SEECCM III
3rd South-East European Conference on Computational Mechanics-
*an ECCOMAS and IACM Special Interest Conference*
M. Papadrakakis, M. Kojic, I. Tuncer (eds.)
Kos Island, Greece, 12–14 June 2013

# WARPING TRANSMISSION IN 3-D BEAM ELEMENT INCLUDING SECONDARY TORSIONAL MOMENT DEFORMATION EFFECT

**Evangelos J. Sapountzakis[1], Vasileios J. Tsipiras[1] and Pavlos G. Gkesos[2]**

[1] School of Civil Engineering, National Technical University
Zografou Campus, GR-157 80 Athens, Greece
e-mail: cvsapoun@central.ntua.gr, tsipiras@gmail.com

[2] Army Corps of Engineers, Hellenic Army General Staff, Ministry of Defence, Greece
Mesogeion Street 227-231, Holargos, PO BOX 15451
e-mail: gessos.paul@gmail.com

**Keywords:** Warping transmission, warping transfer, shear deformation effect, secondary torsional moment deformation effect, exact stiffness matrix, 3-D beam element.

**Abstract.** *In this paper a 3-D beam element including shear and secondary torsional moment deformation effects is formulated with general warping and bimoment transmission capabilities. The beam's cross section is an arbitrarily shaped doubly symmetric constant one, while its shape is assumed to remain undistorted. Shear deformation (due to shear forces) is taken into account employing shear correction factors determined through an energy approach. Secondary torsional moment deformation effects are modelled by introducing the primary angle of twist per unit length as an additional degree of freedom at each end node of the element and employing a torsional shear correction factor. An exact 14x14 stiffness matrix and the corresponding nodal load vector are formulated based on the solution of the global equilibrium differential equations of the beam. At joints connecting space members, all kinds of warping and bimoment transmission (not involving cross sectional distortion) are modelled, including free warping, full or partial warping restraint, full warping and bimoment continuity, complete direct and inverse warping transmission and general warping and bimoment transmission between intersecting members. A computer program is developed in a mathematical package permitting symbolic algebraic manipulations. Numerical results of great practical interest are worked out to illustrate the efficiency, the accuracy and the range of applications of the developed method.*

## 1 INTRODUCTION

In engineering practice we often come across the analysis of beam assemblages subjected to general loading conditions. Space frames and grid systems are most common examples along with structures such as curved beams and stiffened plates which are sometimes modelled as an assembly of straight members. Such structures often necessitate a rigorous analysis which is not always easy to be achieved due to several reasons.

Firstly, some commercial software packages neglect shear deformation effects (due to shear forces) in the global analysis of the structure. These effects influence notably the behaviour of short beams; often however they could also be significant in long beams of multimaterial cross section exhibiting weak shear connection between different material regions. This could also be the case in beams made of composite materials which are weak in shear. Secondly, in the beam element formulation of most commercial software packages a 12x12 stiffness matrix is considered where 2 degrees of freedom (DOFs) are related to torsion. In such a formulation the effects of nonuniform warping are neglected, thus only the primary (St. Venant) torsional mechanism resisting torsional actions is taken into account. It is well known that nonuniform warping affects significantly the global behaviour of open shaped thin walled cross section bars. Thus, several researchers have employed a 14x14 stiffness matrix with additional DOFs related to warping achieving the inclusion of such effects in the analysis. However, most of the relevant formulations consider the angle of twist per unit length at the element's nodes as additional DOF, thus the secondary torsional moment deformation effect (STMDE) [1] is not taken into account. The aforementioned types of structures influenced by shear deformation are affected similarly by secondary torsional moment deformation as well. Moreover, STMDE should be considered in the analysis of closed shaped thin walled cross section beams [1-3]. Regarding the formulation of the stiffness matrix and the corresponding nodal load vector, several researchers employ numerical methods such as the Finite Element Method (FEM) exploiting simple shape functions and taking proper care in order to avoid shear locking phenomena [4]. In the problem under examination, this procedure results in an approximate formulation, thus more than one element is usually required to model accurately a single straight span. However, the exploitation of the global equilibrium equations of the beam results in an exact stiffness matrix [5] and an inherently locking free formulation where no more than one element is required for each single straight span. It is also pointed out that in most cases the geometric constants (along with the shear correction factors required to accurately model shear and STMD effects) included in the beam element formulation are computed through Thin Tube Theory (TTT) (see for example [5]). The accuracy of this theory depends on the thickness of the shell elements comprising the thin walled beam and sometimes leads to inaccuracies [2].

