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# CESSATION FLOWS OF BINGHAM PLASTICS WITH SLIP AT THE WALL

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**Abstract.** *We use finite elements in space and a fully implicit scheme in time in order to solve the cessation of axisymmetric Poiseuille flow of a Bingham plastic under the assumption that slip occurs along the wall. The Papanastasiou regularization of the constitutive equation is employed and a power-law expression is used to relate the wall shear stress to the slip velocity. The numerical results show that the velocity becomes and remains uniform before complete cessation and that the stopping time is finite only when the exponent* $s<1$. *In the case of Navier slip* ($s=1$)*, the stopping time is infinite for any non-zero Bingham number and the volumetric flow rate decays exponentially. When* $s>1$*, the decay is much slower, i.e. polynomial. The asymptotic expressions for the volumetric flow rate in the case of full-slip are also derived.*

## 1 INTRODUCTION

If the pressure-gradient applied in fully-developed Poiseuille flow of a Newtonian fluid is suddenly set to zero, the velocity decays to zero exponentially, i.e. the theoretical stopping time is infinite [1]. This is not the case for materials exhibiting yield stress, e.g. Bingham plastics. In fact, theoretical upper bounds on the stopping time have been derived [2, 3]. These bounds depend on the density, the viscosity, the yield stress, a new geometric constant, and the leading eigenvalue of the second-order linear differential operator for the interval under consideration. The above asymptotic bounds have been verified by means of numerical simulations. Chatzimina et al. [4] carried out finite element calculations with the regularized Papanastasiou model for Couette and Poiseuille flows of Bingham plastics. The numerical simulations showed in particular that the decay of the volumetric flow rate, which is exponential in the Newtonian case, is accelerated and eventually becomes linear as the yield stress is increased. More recently, the Poiseuille flows of ideal Bingham plastics have been solved numerically using the Augmented Lagrangian method [5].

The objective of the present work is to compute numerically the stopping times in the case of the cessation of the axisymmetric Poiseuille flow of a Bingham fluid allowing slip of the material along the wall (Fig. 1). However, in the discussion of the steady-state solutions we consider the more general Herschel-Bulkley model in which the stress tensor $\underline{\underline{\tau}}$ is given by

$$\begin{cases} \underline{\underline{\dot{\gamma}}} = \underline{\underline{0}}, & \tau \le \tau_0 \\ \underline{\underline{\tau}} = \left( \dfrac{\tau_0}{\dot{\gamma}} + \dot{\gamma}^{n-1} \right) \underline{\underline{\dot{\gamma}}}, & \tau \ge \tau_0 \end{cases} \tag{1}$$

where $\tau_0$ is the yield stress, $k$ is the consistency index, $n$ is the power-law exponent, and

$$\underline{\underline{\dot{\gamma}}} \equiv \nabla \underline{u} + (\nabla \underline{u})^T \tag{2}$$

is the rate of strain tensor, where $\underline{u}$ is the velocity vector and the superscript $T$ denotes the transpose. The magnitudes of $\underline{\underline{\dot{\gamma}}}$ and $\underline{\underline{\tau}}$, denoted respectively by $\dot{\gamma}$ and $\tau$, are defined by

$$\dot{\gamma} \equiv \sqrt{\frac{1}{2} \mathrm{II}_{\dot{\gamma}}} = \sqrt{\frac{1}{2} \underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}}} \quad \text{and} \quad \tau \equiv \sqrt{\frac{1}{2} II_{\tau}} = \sqrt{\frac{1}{2} \underline{\underline{\tau}} : \underline{\underline{\tau}}} \tag{3}$$

where the symbol $II$ stands for the second invariant of a tensor. The power-law fluid and the Bingham plastic are special cases of the Herschel-Bulkley fluid, recovered by setting $\tau_0$=0 and $n$=1, respectively.

