SEECCM III
3rd South-East European Conference on Computational Mechanics-
*an ECCOMAS and IACM Special Interest Conference*
M. Papadrakakis, M. Kojic, I. Tuncer (eds.)
Kos Island, Greece, 12–14 June 2013

# A METRIC THEORY OF RATE INDEPENDENT AND RATE DEPENDENT PLASTICITY: THEORETICAL AND COMPUTATIONAL ASPECTS

**Vassilis P. Panoskaltsis**[1] and **Dimitris Soldatos**[1]

[1]Department of Civil Engineering, Demokritos University of Thrace,
12 Vassilissis Sofias Street, Xanthi 67100, GREECE
email: vpanoska@civil.duth.gr

**Keywords:** Internal Variables, Finite Deformations, Characteristic Times, Generalized Plasticity, Viscoplasticity, Covariance, Material Metric, Lie Derivative, Loading Unloading Conditions, Time Discretization, Backward Euler, Predictor Corrector Algorithm.

**Abstract.** *A new internal variable theory for the description of solid materials having mechanisms with different characteristic times, is developed within a finite deformation framework. The theory relies crucially on the consistent combination of a general viscoplastic theory and a new version of rate – independent generalized plasticity theory. In this new version of rate – independent generalized plasticity the concepts of viscous range and viscous process have been introduced, while, as in classical generalized plasticity, the notion of yield surface, as a basic ingredient, is not involved. The formulation is developed initially in a material setting and then is extended to a covariant one by applying some basic elements and results from the tensor analysis on manifolds. By introducing the material (intrinsic) metric as a primary internal variable, accounting for both rate dependent and rate independent phenomena within the body, a constitutive model is proposed. The ability of the model in simulating several patterns of the complex response of metals under quasi – static and dynamic loadings is assessed by representative numerical examples, after appropriate approximations of the Lie derivatives of tensorial quantities have been derived.*

## 1 INTRODUCTION

The first author of this publication has gotten interested earlier (in the nineties) in the important subject of modeling materials with different characteristic times, namely with characteristic times very short and of the same order compared to a loading process. The first type of mechanisms gives rise to instantaneous plastic strains and the second type to creep strains, which are developed slowly. He and his co-workers have developed models for the description of the behavior of concrete materials by combining in series internal variable theories of viscoelasticity and plasticity, Panoskaltsis et al. [1, 2, 3, 4, 5]. In the first four publications a plasticity model is combined with a viscoelasticity one, while in the fifth generalized plasticity formulations are combined with viscoplasticity ones. All these models were developed within the realm of small strains.

It should be mentioned that it came as a surprise to this author that very few works had been published on this subject, earlier than his. These are works that have been published in the early sixties (Landau et al. [6], Ivlev [7] and Naghdi and Murch [8]). In those publications, several combinations of a rate – dependent theory with a rate – independent one have been proposed, again within the context of infinitesimal deformation.

Nevertheless, the mechanisms responsible for instantaneous plastic strains as well as viscoelastic strains are usually introducing large deformations and can be found in different classes of materials such as metals (e.g., metal forming processes, high – velocity impact, penetration mechanics), shape memory alloys (e.g., finite deformations occurring during phase transformations) and soils (e.g., liquefaction and cyclic mobility in sands). Therefore, a more general development within the context of a large deformation theory, is definitely needed. This is the reason for revisiting this subject. Our new development includes the following basic characteristics:

*(a) A local internal variable theory presented in a covariant setting.* The internal variable vector is assumed to consist of two components. The first of them, which is related to the *rate – dependent* properties of the material, is described by a general viscoplastic formulation which leaves the kinematics of the problem and the number and the nature of the internal variables entirely unspecified. In this sense, the formulation includes both, a number of models which have been developed in order to describe time dependent phenomena in metals and in particular highly nonlinear viscoelastic behavior (e.g., Bodner and Partom [9], Rubin [10, 11, 12]), as well as, the classical overstress models (e.g., Malvern [13], Perzyna, [14, 15], Phillips and Wu [16], Chaboche [17]). The second component, which is related to the *rate – independent* material properties is described by the framework of generalized plasticity (Lubliner [18, 19, 20]) which includes *classical plasticity as a special case* (e.g., see Lubliner [20], Panoskaltsis et al. [21, 22]). The crucial advantage of this approach resides in the compatibility of the two theories, in the sense that neither viscoplasticity nor generalized plasticity employs the concept of the yield surface as a basic ingredient. Unlike the classical formulations of viscoplasticity and rate – independent plasticity, which usually are developed in the stress space, the proposed formulation is developed in the strain (deformation) space. Such formulation, besides inheriting the advantages of the strain space over the stress space (Naghdi [23]), seems more essential for a *covariant formulation* since it employs as a control variable the right Cauchy – Green deformation tensor, which is defined as the pull – back of the spatial metric by the deformation. As in our previous approaches (see Panoskaltsis et al. [22, 24, 25]) within the context of the rate – independent theory, the covariant formulation is achieved naturally by introducing manifold spaces (see, for instance, Bishop and Goldberg [26], Lovelock