Thirdly, in most modelling efforts regarding beam assemblages taking into account nonuniform warping effects, the warping and bimoment transmission between nonaligned intersecting members is treated in a simplified manner. Often, the warping DOFs of the elements' nodes at the joint are considered as fully free or fully restrained. Another frequently employed consideration is the assumption of full warping and bimoment continuity at the joint. This assumption is reliable for example in the modelling of straight members with intermediate torsional supports. The actual transmission conditions depend on the geometry of the joint of intersecting members [6]. However, in many geometric configurations encountered in engineering practice [6-9] the aforementioned treatments are not realistic. Shell or 3-D finite element idealizations permit very accurate modelling of every possible geometric design, however their use is costly and complex as compared to beam element models. Thus the beam elements to be used should be enhanced with general warping and bimoment transmission capabilities.

Several researchers have investigated in the past the problem of warping transmission at joints of intersecting members. Baigent and Hancock [10] investigated among others the case of members of channel shaped cross section attached by the web to a stiffened joint plate. Fully free warping for both members at the joint region is a reliable modelling consideration in this case. Krenk and Damkilde [11] investigated among others the case of members of I-shaped cross section intersecting at a joint with three stiffeners. This configuration is modelled accurately by employing fully restrained warping conditions at the joint region for both members. Yang and McGuire [12] proposed a procedure based on warping springs for analyzing space frames with partial warping restraint. Basaglia et al. [6] analyzed among others several geometric configurations which are accurately modelled with complete direct or inverse warping transmission considerations. Tong et al. [13] investigated the joint of intersecting I-shaped cross section beams having a diagonal stiffener. By taking into account the bending and torsional stiffness of the stiffener, they formulated general relations linking the bimoments and warping DOFs of the intersecting members. Vacharajittiphan and Trahair [14] investigated the coupling of warping and distortion in nonaligned intersecting members. To the authors' knowledge, all studies related to the investigation of warping transmission in which beam elements are employed, neglect shear and STMD effects, while TTT is most commonly employed to determine the torsional geometric constants.

In this paper a 3-D beam element including shear and secondary torsional moment deformation effects is formulated with general warping and bimoment transmission capabilities. The beam's cross section is an arbitrarily shaped doubly symmetric constant one, while its shape is assumed to remain undistorted. Shear deformation (due to shear forces) is taken into account employing shear correction factors determined through an energy approach. Secondary torsional moment deformation effects are modelled by introducing the primary angle of twist per unit length as an additional degree of freedom at each end node of the element and employing a torsional shear correction factor. An exact 14x14 stiffness matrix and the corresponding nodal load vector are formulated based on the solution of the global equilibrium differential equations of the beam. At joints connecting space members, all kinds of warping and bimoment transmission (not involving cross sectional distortion) are modelled, including free warping, full or partial warping restraint, full warping and bimoment continuity, complete direct and inverse warping transmission and general warping and bimoment transmission between intersecting members. A computer program is developed in a mathematical package permitting symbolic algebraic manipulations. Numerical results of great practical interest are worked out to illustrate the efficiency, the accuracy and the range of applications of the developed method.

## 2 STATEMENT AND SOLUTION OF THE PROBLEM

### 2.1 Local stiffness matrix and nodal load vector formulation

Consider a 3-D beam element of length $l$ (Fig.1) with an arbitrarily shaped doubly symmetric constant cross section, occupying the two dimensional multiply connected region $\Omega$ of the $y,z$ plane bounded by the $\Gamma_j \left( j=1,2,...,K \right)$ boundary curves, which are piecewise smooth, i.e. they may have a finite number of corners. The material of the bar is homogeneous isotropic and linearly elastic with modulus of elasticity $E$ and shear modulus $G$, while the effects of cross sectional distortion and geometrical nonlinearity are ignored. In Fig.1a $Syz$ is the coordinate system through the cross section's shear center. The bar is subjected to the combined action of arbitrarily distributed or concentrated transverse loads $p_y = p_y(x)$, $p_z = p_z(x)$ and bending moments $m_y = m_y(x)$, $m_z = m_z(x)$ acting in the $y$ and $z$ directions, respectively, axial load $p_x = p_x(x)$, twisting $m_x = m_x(x)$ and warping $m_w = m_w(x)$ moments acting in the $x$ direction (Fig.1b).