In the present work, a generalized slip model is employed of the form

$$\begin{cases} u_w = 0, & \tau_w \le \tau_c \\ \tau_w = \tau_c + \beta \mathrm{u}_w^s, & \tau_w > \tau_c \end{cases} \tag{4}$$

where $\tau_w$ is the wall shear stress, $u_w$ is the slip or sliding velocity (i.e. the relative velocity of the fluid with respect to that of the wall), $\tau_c$ is the slip yield stress, $\beta$ is the slip coefficient and $s$ is the exponent. When $\tau_c$=0 and $s$=1 the classical Navier slip is recovered. When $\tau_c$=0, the no-slip and full-slip limiting cases are recovered when $\beta \to \infty$ and 0, respectively.

The slip yield stress $\tau_c$ may be smaller or greater than the yield stress $\tau_0$ depending on the material. For example, Seth et al. [6] reported that the slip yield stress is much lower in the case of concentrated suspensions of soft deformable particles.

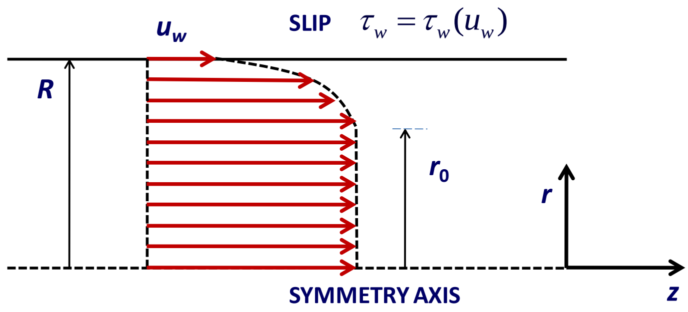



Figure 1: Axisymmetric Poiseuille flow of a Herschel-Bulkley fluid

## 2 GOVERNING EQUATIONS

We consider the laminar flow of a Herschel-Bulkley fluid in a tube of radius $R$, as shown in Fig. 1. The constitutive equation of the material is simplified as follows:

$$\begin{cases} \dfrac{\partial u_z}{\partial r} = 0, & |\tau_{rz}| \le \tau_0 \\ \tau_{rz} = -\tau_0 - k\left(-\dfrac{\partial u_z}{\partial r}\right)^n, & |\tau_{rz}| \ge \tau_0 \end{cases} \tag{5}$$

where $\tau_{rz}$ is the shear stress and $u_z$ is the axial velocity.

Under the assumptions of unidirectionality and zero gravity, the $z$-momentum equation for any fluid becomes

$$\rho \frac{\partial u_z}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) \tag{6}$$

where $\rho$ is the density.

To dedimensionalize the governing equations we scale lengths by the tube radius, $R$, the velocity by the mean steady-state velocity, $V$, in the capillary, the pressure and the stress components by $kV^n/R^n$, and time by $\rho R^{n+1}/(kV^{n-1})$, where $\rho$ is the constant density of the fluid. With these scalings the dimensionless forms of Eqs. (5) and (6) are as follows:

$$\begin{cases} \dfrac{\partial u_z}{\partial r} = 0, & |\tau_{rz}| \le Bn \\ \tau_{rz} = -Bn - \left(-\dfrac{\partial u_z}{\partial r}\right)^n, & |\tau_{rz}| \ge Bn \end{cases} \tag{7}$$

and

$$\frac{\partial u_z}{\partial t} = G + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) \tag{8}$$

where all variables are now dimensionless (for simplicity, the same symbols are used),

$$Bn \equiv \frac{\tau_0 R^n}{kV^n} \tag{9}$$

is the Bingham number, and $G$ is the dimensionless pressure gradient.

The dimensionless form of the slip equation is

$$\begin{cases} u_w = 0, & \tau_w \le B_c \\ \tau_w = B_c + B u_w^s, & \tau_w > B_c \end{cases} \tag{10}$$

where

$$B_c \equiv \frac{\tau_c R^n}{k V^n} \tag{11}$$

is the slip yield stress number and

$$B \equiv \frac{\beta R^n}{k V^{n-s}} \tag{12}$$

is the usual slip number. (The no-slip and full-slip limiting cases are recovered when $B \to \infty$ and 0, respectively.) In the special case $\tau_c = \tau_0$, the slip yield stress number, $B_c$, coincides with the Bingham number, *Bn*.