and Rund [27]), not only for the body of interest and the ambient space, but also for the state space, that is the set of all realizable states over a material point. Accordingly, the motion of the body which is considered as a time dependent mapping within the ambient space, is extended to a local dynamical process by considering the state space as a fiber over the body particles (e.g., see Panoskaltsis et al. [22, 24, 25]). In turn, the involvement of the standard pull – back/push – forward operations of the tensor analysis on manifolds (e.g., Marsden and Hughes [28, p.67], Stumpf and Hoppe [29]) leads to the introduction of the convected Lie derivative (e.g., see Marsden and Hughes [28, p. 95], Stumpf and Hoppe [29]), which eventually leads to a covariant formulation of the theory.

*(b) A constitutive model which is based on the concept of the "physical" metric.* This is a rather new and powerful concept that has been introduced in the phenomenological description of solid materials by Valanis [30] and was extended by Valanis and Panoskaltsis [31] (see also Panoskaltsis et al. [22, 25]). According to this theory, the intrinsic material ("physical") metric, namely the (body) metric in the material configuration, can be considered as a *basic internal variable*, modeling non – affine deformation. In his original work Valanis [30], by considering the physical metric's time derivative with respect to either the Newtonian time or the intrinsic time, derived hereditary constitutive equations for viscoelastic and plastic solids, respectively. Within the present study the concept is revisited and it is used for the construction of a material model. This model is based on a hyperelastic extension of a $J_2$ flow theory to the finite deformation regime. In particular, it is shown how the "physical" metric can be used as a measure of viscoplastic deformation which is due to mechanisms with different characteristic times within the material substructure. The proposed model belongs to the class of *multi – mechanism models*, which constitutes at the present time a very active area of research (e.g., see Taleb and Cailletaud [32], Sai [33]). This model is based on the following basic ingredients:

i. A hyperelastic constitutive equation for the characterization of the stress response.
ii. A von – Mises type of expression, with isotropic hardening, for the (suitably defined) yield surface.
iii. A normality flow rule in terms of the material metric.

The ability of the proposed model in simulating several patterns of the extremely complex response of metals under quasi – static and dynamic loadings is assessed by representative numerical examples.

## 2 DEVELOPMENT OF CONSTITUTIVE THEORY

We follow the geometrical approach of Marsden and Hughes [28], proposed within the context of non – linear elasticity (see also Stumpf and Hoppe [29]). Approaches of this type have been also considered within the context of isothermal formulations of classical plasticity by Simo [34], Miehe [35], Panoskaltsis et al. [22, 24, 25], and within a non – isothermal framework by Duszek and Perzyna [36] and Le and Stumpf [37]. Accordingly, we consider both the body of interest and the ambient space *S*, as three dimensional Riemannian manifolds and we denote by *B* the reference configuration of the body with points labeled by $\mathbf{X}(X^1, X^2, X^3)$. A motion of *B* within the ambient space is defined as a time dependent mapping $\mathbf{x}$: $B \rightarrow S$ given as

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \tag{1}$$

which maps the points of the reference configuration *B* onto the points $\mathbf{x}(x^1, x^2, x^3)$ of the ambient space *S*. The deformation gradient is defined to be the tangent map of the motion, i.e.,

$$\mathbf{F}(\mathbf{X},\mathrm{t}) = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \tag{2}$$

with determinant $J = \det\mathbf{F}(\mathbf{X}, t) > 0$.
In the absence of thermal effects, the material state at the referential point **X** may be determined by the right Cauchy – Green deformation tensor defined as the pull – back of the spatial metric **g,** by the motion (1), that is

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{g}\mathbf{F}, \tag{3}$$

which is assumed to lie on a manifold *C*, and the internal variable vector **Q**, which in turn is assumed to lie on another manifold *Q*. The *state space,* that is the product manifold $D = C \times Q$ is assumed to be attached at the point **X** so that the set $\{\mathbf{X}\} \times D$ is a fiber of **X**. Furthermore, we note that the set $\{\mathbf{X}\} \times D$ is an open subset of $B \times D$ and so it is a local manifold. Since we study materials *with both rate – dependent and rate – independent characteristics* we assume that the internal variable vector be composed of two internal variable vectors **V** and **P**, which belong to the manifolds *V* and *P* respectively; the former of them is associated with the material rate – dependent properties while the latter is associated with the material rate – independent ones.

A *local process* $\mathbf{\Psi}$ in the state space *D* is defined as a curve in *D*, i.e., as a mapping from the time interval of interest *I* to the state space *D,* i.e.,

$$\mathbf{\Psi} : I \in R \to D,$$

defined as

$$\mathbf{\Psi}(t) = \begin{bmatrix} \mathbf{C}(t) \\ \mathbf{V}(t) \\ \mathbf{P}(t) \end{bmatrix}.$$

The direction and the speed of such a process are determined by the tangent vector $\dot{\mathbf{\Psi}}(t) = \begin{bmatrix} \dot{\mathbf{C}}(t) \\ \dot{\mathbf{V}}(t) \\ \dot{\mathbf{P}}(t) \end{bmatrix}$, where the superimposed dot stands for the time derivative.

