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THE CHOICE OF QUADRATURE IN NURBS-BASED ISOGEOMETRIC ANALYSIS

Francesco Calabrò[1] **and Carla Manni**[2]

[1]Dipartimento DIEI
Università di Cassino e del Lazio Meridionale, Italy
e-mail: calabro@unicas.it

[2] Dipartimento di Matematica
Università di Roma Tor Vergata, Italy
e-mail: manni@mat.uniroma2.it

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Abstract. *The construction of ad-hoc quadrature rules for Isogeometric Analysis is an issue that has been considered as the greater inter-element regularity can give considerable savings in terms of computational effort. In NURBS-based Isogeometric Analysis the request for the quadrature is to give exact result for piece-wise polynomials (on the elements) of degree* p *with* r *continuous derivates. A choice for the basis for this space are the B-splines, that are considered because of their properties related to support, refinability and regularity. In this work we intend to present some preliminary results on the use of global quadrature rules capable of exact results with respect to these basis functions, focusing on differences and potentiality.*

1 Introduction

The construction of ad-hoc quadrature rules for Isogeometric Analysis (IgA) is an issue that has been considered as the greater inter-element regularity can give considerable savings in terms of computational effort [1, 6].
In NURBS-based IgA the request for the quadrature is to give exact result for piece-wise polynomials of degree p with r continuous derivatives. A choice for the basis for this space are the B-splines, that are considered because of their properties related to support, refinability and regularity. In this work we intend to review the possible choices of quadrature rules when one intends to calculate integrals involving B-splines on the whole support of these functions for uniform knots sequences. This changes the usual prospective because actual software constructs the discrete problem element-by-element. In the authors' opinion what presented in [1, 6] gives a good way to treat cases of interest, while in this new prospective there are many roads to be explored, especially in the uniform case, where many properties can be used.
The results presented are preliminary, and consider only the calculation of integrals of the type

$$\int_0^{p+1} \mathcal{N}_p(x) f(x)\, dx \tag{1}$$

where $\mathcal{N}_p(x)$ is the cardinal B-spline of degree p. These integrals arise when the right hand side (the forcing term) is considered in IgA with maximal regularity ($r = p - 1$). All the others that have to be computed[1] are made with the translated copies of function $\mathcal{N}_p$ thus are of the same type.
Obviously, (1) are not the only integrals that have to be considered for the construction of the discrete counterpart of the differential problems, and the question related to the other cases such as Mass and Stiffness matrices will be considered in a forthcoming work.
Since now, the most common technique to calculate these integrals with a quadrature rule is to consider a proper composite Gauss formula: using the piece-wise polynomial expression of B-splines one can consider the Gauss quadrature that integrates exactly polynomials of degree p on each subinterval in order to gain exactness. This choice is not optimal from a computational point of view, but is in line with what is done in finite element software.
What we want to focus on is that many properties of the B-splines can be considered in order to construct quadrature formulae that achieve some exactness requirement. The intent of this paper is to report and compare some of these quadrature rules.
We begin with an introduction on B-splines and some of their properties. The construction of the different quadrature rules is outlined in Section 3. At last, we collect some conclusions.

2 Preliminars on B-splines

We will denote by $\mathcal{N}_p(x)$ the cardinal B-spline of degree p over the uniform knots sequence $\{0, 1, \ldots, p+1\}$ which is defined recursively as follows:

$$\mathcal{N}_0(x) := \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and:

$$\mathcal{N}_p(x) := \frac{x}{p}\mathcal{N}_{p-1}(x) + \frac{p+1-x}{p}\mathcal{N}_{p-1}(x-1)\,,\ \forall p \ge 1\,.$$

[1] We will consider here the case of periodic uniform knot vector.

