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INVARIANCE OF EIGENFREQUENCIES AND EIGENMODES UNDER GEOMETRIC TRANSFORMATION IN ELONGATED ELASTIC STRUCTURES

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Abstract. *The problem of transformation optics for longitudinal and flexural waves in monodimensional elastic systems is analyzed. System of finite dimensions are considered and it is shown that, under appropriate interface conditions, eigenfrequencies in finite systems remain unchanged while eigenmodes can be tuned depending on the applied geometric transformation. Eigenfrequency analysis can be used in cloaking problem in order to demonstrate the quality of the cloak.*

1 INTRODUCTION

Transformation optics has been applied for the design of cloaking models for acoustic and electromagnetic waves [1, 2, 3, 4, 5, 6, 7, 8, 9].

Cloaking for elastic waves brings new challenges regarding the physical interpretation of equations and boundary conditions in the cloaking region. In particular, the governing equations for elastic waves in the vector problem of elasticity and for flexural waves governed by fourth-order differential equations, are not invariant under transformation. The vector problem of elasticity have been analyzed in [10, 11, 12, 13] and it has been shown that in order to enforce invariance of transformed equation, polar materials involving non symmetric Cauchy stress are needed.

The theoretical analysis, the design and the physical interpretation for the cloaking of flexural waves has been addressed in [14, 15, 16, 17, 18, 19]. In particular, [17, 18] have shown that transformed equation for Kirchhoff plates corresponds to an inhomogeneous anisotropic material in presence of pre-stress. The results has also been obtained following an asymptotic procedure from three-dimensional vector elasticity. In [19] the effect of internal boundary conditions for nearly perfect cloak has been detailed.

In general, the quality of the cloak is addressed qualitatively by implementing numerically or experimentally a plane wave, or a wave generated by a point source in an infinite medium, and checking on the shadow generated by a cloaked defect. In [20] the quantitative scattering measure has been introduce in order to evaluate the effectiveness of the cloak.

Here, we suggest the analysis of eigenfrequencies and eigenmodes as an advanced and simplified tools in order to demonstrate the quality of the cloaking and of the transformation. Such a vision drastically simplify the experimental and numerical implementation avoiding the necessity to introduce Perfectly Matched Layers, which can only been obtained approximatively for flexural waves. In particular, we consider a simple monodimensional Euler-Bernoulli beam subjected to longitudinal and flexural waves and we demonstrate that under transformation eigenfrequencies remain unchanged and eigenmodes can be perturbed simply changing the adopted transformation. Again, two possible quantitative measures can be considered in order to check the quality of the effect. The perturbation of the field in the untransformed domain as in the scattering measure or the values of the eigenfrequencies after transformation.

Also, the analysis within a finite domain is important because it enhances the contribution of the non-propagating part of the waves which is often disregarded in cloaking problems. Flexural waves are governed by fourth-order differential equations, ordinary for beam and partial for plates. The solution is the superposition of propagating waves, solution of Helmholtz equation and non propagating waves, solution of the modified Helmholtz equation [21, 22, 23, 24, 25]. The boundary conditions in finite system couple the solutions of the Helmholtz and modified Helmholtz equations.

The paper is organized as follows, In Section 2 we report the time-harmonic equations of motion for longitudinal and flexural waves, in Section 3 we report the transformed equations and we discuss the conditions for the automatic satisfaction of interface conditions. In Section 4 we report the anlytical and numerical eigenfrequency analysis for longitudinal waves in a rod and flexural waves in a beam. Different transformations are discussed and detailed.

2 EQUATIONS OF MOTION IN ELONGATED BEAM STRUCTURES

We apply the transformation to one-dimensional elongated structures. We restrict the attention to time-harmonic regime. We consider slender structure such that the behavior can be described within the classical Euler-Bernoulli beam model. The longitudinal displacement is $U(X)$ and the transverse displacements are $V(X)$ and $W(X)$. The structure has Young's modulus E, density ρ, cross sectional area A, second moments of inertia J_Y and J_Z.