Since, the component $\dot{\mathbf{C}}$ of $\dot{\mathbf{\Psi}}$, is always known under deformation control, the components $\dot{\mathbf{V}}$ and $\dot{\mathbf{P}}$ have to be determined. By considering just a first order differential equation, that is by imposing some limitations concerning the memory of the material, the rate – dependent component of the internal variable vector may be assumed to be given as

$$\dot{\mathbf{V}} = \mathbf{A}\left(\mathbf{C},\mathbf{V}\right), \tag{4}$$

where $\mathbf{A}: C\times V \to \mathrm{T}D$ is a vector field to be interpreted as a tensorial function of the denoted arguments, associated with the rate – dependent material properties. A local process may be defined as *elastic* if it lies entirely in a six dimensional submanifold of *D* (actually the deformation space *C*) defined by **V=**const., otherwise is defined as *inelastic*. Accordingly, an *elastic domain* (at **V**=const.) may be defined as a submanifold of the deformation space *C* comprising the deformation points which can be reached by an elastic process from the current deformation point. If the elastic domain has a non empty interior, its boundary is a five dimensional manifold, the points of which have a coordinate neighborhood on it, which is attached to the interior in much the same way as the surface of a solid is attached to its interior. The latter manifold may be defined as a *yield surface*. We are now stating the following important remarks, that underscore our theory which is developed here.

*REMARK 2.1.* The *existence of a yield surface* classifies the viscoplastic models frequently met in the literature (e.g., see the review paper by Naghdi [23]) into two major categories as follows: In view of Equation (4) one can conclude that an elastic domain can be defined at any material state in $C\times V$, by the equation

$$\mathbf{A}\left(\mathbf{C},\mathbf{V}\right) = \mathbf{0}.$$

The case in which the function **A** is *a* non – vanishing function of its arguments corresponds to a viscoplastic model, where an elastic domain does not exist and every process results in inelastic behavior at any deformation level, no matter how small it is. A viscoplastic model of this type is termed as a "unified" constitutive model (e.g., Bodner and Partom [9], Rubin [10, 11, 12]). On the other hand, if it is assumed that a yield (hyper) surface exists and is given by an expression of the form

$$G(\mathbf{C},\mathbf{V}) = 0,$$

where $G\colon C\times V \to R$ is a scalar function, then by considering for the function **A** an expression of the form

$$\mathbf{A} = \frac{1}{\eta}\left\langle \Gamma(G)\right\rangle \mathbf{B}(\mathbf{C},\mathbf{V}),$$

where the function $\left\langle \Gamma(G)\right\rangle$ is defined as

$$\left\langle \Gamma(G)\right\rangle = \begin{cases} 0 \text{ for } G\leq 0 \\ \Gamma(G) \text{ for } G>0, \end{cases}$$

where $\eta$ is a (temperature dependent) viscosity, and **B** a vector field in T*D*, the proposed viscoplastic formulation is reduced to the classical overstress concept due to Perzyna [14, 15]. Models of this kind have been also proposed among others, by Malvern [13], Phillips and Wu [16] and Chaboche [17].

*REMARK 2.2.* Within the present formulation it is implied that if an elastic domain exists, then the values of the plastic internal variables **P** are constant within this domain, since the

rate – independent material mechanisms are always activated after the rate – dependent ones have done so.
The *characterization of the rate – independent response* of the material*, in conjunction to the rate – dependent one,* will be described by appropriately *extending* the generalized plasticity framework (for the rate – independent generalized plasticity framework see Lubliner [18, 19, 20] and Panoskaltsis et al. [21, 22, 24]) as follows. The *viscous* (quasistatic) *range* will be introduced and it will be defined as the submanifold of $C \times V$ which *encompasses the states* that *can be reached by a process which does not activate the rate – independent mechanisms within the material,* that is one with **P** = constant. This process may be either an elastic process –belonging to *C* – or an inelastic process belonging to $C \times V$, that is one with a tangent vector of the form $\dot{\mathbf{\Psi}}(t) = \begin{bmatrix} \dot{\mathbf{C}}(t) \\ \dot{\mathbf{V}}(t) \\ \mathbf{0} \end{bmatrix}$.