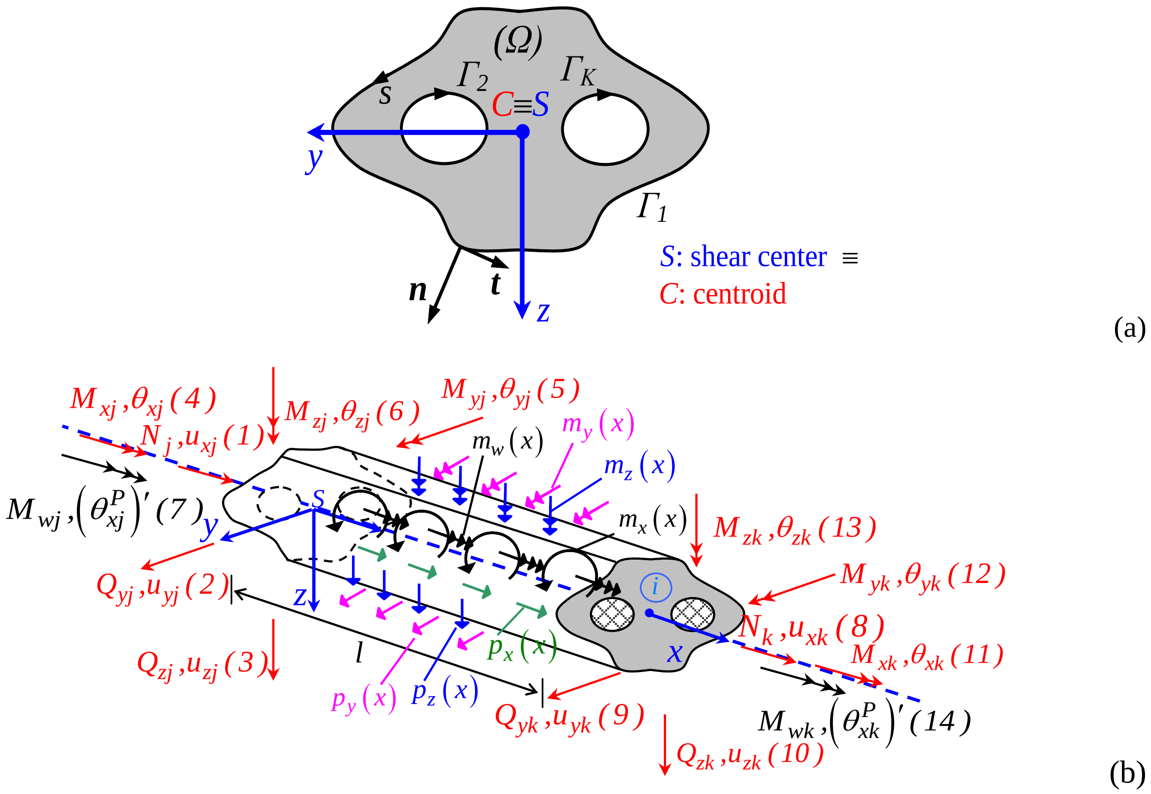


Figure 1: 3-D prismatic beam element with arbitrarily shaped doubly symmetric cross section occupying region $\Omega$ (a) subjected to general loading (b).

In order to take into account shear deformation effects in the study of the aforementioned element, the Timoshenko beam theory is employed along with the bending rotations $\theta_y$, $\theta_z$ which are used as DOF at each end node (Fig.1b). $\theta_y$, $\theta_z$ substitute the transverse displacement components per unit length $u'_y$, $u'_z$ which are employed in the classic 3-D beam element based on the Euler - Bernoulli beam theory. Moreover, in order to take into account nonuniform (torsional) warping effects, an additional warping DOF is included at each end node of the element. In classic (Vlasov) nonuniform torsion theory, this DOF is the angle of twist per unit length $\theta'_x$. In this paper, the primary angle of twist per unit length $\left(\theta_x^P\right)'$ is employed as additional DOF [1-2], permitting the inclusion of STMDE in the study of the element. This quantity is a 1-D measure of the intensity of warping of the cross section and its use is advantageous in the application of torsional boundary conditions, while in general $\left(\theta_x^P\right)' \neq \theta'_x$ with $\left(\theta_x^P\right)' - \theta'_x$ being a 1-D measure of the intensity of warping shear stresses.

From the above considerations, the local nodal displacement vector is expressed as (Fig.1b)

$$\left\{D^i\right\}^T = \left\{u_{xj} \quad u_{yj} \quad u_{zj} \quad \theta_{xj} \quad \theta_{yj} \quad \theta_{zj} \quad \left(\theta_{xj}^P\right)' \quad u_{xk} \quad u_{yk} \quad u_{zk} \quad \theta_{xk} \quad \theta_{yk} \quad \theta_{zk} \quad \left(\theta_{xk}^P\right)'\right\} \tag{1}$$

along with the corresponding nodal load vector

$$\left\{F^i\right\}^T = \left\{N_j \quad Q_{yj} \quad Q_{zj} \quad M_{xj} \quad M_{yj} \quad M_{zj} \quad M_{wj} \quad N_k \quad Q_{yk} \quad Q_{zk} \quad M_{xk} \quad M_{yk} \quad M_{zk} \quad M_{wk}\right\} \tag{2}$$