## 3 STATE –STATE HERSCHEL-BULKLEY FLOWS WITH SLIP

The solution of the steady, incompressible Poiseuille flow of a Herschel-Bulkley fluid in the special case of Navier slip ($B_c$=0, $s$=1) has been provided under different forms by Kalyon et al. [7]. Taking into account the slip yield stress, leads to some interesting flow regimes, which are discussed below.

In steady-state, the dimensionless wall shear stress for any generalized Newtonian fluid is $\tau_w = G/2$. Depending on the value of the applied pressure gradient, $G$, and the relative values of $B_c$ and $Bn$, there are four different possibilities:

(i) If $G \le \min\{2B_c, 2Bn\}$, then no flow occurs.

(ii) If $G \ge \max\{2B_c, 2Bn\}$, then we have non-uniform flow with wall slip and yielded/unyielded regions in the flow domain. The dimensionless velocity profile is given by

$$u_z(r) = u_w + \frac{n}{2^{1/n}(n+1)} G^{1/n} \begin{cases} (1-r_0)^{1/n+1}, & 0 \le r \le r_0 \\ \left[(1-r_0)^{1/n+1} - (r-r_0)^{1/n+1}\right], & r_0 \le r \le 1 \end{cases} \tag{13}$$

where

$$u_w = \left(\frac{G-2B_c}{2B}\right)^{1/s} \tag{14}$$

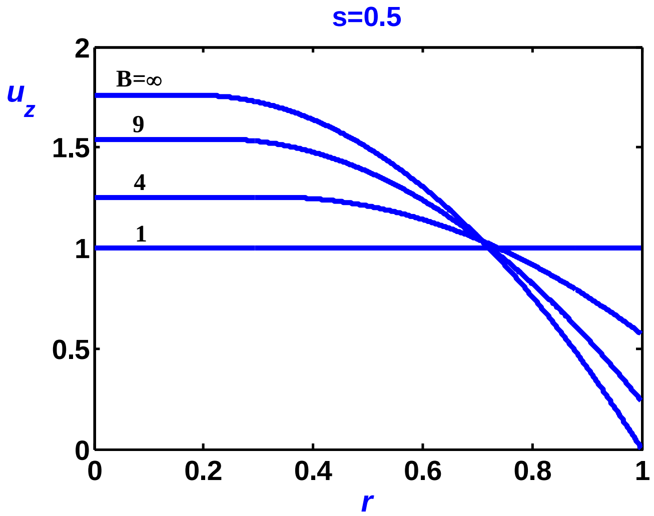
and

$$r_0 = \frac{2Bn}{G} \le 1 \tag{15}$$

is the yield point. The dimensionless pressure-gradient is a solution of the following equation:

$$2^{1/n}\frac{3n+1}{n}(1-u_w)G^3 = (G-2Bn)^{1/n+1}\left[G^2 + \frac{4nBn}{2n+1}G + \frac{8n^2Bn^2}{(n+1)(2n+1)}\right] \tag{16}$$

(iii) If $2Bn < G < 2B_c$, then the fluid flows without slip ($u_w$=0) with distinct yielded and unyielded regions. Equations (10) and (13) still apply.

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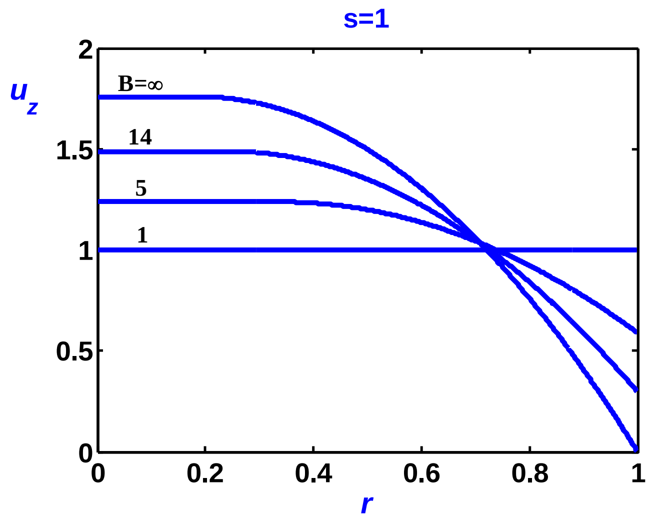