Such an *inelastic process* will be defined as *viscous*. The boundary of the viscous range will be defined as a *loading surface* (e.g., Eisenberg and Phillips [38], Lubliner [39, 20]). On the loading surface we may construct a coordinate system similar to a coordinate system on the yield surface. A state lying on a loading surface may be defined as a *viscoplastic state*, while an *inelastic process*, which results in a change of the plastic internal variable vector **P** will be defined as *viscoplastic process*. The rate equation for the evolution of **P** is closely related to the notion of the viscoplastic process, which in turn may be systematically studied on the basis of the *loading rate concept* (see Lubliner [40, 18, 20]). This is defined as the inner product of the outward normal to the loading surface (a one – form on the cotangent space $T^{*}C \times V$), with the projection of the tangent to the process vector onto the space $C \times V$, that is

$$L(\mathbf{C}, \mathbf{V}, \dot{\mathbf{C}}, \dot{\mathbf{V}}) = \frac{\partial F}{\partial \mathbf{C}} : \dot{\mathbf{C}} + \frac{\partial F}{\partial \mathbf{V}} \dot{\mathbf{V}}, \tag{5}$$

where $F : D \rightarrow R$ is the mathematical expression for the loading surface in *D*. The loading rate determines the speed and the direction of a process from a *viscoplastic state relatively to its viscous range.* If $L \leq 0$, then the viscous range is *invariant* under the flow of $(\dot{\mathbf{C}}, \dot{\mathbf{V}})$ (e.g., see Abraham et al. [41, pp. 256 – 258]) and the process is *viscous.* If $L > 0$, then the viscous range *is not invariant* anymore and a new viscoplastic state at a new value of **P** is initiated. Accordingly, the rate equation for the evolution of the plastic internal variable vector **P** (see also Lubliner [18, 19] for the rate – independent case), may be stated as

$$\dot{\mathbf{P}} = H(\mathbf{C}, \mathbf{V}, \mathbf{P}) \mathbf{D}(\mathbf{C}, \mathbf{P}) \langle L \rangle, \tag{6}$$

in which $H$ is a scalar function of the state variables, which enforces the defining property of a viscoplastic state, i.e., the value of $H$ must be positive at any viscoplastic state and zero in any other one. Finally, **D** is another vector field in T*D*, enforcing the rate – independent properties of the material and $\langle \cdot \rangle$ stands from now on for the McCauley bracket defined as

$$\langle x \rangle = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

The formulation is supplemented by the general loading – unloading conditions of the new theory, which follow by using the basic Equations (4) and (6), as:

$$\begin{cases} \text{If } \mathbf{A}(\mathbf{C},\mathbf{V})=\mathbf{0}: \text{ Elastic domain.} \\ \text{If } \mathbf{A}(\mathbf{C},\mathbf{V})\neq\mathbf{0} \text{ and } H(\mathbf{C},\mathbf{V},\mathbf{P})=\mathbf{0}: \text{ Viscous process.} \\ \text{If } \mathbf{A}(\mathbf{C},\mathbf{V})\neq\mathbf{0} \text{ and } H(\mathbf{C},\mathbf{V},\mathbf{P})\neq\mathbf{0}: \text{ If } L\leq 0\text{: Viscous process,} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{If } L>0: \text{ Viscoplastic process.} \end{cases} \tag{7}$$

A particular case of interest appears when a yield surface exists and coincides with the initial loading surface (see the concise discussion of Eisenberg and Phillips [38]). This case corresponds to a material which exhibits both rate – dependent and rate – independent characteristics, with the *yield criterion of the instantaneous rate – independent behavior governing both*. Another particular case of interest arises when both functions **A** and $H$ in Equations (4) and (6) are non – vanishing. In this case every state is a viscoplastic one, every process results in viscous response and the material possesses a rate – dependent quasi – yield surface (see Lubliner [39], Panoskaltsis et al. [24]).

An equivalent description in the spatial configuration can be derived by extending the motion (1) to a dynamical process $\underset{\sim}{\mathbf{P}}$ by considering the local vector bundle mapping (e.g., Abraham et al. [41, p. 67])

$$\underset{\sim}{\mathbf{P}}: B\times D \rightarrow S\times D',$$

where $D'$ is the state space "as seen" in the current configuration. This mapping is defined as

$$\underset{\sim}{\mathbf{P}}\left(\mathbf{X},\mathbf{C},\mathbf{Q},t\right)=\left((\mathbf{x}),\mathbf{x}_*(\mathbf{C}),\mathbf{x}_*(\mathbf{Q})\right)=\left((\mathbf{x}),\mathbf{g},\mathbf{q}\right), \tag{8}$$

where $\mathbf{x}_*(\cdot)$ stands for the push – forward operator and **q** denotes the push – forward of the internal variable vector **Q** and is defined on the basis of the general transformation law (e.g., Lovelock and Rund [27, p. 66]) in component form as

$$q^{j_1\ldots j_r}{}_{i_1\ldots i_s}=\frac{\partial x^{j_1}}{\partial X^{I_1}}\cdots\frac{\partial x^{j_r}}{\partial X^{I_r}}\frac{\partial X^{K_1}}{\partial x^{i_1}}\cdots\frac{\partial X^{K_s}}{\partial x^{i_s}}Q^{I_1\ldots I_r}{}_{K_1\ldots K_s}. \tag{9}$$