Time-harmonic equation of motion for longitudinal waves in a rod is the following Helmholtz equation for the longitudinal displacement $U(X)$

$$[EA\,U'(X)]' + \rho A\,\omega^2 U(X) = 0, \tag{1}$$

where ω is the radian frequency.

The time-harmonic equations of motion for flexural waves are the following equations for the transverse displacements $V(X)$ and $W(X)$

$$\begin{aligned} [EJ_Z\,V''(X)]'' - \rho A\,\omega^2 V(X) = -T'_Y(X) - \rho A\,\omega^2 V(X) = 0,\\ [EJ_Y\,W''(X)]'' - \rho A\,\omega^2 W(X) = -T'_Z(X) - \rho A\,\omega^2 W(X) = 0. \end{aligned} \tag{2}$$

Note that the transverse shears are given by $T_Y(X) = -\,[EJ_Z\,V''(X)]'$ for the component along Y and $T_Z(X) = -\,[EJ_Y\,W''(X)]'$ for the shear component along Z.

3 TRANSFORMED EQUATION

In order to transform the domains we apply a coordinate transformation $x = G(X)$, having inverse $X = g(x)$. Capital letters indicate quantities defined in the untransformed domain, lower case letters refer to quantities in the transformed domain. Implementation of coordinate transformation within the equations of motion (1) and (2) lead to the transformed equations of motion.

The transformed equation of motion for longitudinal waves has the form

$$\left[\overline{EA}u'(x)\right]' + \overline{\rho A}\,\omega^2 u(x) = 0. \tag{3}$$

In Eq. (3) $u(x)$ is the transformed longitudinal displacement. We assume $u(x) = U(X)$. The transformed Eq. (3) involves non-homogeneous longitudinal stiffness $\overline{EA} = EA/g'(x)$ and non-homogeneous linear density $\overline{\rho A} = g'(x)\rho A$. The axial force $N(X) = EAU'(X)$ transforms to $n(x) = \overline{EA}u'(x)$, which results to be equal to $N(X)$.

The transformed equations of motion for flexural waves have the form

$$\begin{aligned} [t_y(x) + EJ_Z\,n(x)v'(x)]' + \overline{\rho A}(x)\,\omega^2 v(x) = 0,\\ [t_z(x) + EJ_Y\,n(x)w'(x)]' + \overline{\rho A}(x)\,\omega^2 w(x) = 0. \end{aligned} \tag{4}$$

In Eqs. (4) $v(x)$ and $w(x)$ are the transformed transverse displacements, that we assume such that $v(x) = V(X)$ and $w(x) = W(X)$. The transformed shear forces and axial force have the form

$$\begin{aligned} t_y(x) = m'_z(x) = -\left[\overline{EJ_Z}(x)v''(x)\right]',\\ t_z(x) = m'_y(x) = -\left[\overline{EJ_Y}(x)w''(x)\right]',\\ n(x) = \frac{3(g''(x))^2 - g'''(x)g'(x)}{(g'(x))^5}, \end{aligned} \tag{5}$$

respectively. The transformed bending moments are $m_z(x) = -\overline{EJ_Z}(x)v''(x)$ and $m_y(x) = -\overline{EJ_Y}(x)w''(x)$. Eqs. (4) involve non-homogeneous bending stiffnesses $\overline{EJ_Z} = EJ_Z/(g'(x))^3$ and $\overline{EJ_Y} = EJ_Y/(g'(x))^3$ and non-homogeneous linear density $\overline{\rho A} = g'(x)\rho A$.

3.1 Interface boundary conditions

Transformation affects also boundary conditions, which have to be checked in order to avoid any perturbation of the fields corresponding to the homogeneous problems.

At interface points X_i between untransformed and transformed domains the condition $X_i = g(x_i) = x_i$ assures that the same interface point is shared between the different domains.

Concerning longitudinal waves the essential interface condition on displacement $U(X_i) = u(x_i)$ and the natural condition on axial force $N(X_i) = n(x_i)$ are automatically satisfied.