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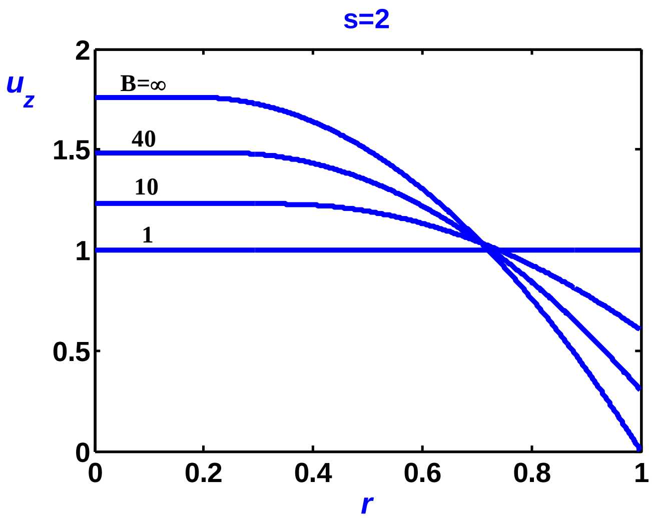



(c)

Figure 2: Velocity profiles in axisymmetric Poiseuille flow of a Bingham plastic ($Bn$=1) with zero slip yield stress ($B_c$= 0) and different values of the slip number: (a) $s$=0.5, (b) $s$=1, and (c) $s$=2.