These vectors are linked with the 14x14 local exact stiffness matrix $\left[K^i\right]$ of the 3-D beam element [2]. This stiffness matrix is readily determined through a mathematical package [15] by solving analytically one global differential equation of equilibrium of axial forces and three systems of global differential equations of equilibrium of bending moments, transverse shear forces acting in the $y$ and $z$ directions, torsional and warping moments (see for example [1, 5]). This procedure results in a locking free element formulation [4]. The coefficients of these equations depend on the geometric constants of the element along with its shear correction factors which are employed to model shear and STMD effects [1-2]. These factors are determined through an energy approach [1-2]. The primary torsion constant, the warping constant and the shear correction factors depend on the primary and secondary warping functions and two stress functions which are determined from the numerical solution of four boundary value problems through BEM [2]. The adopted procedure is free from the assumptions of TTT. The aforementioned three systems of equations may be studied independently due to the assumption of geometrically linear conditions and the doubly symmetric shape of the cross section [16]. It is also worth noting that the nodal load vector is also readily computed analytically for frequently encountered distributions of externally applied actions along the length of the element. Finally, the local exact stiffness matrices and the corresponding nodal load vectors of the classic theories of nonuniform torsion (Vlasov) and flexure (Euler- Bernoulli) have also been formulated for comparison purposes.

After the solution of the problem and the determination of $\left\{D^i\right\}$, the stress resultants of the element may be readily computed through $\left[K^i\right]$. The primary and secondary twisting moments $M_{xj}^P$, $M_{xj}^S$ are determined employing $M_{xj}$, $\left(\theta_{xj}^P\right)'$ as

$$M_{xj}^P = I_t^P\left(M_{xj} + GI_t^S\left(\theta_{xj}^P\right)'\right)/\left(I_t^P + I_t^S\right) \qquad M_{xj}^S = M_{xj} - M_{xj}^P \tag{3a,b}$$

where $I_t^P$ and $I_t^S$ are the primary and secondary torsion constants [17], respectively, while it is noted that $M_{xk}^P$, $M_{xk}^S$ may be computed through similar relations.

## 2.2 Global stiffness matrix formulation

The global stiffness matrix $\left[\bar{K}^i\right]$ linking the nodal displacement and load vectors at the global coordinate system is determined as

$$\left[\bar{K}^i\right] = \left[\Lambda_{SSFw}\right]^T\left[K^i\right]\left[\Lambda_{SSFw}\right] \tag{4}$$

where $\left[\Lambda_{SSFw}\right]$ is a 14x14 transformation matrix given as

$$\left[\Lambda_{SSFw}\right]=\begin{bmatrix} [\Lambda] & & & & & sym. \\ & [\Lambda] & & & & \\ & & 1 & & & \\ & & & [\Lambda] & & \\ \mathbf{0} & & & & [\Lambda] & \\ & & & & & 1 \end{bmatrix} \tag{5}$$

with $\left[\Lambda\right]$ being the usual 3x3 transformation matrix employed in the matrix analysis of space frames [18]. It is noted that $\left(\theta_{xj}^{P}\right)'$, $M_{wj}$, $\left(\theta_{xk}^{P}\right)'$, $M_{wk}$ are not transformed since they are not vectorial quantities.

### 2.3 Warping transmission

Warping transmission conditions at a joint connecting space members depend on the geometry of the joint [6]. In some cases such as unstiffened joints, the coupling of distortional and warping deformations of intersecting members is significant [6, 14]. The aforementioned 3-D beam element is enhanced with general warping and bimoment transmission capabilities. Having in mind that distortional deformations at the element level are neglected in the present work, the following cases previously reported in the literature are modelled:

i. Fully free or fully restrained warping (see for example Baigent and Hancock [10]) at an element's node. In the first case $M_w$ is known, whereas in the second case $\left(\theta_{x}^{P}\right)'$ is known. In these cases the warping DOF remains unlinked with other DOFs of the structure, thus it is treated in the usual manner as if the node was either a free or a fully restrained end.

ii. Partial warping restraint at an element's node (see for example Yang and McGuire [12]). This case is particularly useful for modelling member supports. In this case the warping DOF is linked with another DOF, the value of which is prescribed. A link element coupling the two DOFs is employed with a stiffness matrix of the form

$$\left[K_{link}\right]=\begin{bmatrix} k_w & -k_w \\ -k_w & k_w \end{bmatrix} \tag{6}$$

where $k_w$ is the stiffness of the warping spring [12].

iii. Full warping and bimoment continuity. This case is encountered in straight members with intermediate torsional supports. Here, the elements' nodes at an intermediate torsional support share a common warping DOF. Thus, the usual assembly procedure of the Direct Stiffness Method related to common DOFs of beam structures is followed.