As a result, by applying a push – forward operation to Equations (4) and (6) the equivalent assessment of the basic equations in the spatial description is obtained as

$$\mathrm{L}_{\mathbf{V}}\mathbf{v}=\mathbf{a}(\mathbf{g},\mathbf{v},\mathbf{F}), \tag{10}$$

$$\mathrm{L}_{\mathbf{V}}\mathbf{p}=h(\mathbf{g},\mathbf{v},\mathbf{p},\mathbf{F})\mathbf{d}(\mathbf{g},\mathbf{p},\mathbf{F})\langle l\rangle, \tag{11}$$

where **v**, **p**, **a**, **d** are the push – forwards onto the spatial configuration of the material tensorial quantities **V**, **P**, **A**, **D**; $h$ is the equivalent expression for the (scalar invariant) state function $H$ and $l$ stands for the *loading rate* in the spatial configuration defined as

$$l = \frac{\partial f}{\partial \mathbf{g}} : \mathrm{L}_{\mathbf{v}} \mathbf{g} + \frac{\partial f}{\partial \mathbf{v}} \mathrm{L}_{\mathbf{v}} \mathbf{v}. \tag{12}$$

In this equation, $f$ is the expression for the loading surfaces in the spatial configuration, i.e., $f : D' \to R$ with $f(\mathbf{g}, \mathbf{v}, \mathbf{p}, \mathbf{F}) = 0,$ while $\mathrm{L}_{\mathbf{v}}(\cdot)$ denotes the Lie derivative, defined as the convected derivative relative to the current configuration (e.g., Stumpf and Hoppe, [29]). We note the dependence of the state functions on the deformation gradient **F,** which is due to the forward operation by which Equations (10) and (11) are derived from basic Equations (4) and (6). The loading – unloading conditions (7) in the current configuration in view of Equations (10), (11) and (12) can be stated as:

$$\begin{cases} \text{If } \mathbf{a}(\mathbf{g},\mathbf{v},\mathbf{F})=\mathbf{0}: \text{ Elastic domain.} \\ \text{If } \mathbf{a}(\mathbf{g},\mathbf{v},\mathbf{F}) \neq \mathbf{0} \text{ and } h(\mathbf{g},\mathbf{v},\mathbf{p},\mathbf{F})=\mathbf{0}: \text{ Viscous process.} \\ \text{If } \mathbf{a}(\mathbf{g},\mathbf{v},\mathbf{F}) \neq \mathbf{0} \text{ and } h(\mathbf{g},\mathbf{v},\mathbf{p},\mathbf{F}) \neq \mathbf{0}: \text{ If } l \leq 0\text{: Viscous process,} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{If } l > 0: \text{ Viscoplastic process.} \end{cases} \tag{13}$$

*REMARK 2.3.* Unlike our approaches to the rate – independent theory (see Panoskaltsis et al. [21, 22, 24]) we refrain from characterizing conditions (7) (or (13)) as *loading – unloading criteria*, since the loading rate $L$ (or equivalently $l$) is dependent on the viscous internal variable and its rate, which are in general *non – controllable* quantities.

*REMARK 2.4.* As it has been already mentioned the case of combined rate – dependent and rate – independent behavior has been studied by Naghdi and Murch [8]. In their approach Naghdi and Murch, propose a general formulation on the basis of a combination of a theory of linear viscoelasticity, expressed in terms of convolution integrals, in series with a theory of classical plasticity. As a result, unlike the present approach where the non – necessity of the existence of a yield surface has led to the absence of the plastic consistency parameter, in their approach the consistency parameter is determined from the consistency condition of the instantaneous plasticity. For the earlier works of Panoskaltsis and co-workers we note that in Panoskaltsis et al. [1] a viscoelastic model has been introduced in series with a rate – independent internal variable plastic model, while in Panoskaltsis et al. [2, 3, 4] a viscoelastic model has been introduced in series with an internal variable *rate – dependent plastic* model (i.e., one with the *initial* yield depending on the rate), for modeling of concrete materials.

*REMARK 2.5.* We note the dependence of the loading surface on the internal variable vector **V**, which in turn means that the loading surface even though is related to the rate – independent properties of the material, *is itself rate – dependent.* This consideration is based on the experimentally observed behavior according to which, the activation of the rate – independent mechanisms within the material (yielding) is a rate – dependent phenomenon.

*REMARK 2.6.* The case in which a rate – independent loading surface is involved corresponds to a combination of a viscoplastic theory with rate – independent generalized plasticity in an uncoupled setting. Then the rate – independent formulation reduces to that of standard generalized plasticity (see Lubliner [20]; see also Panoskaltsis et al. [21, 22, 24]) with the term *elastic* replaced throughout by the term *viscous.* Such a formulation has its origins in the work of Landau et al. [6], where a combination of a viscoelastic theory with a perfectly plastic solid obeying a von Mises yield criterion has been proposed within the context of infinitesimal deformation.