In relation to flexural waves essential interface conditions $V(X_i) = v(x_i)$ and $V'(X_i) = v'(x_i)$ are automatically satisfied if $g'(x_i) = 1$; while natural conditions $M_Z(X_i) = m_z(x_i)$ and $T_Y(X_i) = r_y(x_i)$, with M_Z the bending moment in the untransformed domain and $r_y(x) = t_y(x) + EJ_Z\, n(x)v'(x)$, require the additional conditions $g''(x_i) = 0$ and $g'''(x_i) = 0$ in order to be automatically satisfied. Therefore, in every point x_j in which we want to identify the condition in the transformed domain with a condition in the untransformed domain in X_j the relations $g(x_j) = X_j$ plus $g'(x_j) = 1$, $g''(x_j) = 0$, $g'''(x_j) = 0$ have to be imposed to the transformation. The same conditions hold for the flexural waves in the other direction involving displacement $w(x)$, rotation $w'(x)$, bending moment $m_y(x)$ and force component along z, $r_z(x)$.

4 EIGENFREQUENCY ANALYSIS

In this Section we compute eigenfrequencies and eigenmodes for longitudinal and flexural waves in a domain in which we introduce a transformation, and we compare the results with the eigenfrequencies and eigenmodes in a homogeneous system in absence of the transformation.

4.1 Longitudinal waves

We consider an homogenous rod of length $2L$ fixed at his ends. The solution of the Helmholtz equation (1) is

$$U(X) = A_1 e^{i\alpha X} + A_2 e^{-i\alpha X}, \tag{6}$$

where $\alpha = \omega\sqrt{E/\rho}$.

For fixed-fixed boundary conditions $U(\pm L) = 0$ the eigenfrequency need to satisfy the condition $\sin(2\alpha L) = 0$ leading to the well-known results $\omega = \sqrt{\rho/E}(n\pi)/(2L)$, with n positive integer. The corresponding eigenmodes are $\sin[n\pi(X + L)/(2L)]$.

Now, we introduce a second structure having the same homogeneous properties for $-L \leq X \leq 0$, while the right half $0 \leq X \leq L$ is transformed into the domain $0 \leq x \leq l$ by generic transformation $G(X)$ with inverse $g(x)$. The transformation $g(x)$ has to satisfy the conditions $g(0) = 0$ and $g(l) = L$. The problem is solved by

$$\begin{cases} U(X) = B_1 e^{i\alpha X} + B_2 e^{-i\alpha X}, & \text{for } -L \leq X \leq 0, \\ u(x) = B_3 e^{i\alpha g(x)} + B_4 e^{-i\alpha g(x)}, & \text{for } 0 \leq x \leq l. \end{cases} \tag{7}$$

The solution is found by satysfying the boundary conditions

$$U(-L) = u(l) = 0 \tag{8}$$

and the interface conditions

$$U(0) = u(0), \qquad EA\, U'(0) = \overline{EA}(0)u'(0). \tag{9}$$

The system of boundary and interface conditions has the form

$$\begin{bmatrix} e^{-i\alpha L} & e^{i\alpha L} & 0 & 0 \\ 1 & 1 & -1 & -1 \\ i\alpha & -i\alpha & -i\alpha & i\alpha \\ 0 & 0 & e^{i\alpha g(l)} & e^{-i\alpha g(l)} \end{bmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{10}$$

The condition of determinant equal to zero for the matrix in Eq. (10) is

$$-4\alpha \sin(2\alpha L) = 0, \tag{11}$$

which gives exactly the same eigenfrequencies as in the homogeneous case.