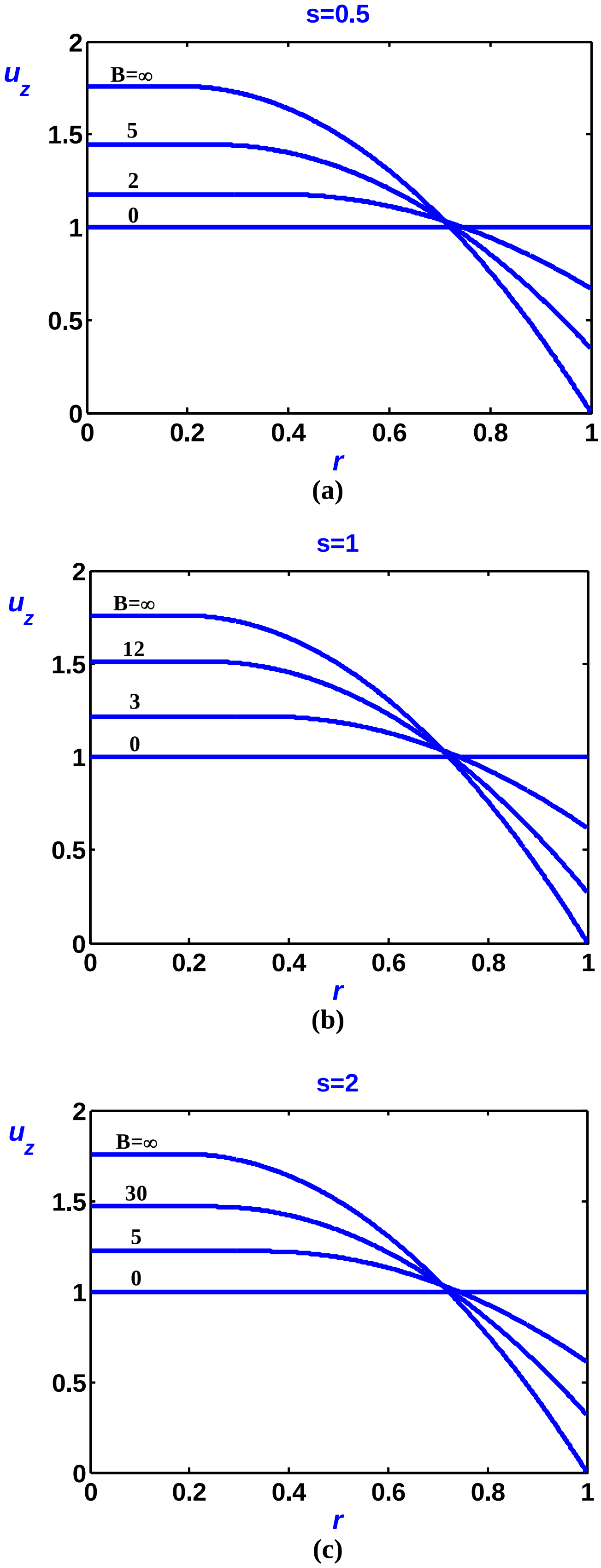



Figure 3: Velocity profiles in axisymmetric Poiseuille flow of a Bingham plastic with $Bn= B_c=1$ and different values of the slip number: (a) $s$=0.5, (b) $s$=1, and (c) $s$=2.

(iv) Finally, if $2B_c < G < 2Bn$, the fluid is unyielded everywhere in the flow domain, sliding with unit velocity.

When no-slip is applied, $r_0$ tends to unity asymptotically as $Bn$ goes to infinity. Otherwise, it is deduced from Eq. (16) that a flat velocity profile ($u_x$=$u_w$=1) is attained when $G$=2$Bn$>2$B_c$. In the Newtonian case ($Bn$=0), the velocity tends to a plug profile ($u_x$=$u_w$=1) in the limit of zero $B$ (full slip). Interestingly, with viscoplastic fluids, the plug velocity profile is attained at a finite non-zero value of $B$ at which the yield distance $r_0$ becomes 1. The critical slip number for this extreme case is obtained from Eq. (14):

$$B_{crit} = Bn - B_c \tag{17}$$

Hence, for given $Bn$ and $B_c$, solutions for slip numbers below $B_{crit}$ are not admissible. Similarly, when $B$ and $B_c$ are given, there exists a critical upper bound for the Bingham number, $Bn_{crit}$=$B$+$B_c$, which cannot be exceeded. At $Bn_{crit}$ both the yield distance and the slip velocity become 1. This implies that the flow becomes plug at a critical wall shear stress, which is consistent with experimental observations on highly filled suspensions [8].

The case of Herschel-Bulkley flow with Navier slip ($B_c$=0 and $s$=1) has been discussed in detail by Damianou et al. [9], who, however, employed a different definition of the slip number ($A_1$ in their paper corresponds to 1/(2$B$)). Here we consider first Bingham flow ($n$=1) with zero slip yield stress ($B_c$=0).

The effect of the exponent $s$ when $B_c$ is still zero is illustrated in Fig. 2, where we plot velocity profiles for Bingham flow ($n$=1) with $Bn$=1 and various slip numbers. The velocity becomes flat at the critical slip number $B_{crit}$=$Bn$.

Let us now focus on the special case when $Bn$=$B_c$, in which only the flow regimes (i) and (ii) are possible. In other words, slip occurs as soon as flow occurs and there are no bounds for the slip and Bingham numbers. For simplicity, we once again restrict ourselves to the Bingham-plastic case ($n$=1).

The effect of the exponent $s$ for $Bn$=$B_c$=1 is illustrated in Fig. 3. The critical value of the slip number for the velocity to become flat is $B_{crit}$=0. As expected, slip effects are delayed, i.e., compared to the $B_c$=0 case, similar changes in the velocity profile are observed at lower values of $B$.

## 4 TIME-DEPENDENT BINGHAM FLOWS WITH SLIP

The time-dependent Bingham plastic flow is solved numerically, since it is not amenable to analytical solution. Instead of the ideal Bingham model (5), in our numerical simulations we use the regularized constitutive equation proposed by Papanastasiou [10], which in dimensionless form is given by

$$\tau_{rz} = \left\{ \frac{Bn\left[1-\exp\left(-M\ \dot{\gamma}\right)\right]}{\dot{\gamma}} + 1 \right\} \frac{\partial u_z}{\partial r} \tag{18}$$

where $\dot{\gamma}$ =$|\partial u_z/\partial r|$, while the stress growth exponent $M$ is given by

$$M \equiv \frac{mV}{H} \tag{19}$$

where $m$ is a stress growth exponent.