## 3 THE PHYSICAL METRIC

The basic objective of this section is the introduction of a tangible strain space constitutive model. The proposed model is based on a consistent implementation of a rather new and powerful concept, namely the “physical” metric concept (see Valanis [30] and Valanis and Panoskaltsis [31]). The basic idea relies on the fact that the *intrinsic material (“physical”) metric*, *namely the (body) metric in the material configuration,* can be considered as a basic internal variable associated with the non – affine deformation. Unlike the original theory of Valanis [30], where the concept was introduced within the context of hereditary constitutive relations, in the present proposal the *metric* concept is introduced within the framework of materials with internal variables; in this sense, the proposed model resembles a model by Panoskaltsis et al. [22] within the context of rate – independent generalized plasticity. Moreover, in contrast to the theoretical presentation given in section 2, where in general it has been assumed that different mechanisms within the material substructure are responsible for rate – dependent (viscous) and rate – independent (plastic) phenomena, within the specific model formulation it is assumed that the same mechanisms are responsible for the combined (viscoplastic) material response, which in turn is described in terms of the physical metric. Since we deal with large scale viscoplastic flow, the kinematics of the problem together with the principle of spatial covariance suggest that a formulation of the model in terms of the *spatial metrics* and *their convected (Lie) derivatives* is more natural. Further, in the current configuration, the spatial metric usually has a diagonal form, which makes the computations simpler than those in the reference configuration where the metric $\mathbf{C}$ is fully populated (e.g., see Miehe [35], Panoskaltsis et al. [24]).

Accordingly, the stress response is assumed to be governed by an isotropic strain energy function which is given in terms of the invariants of the tensor $\mathbf{gb}$, where $\mathbf{b}$ is the left Cauchy – Green tensor, defined as the push – forward of the reciprocal (contravariant) metric $\mathbf{G}^{-1}$,i.e., ( $(\mathbf{b} = \mathbf{x}_* \left(\mathbf{G}^{-1}\right) = \mathbf{F}\mathbf{G}^{-1}\,\mathbf{F}^{\mathbf{T}})$, as

$$\hat{\psi}(I_1, I_3) = \gamma \ln\sqrt{I_3} + \mu I_3^{-\frac{1}{3}}[-\ln\sqrt{I_3} + \frac{1}{2}(I_1 - 3)], \tag{14}$$

where $I_1 = tr(\mathbf{bg})$ and $I_3 = det(\mathbf{bg})$, are the first and third invariants of $\mathbf{bg}$, $\rho_0$ is the referential density $(\rho_0 = \rho J)$ and $\gamma$ and $\mu$ are Lame$'$ type elastic constants. Then, the Kirchhoff stress tensor $\boldsymbol{\tau}$ $(\boldsymbol{\tau} = J\boldsymbol{\sigma})$, is given as (see, e.g., Marsden and Hughes [28, p. 204])

$$\boldsymbol{\tau} = 2\rho_0 \frac{\partial \hat{\psi}}{\partial \mathbf{g}} = \gamma \ln\sqrt{I_3}\mathbf{g}^{-1} + \mu I_3[\mathbf{b} - \mathbf{g}^{-1}]. \tag{15}$$

The loading surfaces are assumed to be given by a von – Mises expression of the form (e.g., Simo [34])

$$f(\boldsymbol{\tau}, \mathbf{g}, \alpha) = \sqrt{\tau^{ij}\tau^{kl} g_{ik} g_{jl} - \tfrac{1}{3}\left(\tau^{kl} g_{kl}\right)^2} - \sqrt{\tfrac{2}{3}}\left(\sigma_y + K\alpha\right), \tag{16}$$

where $\alpha$ is a scalar internal variable which controls the isotropic hardening of the von – Mises loading surface, $\sigma_y$ is a model parameter to be interpreted as the uniaxial yield stress in the absence rate dependent phenomena and $K$ is the isotropic hardening modulus. The yield surface is assumed to be similar to the loading surfaces and it is defined by an expression of the form

$$g(\boldsymbol{\tau},\mathbf{g},\alpha)=\sqrt{\tau^{ij}\tau^{kl}g_{ik}g_{jl}-\tfrac{1}{3}\left(\tau^{kl}g_{kl}\right)^2}-\sqrt{\tfrac{2}{3}}k\left(\sigma_y+K\alpha\right), \tag{17}$$

where $k$ is the similarity ratio ($0\leq k\leq 1$).

The evolution of the (contravariant) metric **b** is assumed to be given *by a normality flow rule on both the yield and the loading surfaces*, which resembles the one derived on the basis of the maximum plastic dissipation by Simo [34], as

$$\mu I_3^{-\frac{1}{3}}dev(\mathrm{L}_{\mathbf{v}}\mathbf{b})=-2\bar{\mu}\frac{\langle g\rangle^x}{\eta}\boldsymbol{\nu}-2\bar{\mu}\frac{\langle f\rangle}{|f|\beta}\mathbf{n}\langle\boldsymbol{\nu}:\mathrm{L}_{\mathbf{v}}\mathbf{g}\rangle, \tag{18}$$

where $dev(\cdot)$ stands for the deviatoric operator in the spatial configuration that is $dev(\cdot)=(\cdot)-\dfrac{1}{3}[\mathbf{g}:(\cdot)]\mathbf{g}^{-1}$, $\bar{\mu}=\mu I_3^{-\frac{1}{3}}I_1$, $\mathbf{n}=\dfrac{\mathrm{dev}\boldsymbol{\tau}}{\|\mathrm{dev}\boldsymbol{\tau}\|}$, $\boldsymbol{\nu}=\dfrac{\partial f}{\partial\mathbf{g}}$, is the normal vector to the loading surfaces in the deformation space, $\eta$ is a viscosity type of parameter, $x$ is a model parameter related to the rheological characteristics of the material and $\beta$ is an additional parameter that is related to the rate independent part of the model (see Panoskaltsis et al. [21, 22, 24]).