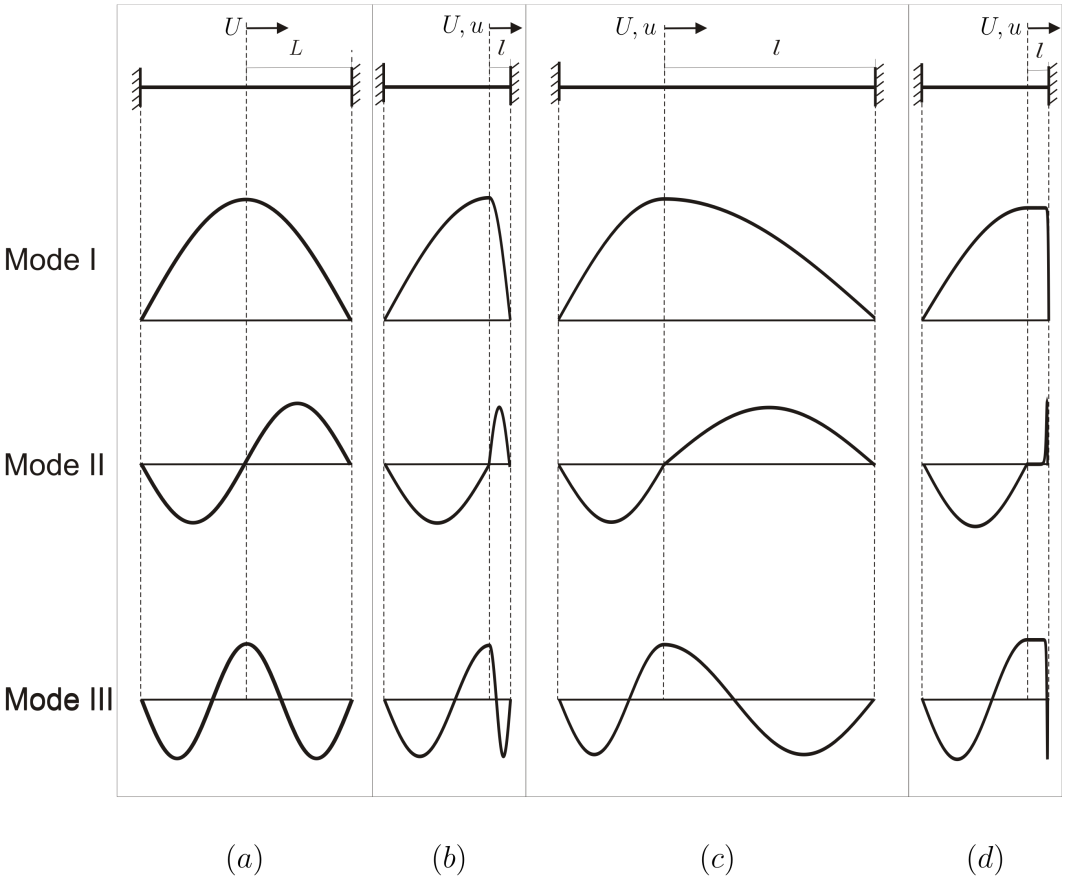

Figure 1: Eigenmodes for longitudinal waves in a rod. (a) Homogeneous rod of length $2L$. (b-d) The rods are subjected to a geometric transformation in the domain $0 \leq X \leq L$. (b) Rod subjected to the linear transformation (12) with $l/L = 0.2$. (c) Rod subjected to the linear transformation (12) with $l/L = 2$. (d) Rod subjected to the non linear transformation (13) with $l/L = 0.2$.

The eigenmodes for the homogeneous problem and the problem with transformation are given in Figure 1. We consider different transformation. We start with two linear transformations given by

$$g_a(x) = \frac{L}{l} x, \tag{12}$$

where we consider the two cases $l/L = 0.2$ and $l/L = 2$. Then, we show the results for the nonlinear transformation

$$g_b(x) = \frac{e^{-100x} - 1}{e^{-100l} - 1} L, \tag{13}$$

From the comparative analysis in Figure 1 it is shown that the eigenmodes are the same in the homogenous domain while $u(x) = u[G(X)] = U(X)$ in the non-homogenous ones. The eigenmodes in the transformed domains can be tuned by changing the transformation $g(x)$, which can be linear or not.

4.2 Flexural waves

Referring to the homogeneous problem as in Section 4.1, the solution of the equation of motion (2)a has the form

$$V(X) = C_1 e^{i\beta X} + C_2 e^{-\beta X} + C_3 e^{-i\beta X} + C_4 e^{\beta X}, \tag{14}$$

in which $\beta = [(\rho A)/(EJ)]^{1/4}\sqrt{\omega}$ is the frequency parameter. If we consider a simply supported beam the boundary conditions $V(-L) = V(L) = 0$ and $M_Z(-L) = M_Z(L) = 0$ lead to the eigenfrequencies

$$\omega = \left(\frac{n\pi}{2L}\right)^2 \sqrt{\frac{EJ_Z}{\rho A}}, \qquad (n \text{ positive integer number}). \tag{15}$$

The corresponding eigenmodes are $V(X) = \sin[n\pi(X + L)/(2L)]$.