Translated copies of cardinal B-splines are a basis for 0-periodic splines of degree p and regularity $p-1$. Among all the properties, we recall the following [3]:

$$\begin{aligned} \textit{Positivity:}\ & \mathcal{N}_p(x) \geq 0\ \forall x \in \mathbb{R}\,, \\ \textit{Minimal support:}\ & \mathcal{N}_p(x) = 0\ \forall x \notin [0, p+1]\,, \\ \textit{Symmetry:}\ & \mathcal{N}_p\left(\frac{p+1}{2} + x\right) = \mathcal{N}_p\left(\frac{p+1}{2} - x\right)\,. \end{aligned}$$

An other relevant property that we will use is the fact that B-splines solve a refinement equation. Define the vector $\mathbf{a}$ (called mask) as:

$$\mathbf{a} := \{a_j\,,\ j = 0, \ldots, p+1\}\,,\ a_j := \frac{1}{2^p}\left(\begin{array}{c} p+1 \\ j \end{array}\right)\,.$$

Then the cardinal B-spline is the only[2] function solution of the refinement equation:

$$\mathcal{N}_p(x) = \sum_j a_j\, \mathcal{N}_p(2x - j)\,. \tag{2}$$

In the next section we will use the following result on the product of B-splines (see Lemma 4 in [4]):

$$\int_{\mathbb{R}} \mathcal{N}_p(x)\mathcal{N}_q(x+k)\,dx = \mathcal{N}_{p+q+1}(p+1+k) = \mathcal{N}_{p+q+1}(q+1-k)\,. \tag{3}$$

3 Quadrature formulae

Given a generic function $\phi(x)$, an *n-point quadrature rule* is a choice of n ordered points and relative weights $(\xi_i, \omega_i)_{i=1,\ldots,n}$ such that

$$\int_a^b \phi(x)\,dx \simeq \sum_{i=1}^n \omega_i \phi(\xi_i).$$

The number of quadrature points n fixes the number of function evaluations needed in order to compute the value of the integral, thus is the parameter to be considered for the computational cost. A quadrature rule is said to be *exact* on the family of functions $\{\phi_j\}_{j=1,\ldots,M}$ if

$$\int_a^b \phi_j(x)\,dx = \sum_{i=1}^n \omega_i \phi_j(\xi_i) \qquad \forall j = 1, \ldots, M. \tag{4}$$

Conditions (4) represent M non-linear equations in the $2n$ unknowns (ξ_i, ω_i). In most cases the resolution of this system is ill-conditioned [8]. Classically, exactness is required on polynomials up to a fixed degree; in particular, the n-point Gauss rule is the (only) n-point rule that is exact for polynomials up to degree $2n-1$.
A notable case is when the exactness is required with test functions being of product type: $\phi_j(x) = \tilde{\phi}_j(x)\Phi(x)$. Then the required quadrature is made to be exact on test functions $\tilde{\phi}_j(x)$ with respect to a weighted measure $\Phi(x)dx$. In this case the theory on Gauss quadrature with

[2] All functions that solve a refinement equation are equal apart from a multiplicative constant: B-splines are obtained renormalizing the L^1 norm to be 1.

respect to a general measure can be applied. An example can be found in [5].
Recently, exactness on general families of functions has been also explored. In [8] conditions are given such that a quadrature rule that integrates exactly M independent test functions using $\lceil \frac{M}{2} \rceil$ points exists and a procedure in order to calculate it is presented. This formulae are named *Generalized Gaussian* with respect to the test functions, and are used in [1, 6] for the construction of rules exact on spline functions.
Varying the exactness requirements or fixing some properties on the distribution of nodes different quadrature rules can be derived. The aim of this section is to present some of these that seem well-fitted for the calculation of (1).
We emphasize that the requirement on the use of an optimal number of points has to be related to the possibility to reuse the function evaluations $\{f(\xi_i)\}$. This because the integrals of type (1) have to be calculated for each basis function that, as seen, are translated copies of the one in $[0, p+1]$. For this reason, if the quadrature points are translant-invariant on the unitary elements $[l, l+1]$ (or on multiples of these) then the function evaluations can be reused.

3.1 Composite Gauss rule

Call $(\xi_i^G, \omega_i^G)_{i=1,\dots,N_G}$ the quadrature rule that exactly integrates polynomials of degree p in $[0, 1]$ using the fewest number of nodes. Then one can cosider the following quadrature:

$$\int_0^{p+1} \mathcal{N}_p(x) f(x)\, dx = \sum_{l=0}^{p} \int_l^{l+1} \mathcal{N}_p(x) f(x)\, dx \approx$$
$$\approx \sum_{l=0}^{p} \sum_{i=1}^{N_G} \omega_i^G \mathcal{N}_p(\xi_i^G + l) f(\xi_i^G + l)\ .$$

This formula is exact only for constant $f(x)$ and uses $p \times N_G$ quadrature points; as seen in the previous section, $N_G = \lceil \frac{p+1}{2} \rceil$.