To solve the problem we used 100 quadratic elements in space and finite differences in time. For the time discretization, we used the standard fully implicit Euler backwarddifference scheme with a dimensionless time step $dt \leq 10^{-4}$. The criteria for convergence of the system of equations were that the norm of the error for the velocities and the norm of the residuals were both less than $10^{-4}$.

The general stopping criterion for the "numerical cessation" of the flow was that the norm of the dimensionless deceleration was less than a small number $\varepsilon$. Here $\varepsilon = (\rho H^2/\mu V)10^{-3}$, where $\mu V/\rho H^2$ is the *characteristic deceleration*. The total dimensionless time, $T_f$, found with the above criterion and with constant $dt = 10^{-4}$, is given in all figures where transient solutions are shown.

As discussed in Section 3, in the case of Bingham flow, steady-state plug velocity profiles are admissible when Navier slip is allowed [9]. The critical slip number for attaining a uniform steady-state velocity profile in axisymmetric Poiseuille flow is $B_{\text{crit}}=Bn\text{-}B_c$.

### 4.1 Results for zero slip yield stress

The numerical simulations of the cessation flow show that the velocity becomes and remains uniform before complete cessation. The evolution of the velocity when $Bn$=1, $B$=5 and $s$=1 is illustrated in Fig. 4. Obviously, as the slip number is increased, the initial velocity profile becomes more flat and a uniform profile is attained earlier during cessation.

Figure 5 shows the evolution of the volumetric flow rate $Q$ for $Bn$=1 and different slip numbers for three representative values of the exponent $s$: 0.5, 1, and 2. One observes that the stopping time is finite only in the case of no slip ($B\rightarrow\infty$) when $s$<1. When $s$ is fixed, the stopping time increases with the slip coefficient. In the case of Navier slip ($s$=1), the stopping time is infinite for any non-zero Bingham number and the volumetric flow rate decays exponentially. When $s$>1, the decay is much slower.

Let $t_c$ denote the critical time at which the velocity becomes uniform, i.e. $u_z = u_w(t_c) = u_{wc}$. Integrating the momentum equation (8) over the tube cross-section leads to the following ODE:

$$\frac{du_w}{dt} = -2\tau_w = -2Bu_w^s \tag{20}$$

Therefore, when $s$=1, the velocity decays exponentially:

$$u_z = u_{wc}\exp[-2B(t-t_c)], \qquad t > t_c \tag{21}$$

where $u_{wc} = u_w(t_c)$. Otherwise,

$$u_z = u_{wc}[1-2(1-s)B(t-t_c)]^{1/(1-s)}, \qquad t > t_c \tag{22}$$

It is clear that the stopping time is finite only if $s$<1. The asymptotic estimate of the stopping time is then

$$t_s = t_c + \frac{1}{2(1-s)B} \tag{23}$$

In Figure 6, we compare the slip velocities predicted using Eqs. (21) and (22) with the numerical solutions taking $Bn$=1 and $B$=1, considering three different values for s (0.1, 1, and 2), and assuming that $u_{wc}$=1 and $t_c$=0. Given that $B$ is close to $Bn$, the numerical results practically coincide with the theoretical estimates.