The evolution equation for the isotropic hardening variable, in accordance with the infinitesimal theory of plasticity (e.g., see Simo and Hughes [42, p. 90]), is assumed to be given as

$$\dot{\alpha}=\sqrt{\frac{2}{3}}\left(\frac{\langle g\rangle^y}{\eta}+\frac{\langle f\rangle}{|f|\beta}\langle\boldsymbol{\nu}:L_{\mathbf{v}}\mathbf{g}\rangle\right), \tag{19}$$

where $y$ is an additional model parameter.

As in the general theory the loading – unloading conditions of the model together with the corresponding rate – equations for the evolution of the internal variables, can be derived directly from Equations (18) and (19) by dropping the McCauley brackets as:

$$\begin{cases} \text{If } g \le 0: \text{ Elastic domain,} \\ \qquad dev(\mathrm{L_v}\mathbf{b}) = \mathbf{0}, \ \dot{\alpha}=0. \\ \text{If } g>0 \text{ and } (f \le 0 \text{ or } \boldsymbol{\nu}:\mathrm{L_v}\mathbf{g} \le 0): \text{ Viscous process,} \\ \qquad \mu I_3^{-\frac{1}{3}} dev(\mathrm{L_v}\mathbf{b}) = -2\bar{\mu}\dfrac{g^x}{\eta}\boldsymbol{\nu}, \\ \qquad \dot{\alpha}=\sqrt{\dfrac{2}{3}}\dfrac{g^y}{\eta}. \\ \text{If } f>0 \text{ and } \boldsymbol{\nu}:\mathrm{L_v}\mathbf{g}>0: \text{ Viscoplastic process,} \\ \qquad \mu I_3^{-\frac{1}{3}} dev(\mathrm{L_v}\mathbf{b}) = -2\bar{\mu}\dfrac{g^x}{\eta}\boldsymbol{\nu} - 2\bar{\mu}\dfrac{1}{\beta}\mathbf{n}(\boldsymbol{\nu}:\mathrm{L_v}\mathbf{g}), \\ \qquad \dot{\alpha} = \sqrt{\dfrac{2}{3}}\left(\dfrac{g^y}{\eta} + \dfrac{1}{\beta}(\boldsymbol{\nu}:L_{\mathrm{v}}\mathbf{g})\right). \end{cases} \tag{20}$$

Finally, the normal vector $\boldsymbol{\nu} = \dfrac{\partial g}{\partial \mathbf{g}}$ to the yield (and the loading surfaces), after lengthy computations (e.g., see Simo [34]), can be found to be

$$\boldsymbol{\nu} = \frac{\partial g}{\partial \mathbf{g}} = \frac{\partial f}{\partial \mathbf{g}} = \bar{\mu}\left(\mathbf{n} + \frac{\|\mathrm{dev}\boldsymbol{\tau}\|}{\bar{\mu}}\mathrm{dev}[\mathbf{n}^2]\right). \tag{21}$$

To this end it is instructive to make the following remarks:

*REMARK 4.11.* The particular case where $k = 1$ corresponds to the material which exhibits both rate – dependent and rate – independent response, with the same yield criterion governing both. Such a case is of extreme importance in the study of large scale viscoplastic flow where the elastic strains are negligible and the material response is governed entirely by that appearing in the post yielding regime.

*REMARK 4.12.* The particular case $k = 0$ corresponds to a material with response being identical to that described by a Maxwell fluid in series with the generalized plasticity model. The limiting case where $k = 0$ and $\sigma_y = 0$ corresponds to a model with *a rate – dependent quasi – yield surface*, which is defined by Equation (16). This surface, besides being a rate – dependent loading surface, serves and as a loading potential for the rate – dependent part of the model.

## 4 COMPUTATIONAL ASPECTS AND NUMERICAL SIMULATIONS

### 4.1 Time integration algorithm

In the last section, we examine the ability of the proposed model in simulating several patterns of the behavior of metals under quasi – static and dynamic loading conditions. The model can be implemented numerically by employing a predictor – corrector scheme, which is based on the ideas developed, within the context of a classical elastic – plastic formulation, by Simo and Hughes [42, pp. 311 – 321]. Nevertheless, in sharp contrast with the classical elastic – plastic case, *the internal variables are no longer constrained to lie within the closure of an elastic domain*, since neither viscoplasticity nor generalized plasticity employ the concept of the yield surface as a basic ingredient (see Lubliner [20], Panoskaltsis et al. [21, 24]). Accordingly, unlike the classical elastoplastic case where the evolution equations define a unilaterally constrained problem of evolution governed by the Kuhn – Tucker conditions (e.g., see Simo and Hughes [42, p. 84]), in the present case the evolution equations form a differential system, which must obey the loading – unloading conditions of the model.