3.2 Half-point rule

In [6] it is proposed to use the rule $(\xi_i^{HP}, \omega_i^{HP})_{i=1,\dots,N_{OPT}}$ where $\omega_i^{HP} = 2$ and $\xi_i^{HP} = 2i$ if p is even, $\xi_i^{HP} = 1/2 + 2i$ if p is odd; $N_{OPT} = \lfloor \frac{p+1}{2} \rfloor$. The nodes are equispaced, not symmetric in $[0, p+1]$ and the weights are equal. This formula is exact for constant integrand functions f. In Table 1 we report some of the calculated rules. Last line is added for rapid comparison with what we will present in next sections.

	$p=2$	$p=3$	$p=4$
Nodes $\{\xi_i^{HP}\}$	$\{2\}$	$\{0.5, 2.5\}$	$\{2, 4\}$
Weights $\{\omega_i^{HP}\}$	$\{2\}$	$\{2, 2\}$	$\{2, 2\}$
B-spline values on nodes $\{\mathcal{N}_p(\xi_i^{HP})\}$	$\{1/2\}$	$\{1/48, 23/48\}$	$\{11/24, 1/24\}$

Table 1: Half-point rules.

When searching for quadrature with the requirement that it is exact only for constant functions one can also consider other simple rules. This choice was introduced in [6] because it turns to be optimal in terms of function evaluations when integration is made in an element-wise fashion.

	$p=3$	$p=4$
Nodes $\{\xi_i\}$	$\{1.05131670195, 2, 2.94868329805\}$	$\{1.427619470524, 2.5, 3.572380529476\}$
Weights $\{\omega_i\}$	$\{5/27, 17/27, 5/27\}$	$\{25/138, 44/69, 25/138\}$

Table 2: Weighed Gauss rule, $n = 3$.

3.3 Weighed Gauss rule

Following the approach in [5] one can search for a quadrature rule that is optimal with respect to the weighted measure $\mathcal{N}_p(x)dx$, solving exactness requirements (4) on monomials. This leads to a nonlinear problem that in general is difficult to solve. In the case of interest we can use the refinability property (2) so that the computation becomes stable. We refer to [5, 9] for the details on the construction of such rules. In Table 2 we report the calculated rules in the case $n = 3$, $p = 3, 4$. The nodes and the weights are symmetric in $[0, p+1]$ and all weights are positive. The formula, moreover, is exact for all f polynomial up to degree $2n-1$.

3.4 Fixed points rule

If preassigned samples of the integrand function f have to be used in order to calculate the integral (or some distribution of nodes has to be considered) the quadrature that attains the maximum polynomial degree of exactness is well known to be the interpolation-based one. In the case of integrals of the type (1) one can consider to use samples of function f and construct a quadrature to be used for integration with respect to the weighted measure $\mathcal{N}_p(x)dx$. The use of the refinement properties of the B-splines leads to a stable procedure for the calculation of these quadrature rules. We refer to [2] for the algorithmic details.
This procedure is very powerful in order to obtain exactness on polynomials maintaining the evaluations of the integrand function in points with requested properties, such as translate invariance and symmetry. In Table 3 we have reported the calculated quadrature in some cases for rapid comparison. In particular, the first two lines report the quadrature that give exactness on linear polynomials constructed on the same nodes considered by the Half-point rule seen before. The next two lines take in account all the points to be used by the formula when used on the translated intervals. The relation with the Half-point rule is evident comparing the calculated weights.

	$p=3$	$p=4$
Fixed Nodes $\{\xi_i\}$	$\{0.5, 2.5\}$	$\{2, 4\}$
Weights $\{\omega_i\}$	$\{1/4, 3/4\}$	$\{3/4, 1/4\}$
	$p=3$	$p=4$
Fixed Nodes $\{\xi_i\}$	$\{0.5, 1.5, 2.5, 3.5\}$	$\{1, 2, 3, 4\}$
Weights $\{\omega_i\}$	$\{1/48, 23/48, 23/48, 1/48\}$	$\{1/24, 11/24, 11/24, 1/24\}$

Table 3: Fixed points rules of optimal polynomial degree.

