{q} + c^* q = Q^*, \quad \frac{dq}{dt} = \dot{q} \tag{5}$$

where $q = \left\| q_1, q_2, q_3 \right\|^T$ is s-dimensional vector of generalized coordinates, describing the state of an absolutely rigid system; a^*, b^*, c^*, g^* are $s\times s$-matrices of absolutely rigid system.

The equation (5) describes a motion of an idealized model of mechanical system. This model corresponds to an approximate system (4′) of 0-level. We shall call it a "limit model". A problem: in what conditions a transition from the initial model (3) to its idealized model (to absolutely rigid system) is possible in qualitative analysis, designing, computing? Using methods of stability theory [1, 2], combined with the singular perturbations methods [7,8] and introducing the differential equations for deviations that respond to non-critical (basic) variables x, we can find out the acceptability conditions for transition validity from system (4) to the system (4′) in concrete dynamical problems. After returning to old variables, taking into account the properties of the considered mechanical system, we receive the corresponding statements.

3.1 Stability problem

When the stability property for reduced model (5) will be ensuring same property for original (full) model (3)?

Theorem 1. If $\left|b_{i,j}^{0}+g_{i,j}^{0}\right|_{i=1,2}^{j=2,3}\neq 0,\ \left|c_{31}^{0}\right|\neq 0$ and all roots (except m zero roots) of characteristic equation of reduced system (5) have negative real parts, then with sufficiently small values of μ (sufficiently high rigidity of the system elements) the zero solution stability of the full system will be succeeding from the zero solution stability of reduced system (5). And reduced system (5) has integral

$$\left\|\begin{matrix}a_1^*\\ a_2^*\end{matrix}\right\|\dot{q}+\left\|\begin{matrix}b_1^{*0}+g_1^{*0}\\ b_2^{*0}+g_2^{*0}\end{matrix}\right\|q+j\,(q,\dot{q})=B$$

and full system (3) has integral of Lyapunov:

$$\left\|\begin{matrix}a_1\\ a_2\end{matrix}\right\|\dot{q}_M+\left\|\begin{matrix}b_1^{0}+g_1^{0}\\ b_2^{0}+g_2^{0}\end{matrix}\right\|q_M+F(q_M,\dot{q}_M)=A\,.$$

3.2 Estimations of approximate solutions

Let $q_i=q_i(t,m),\ \dot{q}_i=\dot{q}_i(t,m),\ (i=1...4)$ be the solution of system (3) with the initial conditions $q_{i0}=q_i(t_0,m),\ \dot{q}_{i0}=\dot{q}_i(t_0,m)$;we shall designate $q_i^*=q_i^*(t)$, $q_i^{\cdot *}=q_i^*(t)$, (i =1,…4) as the solution of approximate system (5), defined by the initial conditions $q_{j0}^*=q_j^*(t_0),\ \dot{q}_{j0}^*=\dot{q}_j^*(t_0)$ (j=1, 2, 3), where $q_4^*\equiv 0,\ \dot{q}_4^*\equiv 0$.

Making use of stability theory methods we can prove the following statement:

Theorem 2. If the characteristic equation for system (5) has all roots in the left half-plane (except m zero roots) for d(0, 0)≠0, then under sufficiently big stiffness of the system elements (i.e. m is sufficiently small) there exists such a m_*-value for x>0, h>0, g>0 given in advance (no matter how small x and g are), that in a perturbed motion: $\left\|q_i-q_i^*\right\|<x,\ \left\|\dot{q}_i-\dot{q}_i^*\right\|<x$ (i=1,…,4) when 0<m<m_* for t≥t_0+g, if $q_{j0}=q_{j0}^*,\ \dot{q}_{j0}=\dot{q}_{j0}^*$, ($j$=1,2,3) $\left\|q_{40}\right\|<h,\ \left\|\dot{q}_{40}\right\|<h$.

It should be pointed out that while demonstrating and using variables z, x we introduce deviations a=z-z^*, b=x-x^* and consider a differential equation for b. The analysis of these equations as well as the integral structure enable to derive the statement of Theorem.

These results, complementing already known [9], justify for the systems, considered here, admissibility of approximate limit model (as asymptotic model of 0-level) and determine the conditions, under which the considered transition is correct (in a meaning, adopted here).

Remark. According to this we can introduce other approximate model (as designing-basic model) for (3). This is asymptotic model of 1-level (m-approximation), that has (s+(n-s)/2) of freedom degrees (if in (4) take into consideration members containing μ in power not more than 1). This model is new one (it is very interesting result).

System (4) belongs to the special critical case, when all eigenvalues of matrix P_{22}, corresponding to the fast x_2, are zero.

4 SYSTEMS WITH FAST ROTORS

Using the same asymptotic approach, we can solve a problem of the transition strict substantiation to a reduced (approximate) model for the mechanical systems with the fast rotors (gyroscopes) [13]. No-giving the formulas and computations here, we note only the some results. In this case original mathematical model is accepted in a form of Lagrange's equations [5]; big parameter is introduced through gyroscopic forces [5, 13]; the necessary transformation of state variables is constructed; the initial equations are reduced to standard form (1). The reduced models are got on our scheme. According to elaborated method, by strict mathematical manner we obtained the known (precessional) model and new («limit») model (as shortened model of 1-st level and 0-level, correspondingly). The conditions of acceptability of these models are determined.