We consider now the second structure having the same homogeneous properties for $-L \leq X \leq 0$, and the transformation $X \in [0, L] \to x \in [0, l]$. The transformation $g(x)$ has to satisfy the conditions $g(0) = 0$, $g(l) = L$, $g'(0) = g'(l) = 1$, $g''(0) = g''(l) = 0$ and $g'''(0) = g'''(l) = 0$. In order to satisfy the 8 conditions a polynomial of degree 7 has been implemented.

The problem is solved by

$$\begin{cases} V(X) = D_1 e^{i\beta X} + D_2 e^{-\beta X} + D_3 e^{-i\beta X} + D_4 e^{\beta X}, & \text{for } -L \leq X \leq 0, \\ v(x) = D_5 e^{i\beta g(x)} + D_6 e^{-\beta g(x)} + D_7 e^{-i\beta g(x)} + D_8 e^{\beta g(x)}, & \text{for } 0 \leq x \leq l. \end{cases} \tag{16}$$

The solution is found by satysfying the boundary conditions

$$\begin{cases} V(-L) = v(l) = 0, \\ M_Z(-L) = m_z(l) = 0, \end{cases} \tag{17}$$

and the interface conditions

$$\begin{cases} V(0) = v(0), \\ V'(0) = v'(0), \\ M_Z(0) = m_z(0), \\ T_Y(0) = r_y(0). \end{cases} \tag{18}$$

The system of boundary and interface conditions takes the form

$$\left[\begin{array}{c|c} \mathbf{A}_1 & \mathbf{A}_2 \\ \hline \mathbf{A}_3 & \mathbf{A}_4 \end{array}\right] \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{19}$$