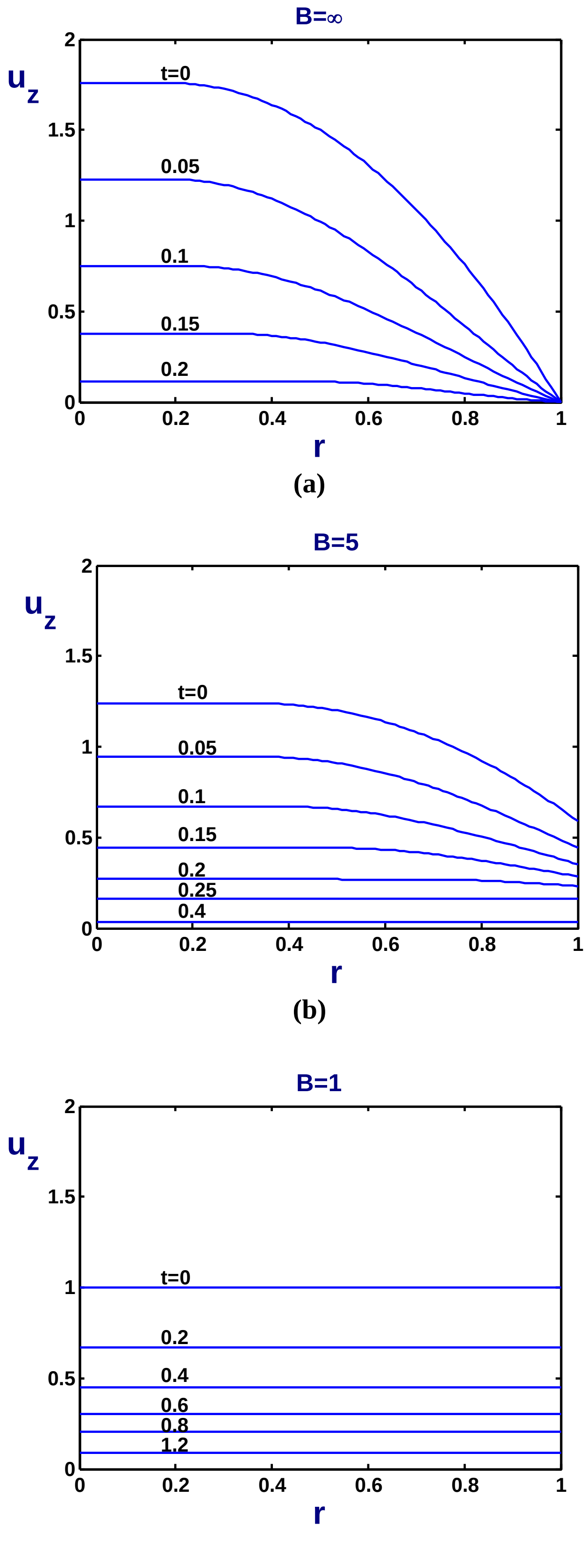
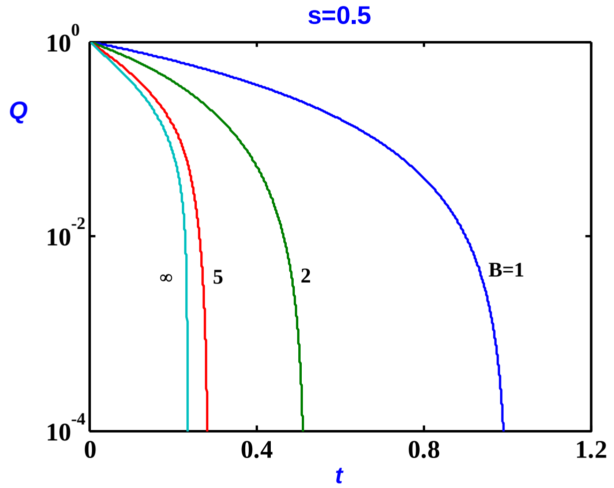



Figure 4: Evolution of the velocity in cessation flow of a Bingham plastic with $Bn$=1, $s$=1 and zero slip yield stress ($B_c$=0): (a) $B$=∞ (no slip); (b) $B$=5; and c) $B$=1 (plug flow).

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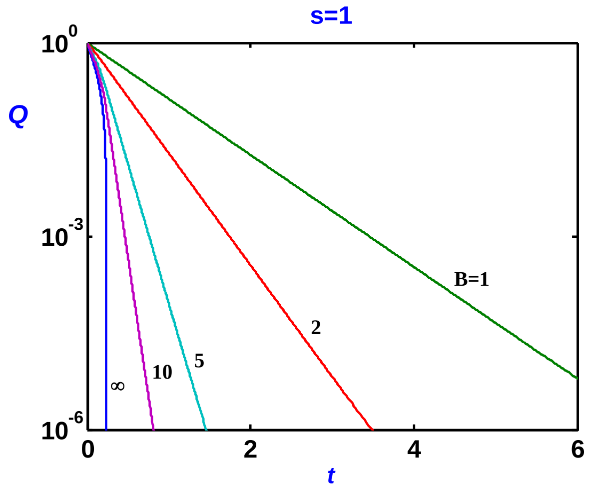