iv. General warping and bimoment continuity (see for example [8, 9, 13]). In this case the warping DOFs of the elements' nodes at a joint of intersecting space members are linked together with general relations. A link element linking the two DOFs is employed with a stiffness matrix of the form

$$\left[K_{link}\right]=\begin{bmatrix} k_{w11} & sym. \\ k_{w21} & k_{w22} \end{bmatrix} \tag{7}$$

where $k_{w11}$, $k_{w21}$, $k_{w22}$ are stiffness parameters determined from the geometry of the joint.

v. Complete direct or indirect warping transmission (see for example [6]). In these cases the warping DOFs of the elements' nodes at a joint of intersecting space members have the same or the opposite value, respectively [6]. Here, the warping DOFs are initially treated as being independent to each other and the usual assembly procedure of the Direct Stiffness Method is followed. Then, the aforementioned relations are added as constraints to the global system of equations by means of the Lagrange multiplier technique [4, 6].

After taking into account the global stiffness matrices of the elements and the warping transmission conditions at joints of the structure under investigation, the usual assembly procedure of the Direct Stiffness Method is followed leading to the formulation of a global system of equations. This system is then solved and the unknown global nodal displacement vector of the structure is computed either symbolically or numerically. Afterwards, the usual post-processing procedure is followed in order to determine the remaining quantities of interest.

## 3 NUMERICAL EXAMPLES

***Example 1***

In the first example, a straight member of a thin walled hollow rectangular cross section previously studied by Murín and Kutis [3] has been analyzed in order to verify the accuracy of the results. The member is supported by two end and two intermediate simple torsional supports and is loaded by two concentrated torsional moments along its length [3]. The member has been modelled with 5 beam elements as in [3], while full warping and bimoment continuity is considered at the nodes of the member. The geometric constants reported by Murín and Kutis [3] and computed through TTT have been employed in order to verify the accuracy of the proposed method ( $I_t^P = 8.9824\times 10^{-2} m^4$ , $I_t^S = 1.107\times 10^{-3} m^4$ , $C_S = 1.930\times 10^{-4} m^6$ ).

In Table 1, the angle of twist and the primary angle of twist per unit length at the nodes are presented as obtained from the proposed method and from Murín and Kutis [3]. It is observed that the agreement between the two methods is excellent. In Table 2, the stress resultants at the beam elements' nodes are reported as obtained from the aforementioned methods, noting the good agreement of the results. In the same table, the corresponding results of the proposed method obtained by employing the geometric constants as they have been computed through BEM ( $I_t^P = 9.133\times 10^{-2} m^4$ , $I_t^S = 1.347\times 10^{-3} m^4$ , $C_S = 2.273\times 10^{-4} m^6$ ) are also presented. Significant discrepancies between the corresponding nonuniform torsional stress resultants obtained from the two methods are observed. Moreover, it is concluded that torsional loading is undertaken mainly through the primary torsional mechanism. However, the warping normal and shear stresses reach significant values locally [3], thus the secondary torsional mechanism and STMDE should not be neglected in stress analysis of bars of closed shaped thin walled cross sections.

***Example 2***

In the second example, a two member right angle frame previously studied by Tong et al. [13] has been analyzed in order to verify the accuracy of the results and demonstrate the range of applications of the developed method. The cross sections of both the beam and the column are thin walled I-shaped ones with total height $h = 0.300m$, total width $b = 0.150m$, flange width $t_f = 0.008m$ and web width $t_w = 0.006m$ ( $A = 4.104\times 10^{-3} m^2$, $A_y = 2.003\times 10^{-3} m^2$,

$A_z = 1.687\times 10^{-3} m^2$ , $I_{yy} = 6.262\times 10^{-5} m^2$ , $I_{zz} = 4.505\times 10^{-6} m^2$ , $I_t^P = 7.094\times 10^{-8} m^4$ , $I_t^S = 4.303\times 10^{-5} m^4$ , $C_S = 9.592\times 10^{-8} m^6$ ), while the joint of the intersecting members has a diagonal stiffener of width $t = 0.008m$ [13]. The right free end of the beam is loaded with a concentrated bimoment load, while the column bottom node is fully clamped. The beam and column have the same length of $l_b = l_c = 2.0m$ and are each modelled with one element, while general warping and bimoment continuity is considered at the joint.

| Node | $\theta_x\left(\times 10^{-3} rad\right)$ | | $\left(\theta_x^P\right)'\left(\times 10^{-3} m^{-1}\right)$ | |
|---|---|---|---|---|
| | Murín and Kutis [3] | Present - geometric constants with TTT | Murín and Kutis [3] | Present - geometric constants with TTT |
| 1 | 0 | 0 | 0.00185 | 0.001851 |
| 2 | 0.00412 | 0.00412 | $-4.801\times 10^{-6}$ | $-4.801\times 10^{-6}$ |
| 3 | 0 | 0 | -0.00195 | -0.001947 |
| 4 | -0.00721 | -0.00721 | $-4.20\times 10^{-6}$ | $-4.200\times 10^{-6}$ |
| 5 | 0 | 0 | 0.00102 | 0.001021 |
| 6 | 0 | 0 | $3.470\times 10^{-6}$ | $3.471\times 10^{-6}$ |