3.5 Quasi-Interpolant-based rules

A recent paper [10] has highlighted that in order to calculate the integrals arising in Isogeometric analysis, one can try to substitute the integrand function with a proper combination of function with the property that the integral of these new functions are known. This general approach is similar to the one that uses modified moments in order to calculate the quadrature rules. In the cited reference quasi-interpolation with B-splines is used and the approach is named *quadrature-free*: in this case we can use equation (3) seen in the previous section (see [4, Lemma 4]). The resulting formulae are very versatile and can gain exactness also in spline spaces [7, 11]. In general one can write:

$$\int_0^{p+1} \mathcal{N}_p(x) f(x)\, dx = \int_{-\infty}^{+\infty} \mathcal{N}_p(x) f(x)\, dx \approx$$
$$\int_{-\infty}^{+\infty} Q(f(x)) \mathcal{N}_p(x)\, dx = \int_{-\infty}^{+\infty} \sum_j \eta_j(f) \mathcal{N}_q(x+j) \mathcal{N}_p(x)\, dx =$$
$$\sum_j \eta_j(f) \int_{-\infty}^{+\infty} \mathcal{N}_q(x+j) \mathcal{N}_p(x)\, dx = \sum_j \eta_j(f) \mathcal{N}_{p+q+1}(p+1+j), \tag{5}$$

where we have denoted by $Q(f)$ a generic quasi-interpolation operator of f and then considered the particular choice of a projection on B-splines of degree q constructed from a local projector [7]:

$$Q(f(x)) := \sum_j \eta_j(f) \mathcal{N}_q(x+j)\,.$$

In this section we will fix the quadratic case $q = 2$. Then we can construct a projection using only points of the type $l/2\,,\ l = 0, \ldots, 2(p+1)$ obtaining a quadrature rule exact for f polynomial up to degree 2 and for B-splines of degree 2.
Fixed $q = 2\,,\ p \geq 2$, we have $j = -2, \ldots, p$ and the projection that gains the best properties of exactness and uses only values of the integrand function in points of the kind $l/2$ is (compare with [7]):

$$\eta_{-2}(f) = (5/2)f(0) - 2f(1/2) + (1/2)f(1);$$
$$\eta_j(f) = 1/2[-f(j+1) + 4f(j+1/2+1) - f(j+2)]\,,\ j = -1, 0, p-1;$$
$$\eta_p(f) = (5/2)f(p+1) - 2f(p+1/2) + (1/2)f(p) \tag{6}$$

With this choice of the projector, we can rearrange (5) in order to write the following the quadrature:

$$\int_0^{p+1} \mathcal{N}_p(x) f(x)\, dx \approx \left[\left(\frac{5}{2}\right) f(0) - 2f(1/2) + \left(\frac{1}{2}\right) f(1)\right] \mathcal{N}_{\tilde{p}}(p-1) +$$
$$\left[\left(\frac{1}{2}\right) f(p) - 2f(p+1/2) + \left(\frac{5}{2}\right) f(p+1)\right] \mathcal{N}_{\tilde{p}}(2p+1) +$$
$$\sum_{j=-1}^{p-1} \frac{1}{2} \left[-f(j+1) + 4f(j+1/2+1) - f(j+2)\right] \mathcal{N}_{\tilde{p}}(p+1+j) \tag{7}$$

where $\tilde{p} = p + 3$. In the case $p = q = 2$ the quadrature is reported in Table 3.5.

4 Conclusions

In this work we have considered some available choices for the computation of integrals arising in Isogeometric methods when considering the integrals on the support of the B-splines. From this preliminary work we can conclude the following.

Nodes $\{\xi_i\}$	$\{0, 0.5, 1, 1.5, 2, 2.5, 3\}$
Weights $\{\omega_i\}$	$\{-7/80, 5/12, -91/240, 11/10, -91/240, 5/12, -7/80\}$

Table 4: Quadrature rule based on the quasi-interpolant of equation (6) in the case $q = p = 2$.

- The use of quadrature on fixed nodes can give good properties both on the exactness and on the computational requirements. This is also empathized by the connection with the Half point rule, that is the optimal one introduced in the element-wise computations.

- The use of quasi-interpolant projector on B-splines can be very useful in order to get exactness on spline spaces. Moreover, the resulting quadrature rules are simple to compute, due to equation (3). From the point of view of the resulting quadrature rule, more analysis is needed in order to study -ad ex.- the quantity $\sum |\omega_i|$ that is crucial when convergence properties are considered.

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