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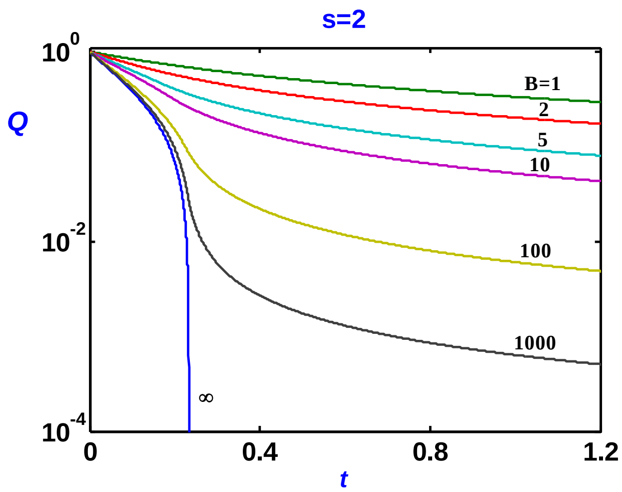


(c)

Figure 5: Evolution of the volumetric flow rate in cessation flow of a Bingham plastic with *Bn*=1, zero slip yield stress ($B_c$=0) and various slip numbers: (a) *s*=0.5; (b) *s*=1; and c) *s*=2.

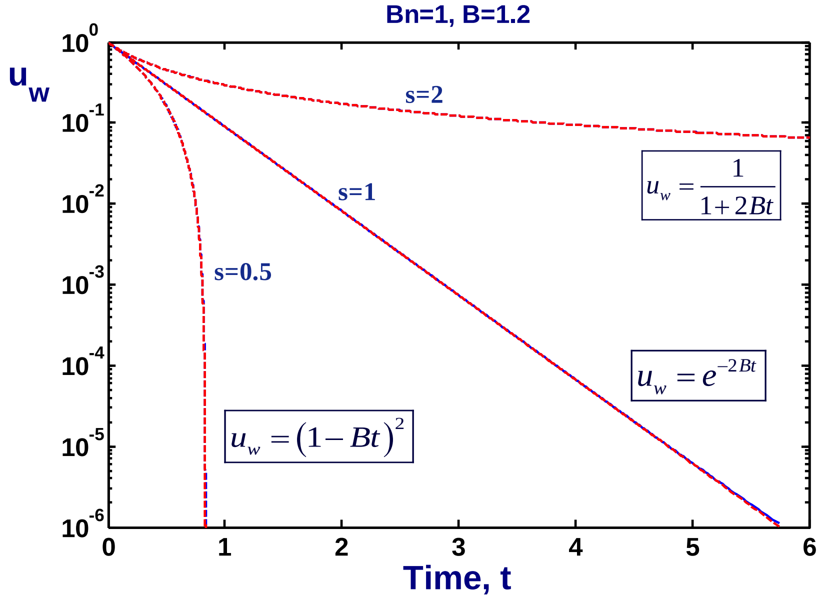


Figure 6: Evolution of the slip velocity in cessation flow of a Bingham plastic with *Bn*=1, B=1, and zero slip yield stress ($B_c$=0): (a) *s*=0.5; (b) *s*=1; and c) *s*=2. The predictions of Eq. (36) taking $u_{wc}$=1 and $t_c$=0 essentially coincide with the numerical solution.

## 5 CONCLUSIONS

We have solved the cessation of axisymmetric Poiseuille flow of a Bingham plastic with wall slip employing a power law slip equation and the Papanastasiou regularization for the constitutive equation. The numerical results for zero slip yield stress show that in this case the velocity becomes and remains uniform before complete cessation. Moreover, the stopping time is finite only when the slip equation exponent *s*<1. The decay of the volumetric flow rate is exponential when *s*=1, and much slower when *s*>1. As for the future plans, we are currently exploring the effect of a non-zero slip yield stress.

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