Table 1: Angle of twist and primary angle of twist per unit length of the member of example 1.

| Elem. side | $M_w\left(kNm^2\right)$ | | | $M_x^P\left(kNm\right)$ | | | $M_x^S\left(kNm\right)$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | Murín and Kutis [3] | Present - geom. const. with TTT | Present - geom. const. with BEM | Murín and Kutis [3] | Present - geom. const. with TTT | Present - geom. const. with BEM | Murín and Kutis [3] | Present - geom. const. with TTT | Present - geom. const. with BEM |
| 1-2 | 0.0 | 0.0 | 0.0 | 14.9824 | 14.9808 | 14.9780 | 0.01907 | 0.02062 | 0.02363 |
| 2-1 | 0.12335 | 0.12333 | 0.14506 | 14.8183 | 14.8184 | 14.7831 | 0.18309 | 0.18305 | 0.21845 |
| 2-3 | 0.12335 | 0.12333 | 0.14506 | -14.8164 | -14.8164 | -14.7809 | -0.1822 | -0.18217 | -0.21750 |
| 3-2 | 0.00571 | 0.00571 | 0.00644 | -14.9880 | -14.9864 | -14.9844 | -0.01059 | -0.01223 | -0.01402 |
| 3-4 | 0.00571 | 0.00571 | 0.00644 | -14.9816 | -14.9800 | -14.9767 | -0.01051 | -0.01215 | -0.01391 |
| 4-3 | -0.12335 | -0.12331 | -0.14504 | -14.8099 | -14.8100 | -14.7731 | -0.18218 | -0.18215 | -0.21746 |
| 4-5 | -0.12335 | -0.12331 | -0.14504 | 14.8248 | 14.8248 | 14.7907 | 0.18311 | 0.18307 | 0.21849 |
| 5-4 | 0.06125 | 0.06124 | 0.07210 | 14.9154 | 14.9146 | 14.8980 | 0.0925 | 0.09335 | 0.11136 |
| 5-6 | 0.06125 | 0.06124 | 0.07210 | 0.07511 | 0.07424 | 0.08899 | -0.09042 | -0.08955 | -0.10701 |
| 6-5 | 0.0 | 0.0 | 0.0 | -0.01482 | -0.01482 | -0.01749 | -0.00049 | -0.00049 | -0.00054 |

Table 2: Warping, primary and secondary twisting moments of the member of example 1.

At first, following the methodology of Tong et al. [13] the stiffness parameters of the link element modelling the joint are computed as $k_{w11} = k_{w22} = 50.780 knm^3$ , $k_{w21} = -47.857 knm^3$ . Subsequently, the structure is analyzed with beam elements takint into account or ignoring both shear and STMD effects. In Figs. 2 and 3, the angle of twist along the beam and the column, respectively, are presented as obtained from the proposed method, from the method of Tong et al. [13] where shear and STMD effects are ignored and from a FEM solution with shell elements reported in [13]. It is observed that the agreement between the corresponding results of the two methods based on beam elements is excellent, verifying the accuracy of the proposed method. Moreover, it is noted that the results of the present method taking into account or ignoring shear and STMD effects practically coincide, demonstrating that these effects are negligible in this example. As expected, it is also deduced that the present formulation is free from locking effects. The agreement of these results with the ones of the FEM solution with shell elements is noteworthy, concluding the reliability of the proposed method. Finally, a solution with symbolic algebraic manipulations has also been established in the examined problem yielding the same results.