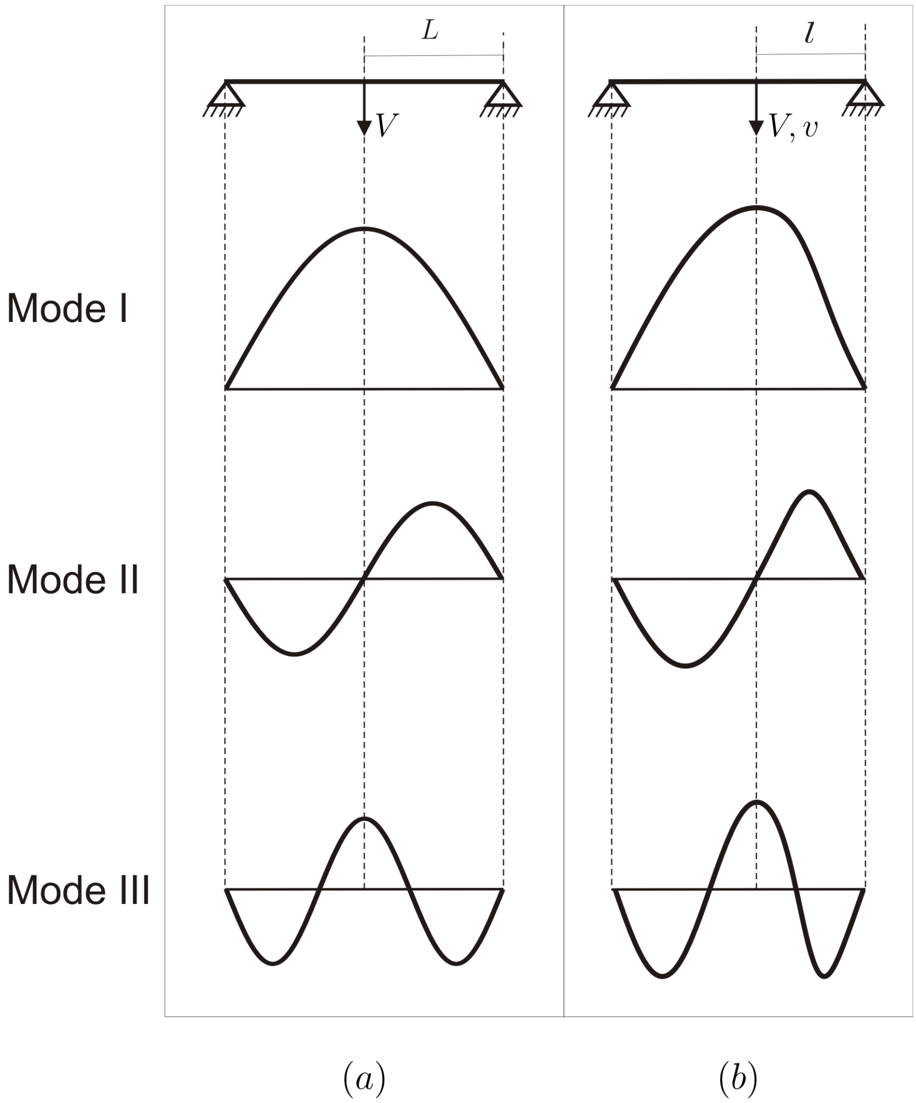

Figure 2: Eigenmodes for transverse waves in a beam. (a) Homogeneous beam of length $2L$. (b) The beam are subjected to a polynomial geometric transformation in the domain $0 \leq X \leq L$, with $l/L = 0.75$.

where

$$\mathbf{A}_1 = \begin{bmatrix} e^{-i\beta L} & e^{\beta L} & e^{i\beta L} & e^{-\beta L} \\ -\beta^2 e^{-i\beta L} & \beta^2 e^{\beta L} & -\beta^2 e^{i\beta L} & \beta^2 e^{-\beta L} \\ 1 & 1 & 1 & 1 \\ i\beta & -\beta & -i\beta & \beta \end{bmatrix} \tag{20}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ -i\beta & \beta & i\beta & -\beta \end{bmatrix} \tag{21}$$

$$\mathbf{A}_3 = \begin{bmatrix} EJ_Z\beta^2 & -EJ_Z\beta^2 & EJ_Z\beta^2 & -EJ_Z\beta^2 \\ iEJ_Z\beta^3 & EJ_Z\beta^3 & -iEJ_Z\beta^3 & -EJ_Z\beta^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

$$\mathbf{A}_4 = \begin{bmatrix} -EJ_Z\beta^2 & EJ_Z\beta^2 & -EJ_Z\beta^2 & EJ_Z\beta^2 \\ -iEJ_Z\beta^3 & -EJ_Z\beta^3 & iEJ_Z\beta^3 & EJ_Z\beta^3 \\ e^{i\beta g(l)} & e^{-\beta g(l)} & e^{-i\beta g(l)} & e^{\beta g(l)} \\ EJ_Z\beta^2 e^{i\beta g(l)} & -EJ_Z\beta^2 e^{-\beta g(l)} & EJ_Z\beta^2 e^{-i\beta g(l)} & EJ_Z\beta^2 e^{-\beta g(l)} \end{bmatrix} \quad (23)$$

Then, setting to 0 the determinant of the matrix in Eqs. (18), we obtain

$$256(EJ)^2\beta^{10}\sinh(2\beta L)\sin(2\beta L) = 0. \quad (24)$$

Eq. (24) gives exactly the same eigenfrequencies of the homogeneous system.

The eigenmodes for an homogeneous beam and a not homogenous beam are given in Figure 2 parts (a) and (b), respectively. Again, the displacement in the transformed domain are $v(x) = v[G(X)] = V(X)$.

5 CONCLUSIONS

Eigenfrequencies and eigenmodes for longitudinal and flexural waves in slender beam structures subjected to geometric transformation are detailed. It is shown that eigenfrequencies are invariant under transformation, while eigenmodes depend on the applied transformation law. Eigenfrequency analysis in finite system can help in the design and implementation of cloaking devices with particular attention to elastic waves.

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