5 SYSTEMS WITH SMALL DELAY TIME

Here the electromechanical systems (EMS), modelling the gyrostabilization systems, are investigated. Original model is presented in general form of Lagrange-Maxwell (Gaponov) equations [14]. The problems: the constructing of reduced model; and their acceptability (in corresponding sense) for these systems. For case of fast-acting systems, interpreting ones as singularly perturbed system, according to our method, we solve these problems. The required transformation is constructed; the reduced models (two types) are obtained; the domains of acceptability are determined.

6 CONCLUSION

In the applications to engineering practice the elaborated methods are very effective, those are enabling to construct the acceptable shortened submodels(as *s-approximations*) by strict mathematical way; to substantiate strongly their correctness in dynamics, including Lyapunov's critical cases; to consider specific cases, inherent for mechanical systems; to evaluate the corresponding errors in such transition-simplifying. It is allowing to realize the using of parallel supercomputing in designing processes on modern computing machines .

The elaborated methods are illustrated on examples from engineering practice, from Mechanics. In framework of this approach it is considered actual problems for fast gyros theory, electromechanical systems, robotic systems; mechanical systems with the friction, non-holonomic systems; Newton's model of point mass dynamics,…

New elegant outcomes are obtained, that are interesting both for theory and for applications, both in general theory of singularly perturbed systems and in applied engineering problems. Also this approach is very perspective from gnosiological view point, for general knowledge theory, with revealing interesting new models, with possibility for investigation of the original complex multi-scale system **with using analytical or computer- analytical parallel methods from first steps of engineering designing**.

Finally as some additional remarks we notice that by analogy with these systems, using the same methods, other singularly perturbed problems and systems of mechanics may be considered.

The subject-matter of investigations is general problem of modelling in mechanics. *The development of mathematical modelling questions of mechanics is closely concerned with actual tasks of mechanical systems dynamics, of differential equations theory* ***with big and small parameters,*** *of stability theory.* In one's turn it is generating new trends in mathematics, interesting mathematical and mechanical problems.

The received results are generalizing and supplementing ones, known in theory of perturbations; these results are developing interesting applications in engineering. With reference to Mechanics and engineering applications the rigorous theoretic justification is

obtained for considered approximate models and theories, both traditional (K.Magnus, A.Andronov, D.Merkin,…) and *new ones*; the separation of state variables on different-frequency groups is developed from first stages of designing; the acceptability of approximate theories, models is discussed, including “exotic” Aristotle’s model in Dynamics.
Deep philosophical aspect of idealization problem in Mechanics is highlighted.

The author is grateful to Russian Foundation of Fundamental Investigation for support in this research.(15-08-00393).

REFERENCES

[1] A.M.Lyapunov, *The general problem of motion stability*. Moscow, USSR AS Coll. of papers, vol.2, pp.7-264, 1956.

[2] N.G.Chetayev, *Appl. Math. and Mech.* Vol.21, No.3, pp.419-421, 1957.

[3] L.K.Kuzmina, Proceedings of 9-th National All-Russian Congress on Theor. and Appl. Mech. Nizh.Novgorod, vol.1, p.76, 2006.

[4] N.N.Moiseev, Mathematical Problems of Systems Analysis. Moscow, Nauka 1981.

[5] L.K.Kuzmina, *Appl. Math. and Mech.* Vol.46, No.3, pp.382-388, 1982.

[6] S.Lefshetz, Differential Equations: geometric theory. New York, 1957.

[7] I.M.Gradstein, *Math. Rev.* Vol.32, No.3, pp.533-544, 1953.

[8] A.N.Tikhonov, *Math. Rev.* Vol.31, No.3, pp.575-586, 1953.

[9] F.L.Chernousko, A.S.Shamaev, *Proc. USSR AS, Mech. of Solid Body*. No.3, pp.33-42, 1983.

[10] Y.Z.Tsypkin, P.V.Bromberg, *Izv.AS USSR, OTS*. No.2, pp.1163-1167, 1945.

[11] A.A.Feldbaum, *Avtomatika and telemech.* No.4, pp.253-279, 1948.

[12] A.A.Voronov, Introduction in dynamics of complex controlled systems. Moscow, Nauka. 1985.

[13] D.R.Merkin, *Gyroscopic systems.* Moscow, 1956.

[14] L.K.Kuzmina, *Appl. Math. and Mech.* Vol.55, No.4, pp.594-601, 1991.

[15] L.K.Kuzmina, *Applied Mechanics in the Americas* Vol.12, AAM, USA, 178-181 2008.

[16]Lyudmila K.Kuzmina. Lyapunov theory methods in stability problems of singular systems. Nonlinear analysis: Theory, Methods and Applications, Elsevier, 2009, v.71, No.12, 2481-2485.