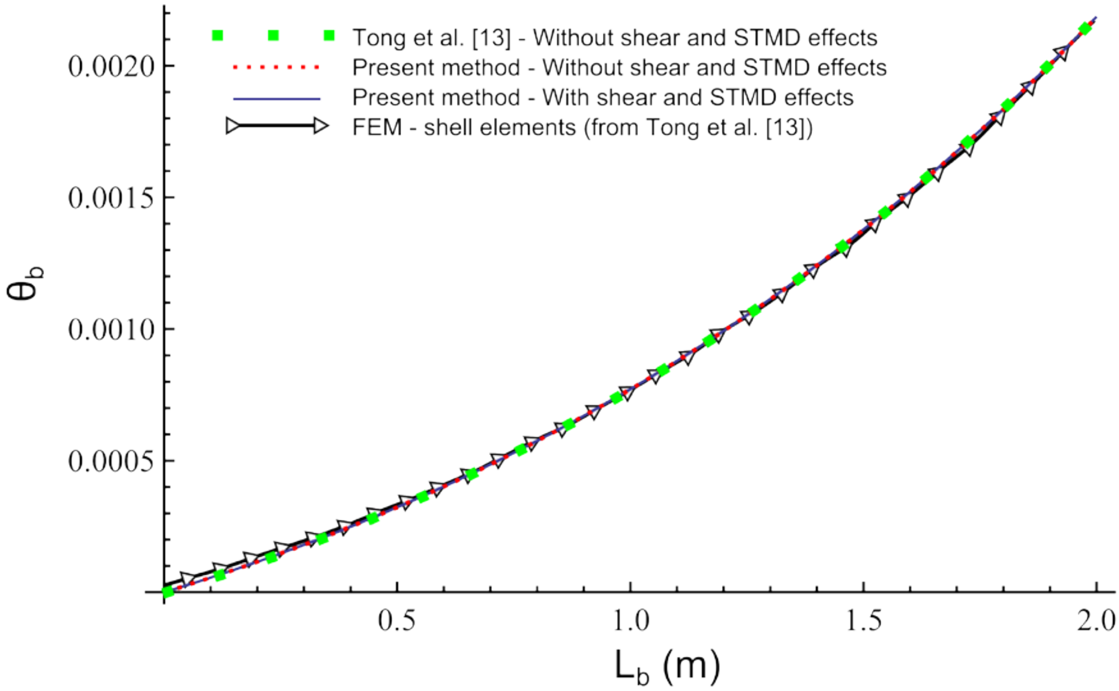


Figure 2: Angle of twist (rad) along the beam of the frame of example 2.

## 4 CONCLUSIONS

The main conclusions that can be drawn from this investigation are

- The proposed method may effectively analyze space beam assemblages and beams of arbitrarily shaped doubly symmetric cross section taking into account general warping and bimoment transmission conditions, shear and secondary torsional moment deformation effects.
- The employed local stiffness matrix is determined analytically through the solution of global differential equations of equilibrium resulting in a locking free element formulation.
- The developed computer program permits the solution of the problem through symbolic algebraic manipulations.
- The agreement of the developed procedure with previously reported results is noteworthy.
- The discrepancy between the torsional geometric constants computed through BEM and TTT affects notably the nonuniform torsional stress resultants of closed shaped thin walled cross section beams.

- The secondary torsional mechanism and secondary torsional moment deformation effects should not be neglected in stress analysis of beams of closed shaped thin walled cross sections.

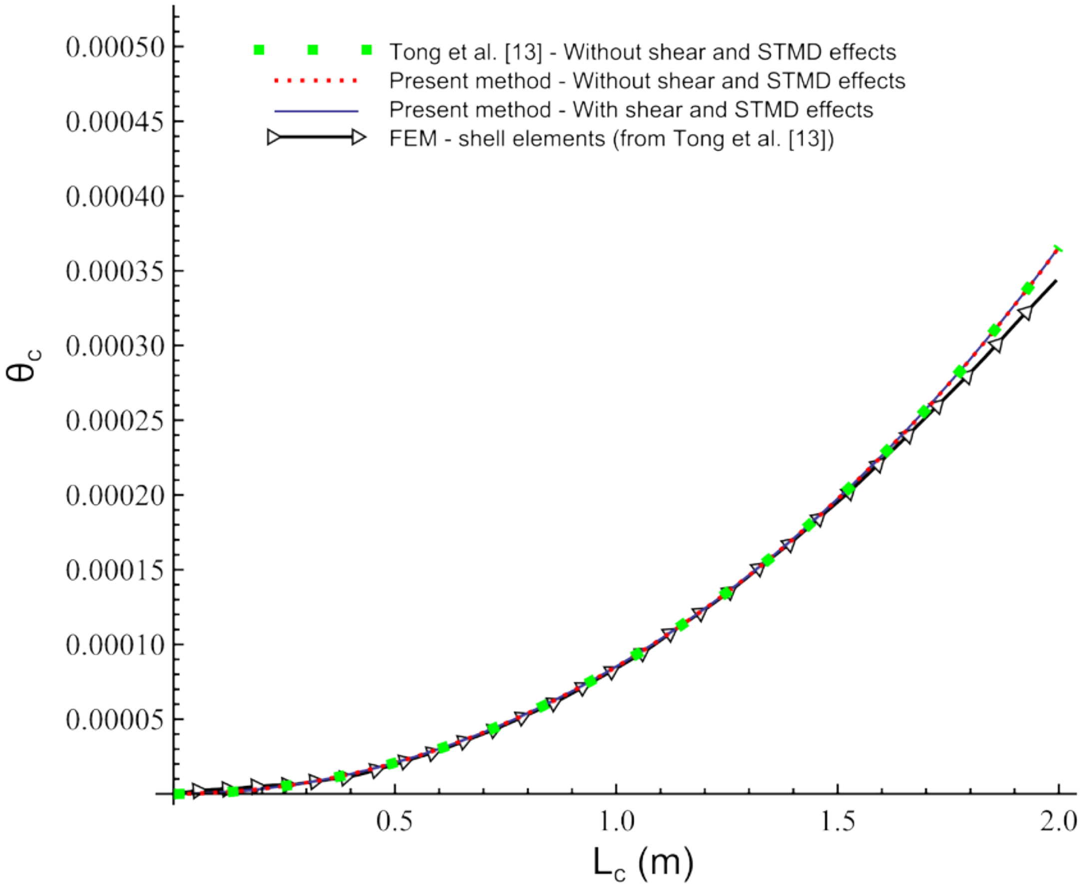


Figure 3: Angle of twist (rad) along the column of the frame of example 2.

## ACKNOWLEDGEMENTS

This research has been co-financed by the European Union (European Social Fund - ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: THALES: Reinforcement of the interdisciplinary and/or inter-institutional research and innovation.

## REFERENCES

[1] V.G. Mokos and E.J. Sapountzakis, Secondary torsional moment deformation effect by BEM. *International Journal of Mechanical Sciences*, **53**, 897-909, 2011.

[2] V.G. Mokos and E.J. Sapountzakis, A three-dimensional beam including torsional warping and shear deformation effects arising from shear forces and secondary torsional moments, *Proceedings of the 13th International Conference on Civil, Structural and Environmental Engineering Computing*, Chania, Crete, Greece, September 6-9, 2011.

[3] J. Murín and V. Kutis, An Effective Finite Element for Torsion of Constant Cross-Sections Including Warping With Secondary Torsion Moment Deformation Effect, *Engineering Structures*, **30**, 2716-2723, 2008.

[4] K.-J. Bathe, *Finite element procedures*, Prentice Hall, 1996.

[5] N. Kim and M. Kim, Exact dynamic/static stiffness matrices of non-symmetric thin-walled beams considering coupled shear deformation effects, *Thin-Walled Structures*, **43**, 701-734, 2005.

[6] C. Basaglia, D. Camotim and N. Silvestre, Torsion warping transmission at thin-walled frame joints: Kinematics, modelling and structural response, *Journal of Constructional Steel Research*, **69**, 39-53, 2012.

[7] P.J.B. Morrell, J.R. Riddington, F.A. Ali and H.A. Hamid, Influence of joint detail on the flexural/torsional interaction of thin-walled structures, *Thin-Walled Structures*, **24**, 97-111, 1996.

[8] S. Choi, G.-W. Jang and Y.Y. Kim, Exact matching condition at a joint of thin-walled box beams under out-of-plane bending and torsion, *Journal of Applied Mechanics, Transactions ASME*, **79**, art. no. 51018, 2012.

[9] I.J. MacPhedran, Frame stability considering member interaction and compatibility of warping deformations, PhD thesis, University of Alberta, Edmonton, Canada, 2009.

[10] A.H. Baigent and G.J. Hancock, Structural analysis of assemblages of thin-walled members, *Engineering Structures*, **4**, 207-216, 1982.

[11] S. Krenk and L. Damkilde, Warping of joints in I-beam assemblages, *Journal of Engineering Mechanics*, **117**, 2457-2474, 1991.

[12] Y.-B. Yang and W. McGuire, Procedure for analysing frames with partial warping restraint, *International Journal for Numerical Methods in Engineering*, **20**, 1377-1398, 1984.

[13] G.S. Tong, X.X. Yan and L. Zhang, Warping and bimoment transmission through diagonally stiffened beam-to-column joints, *Journal of Constructional Steel Research*, **61**, 749-763, 2005.

[14] P. Vacharajittiphan and N.S. Trahair, Warping and distortion at I-section joints, *ASCE J Struct Div*, **100** (ST3), 547-564, 1974.

[15] Mathematica 8, Wolfram, 2011

[16] F. Minghini, N. Tullini and F. Laudiero, Locking-free finite elements for shear deformable orthotropic thin-walled beams, *International Journal for Numerical Methods in Engineering*, **72**, 808-834, 2007.

[17] V.J. Tsipiras and E.J. Sapountzakis, Secondary Torsional Moment Deformation Effect in Inelastic Nonuniform Torsion of Bars of Doubly Symmetric Cross Section by BEM, *International Journal of Non-Linear Mechanics*, **47**, 68-84, 2012.

[18] S. Rao, *The finite element method in engineering*, *4th edition*, Elsevier Science & Technology books, 2004.