The corresponding algorithmic problem is stated as follows: Let I = [0, T], be the time interval of interest. It is assumed that at time $t_n \in \mathrm{I}$, the configuration of the body of interest $b_n \in S$, defined as

$$b_n = \{\mathbf{x}_n = \mathbf{x}_n(\mathbf{X}) \mid \mathbf{X} \in B\},$$

along with the state variables are known, i.e.,

$$\{\mathbf{x}_n, \boldsymbol{\tau}_n, \mathbf{b}_n, \alpha_n\},$$

are the known data at time $t_n$. Assume a time increment $\Delta t_n$, which drives the time to $t_{n+1} = t_n + \Delta t$ and the body configuration to

$$b_{n+1} = \{\mathbf{x}_{n+1} = \mathbf{x}_{n+1}(\mathbf{X}) \mid \mathbf{X} \in B\},$$

where

$$\mathbf{x}_{n+1}(\mathbf{X}) = \mathbf{x}_n(\mathbf{X}) + \mathbf{U}(\mathbf{X}) = \mathbf{x}_n(\mathbf{X}) + \mathbf{u}\left(\mathbf{x}_n\left(\mathbf{X}\right)\right),$$

and **u** is the incremental displacement field, which is assumed to be given.

Then the algorithmic problem in hand is to update the stress tensor and the internal variables to the time step $t_{n+1}$ in a manner consistent with the continuous Equations (15) and (20).

The solution of the problem can be performed by means of a time – discretization of the governing equations of the model on the basis of the backward Euler scheme, which is first – order accurate and unconditionally stable. Because *of the presence of Lie derivatives within the continuous equations adequate approximations for these objects can be derived on the basis of the defining identity of the (convected) Lie derivative and the general tensorial transformation law* (see Panoskaltsis et al. [21, 24]). In particular, the defining identity for the Lie derivative of a tensor **q** of type $\begin{pmatrix} r \\ s \end{pmatrix}$ in the $b_{n+1}$ configuration is

$$L_{\mathrm{v}}\mathbf{q}_{n+1} = \mathbf{x}_{n+1*}\left(\frac{\partial}{\partial t}\mathbf{x}_{n+1}{}^{*}(\mathbf{q})\right). \tag{22}$$

By performing a pull – back operation Equation (22) can be stated as

$$\mathbf{x}_{n+1}{}^{*}(\mathrm{L}_{\mathbf{v}}\mathbf{q}_{n+1}) = \frac{\partial}{\partial t}\left(\mathbf{x}_{n+1}{}^{*}\left(\mathbf{q}_{n+1}\right)\right) = \dot{\mathbf{Q}}_{n+1} \approx \frac{1}{\Delta t}(\mathbf{Q}_{n+1} - \mathbf{Q}_{n}), \tag{23}$$

which in turn may be written in component form on the basis of the general tensorial transformation law (see Equation (9)) as

$$\begin{aligned}&[\frac{\partial X^{I_1}}{\partial x^{i_1}_{n+1}}\cdots\frac{\partial X^{I_r}}{\partial x^{i_r}_{n+1}}\frac{\partial x^{j_1}_{n+1}}{\partial X^{J_1}}\cdots\frac{\partial x^{j_s}_{n+1}}{\partial X^{J_s}}]\mathrm{L}_{\mathbf{v}}\left(q^{i_1\cdots i_r}{}_{j_1\cdots j_s}\right)_{n+1} = \frac{1}{\Delta t}[\left(Q^{I_1\cdots I_r}{}_{J_1\cdots J_s}\right)_{n+1} - \left(Q^{I_1\cdots I_r}{}_{J_1\cdots J_s}\right)_{n}] = \\ &= \frac{1}{\Delta t}[\frac{\partial X^{I_1}}{\partial x^{k_1}_{n+1}}\cdots\frac{\partial X^{I_r}}{\partial x^{k_r}_{n+1}}\frac{\partial x^{l_1}_{n+1}}{\partial X^{J_1}}\cdots\frac{\partial x^{l_s}_{n+1}}{\partial X^{J_s}}\left(q^{k_1\cdots k_r}{}_{l_1\cdots l_s}\right)_{n+1} - \frac{\partial X^{I_1}}{\partial x^{k_1}_{n}}\cdots\frac{\partial X^{I_r}}{\partial x^{k_r}_{n}}\frac{\partial x^{l_1}_{n+1}}{\partial X^{J_1}}\cdots\frac{\partial x^{l_s}_{n+1}}{\partial X^{J_s}}\left(q^{k_1\cdots k_r}{}_{l_1\cdots l_s}\right)_{n}],\end{aligned} \tag{24}$$

from which $\mathrm{L}_{\mathbf{v}}\left(q^{i_1\cdots i_r}{}_{j_1\cdots j_s}\right)_{n+1}$ can be determined as

$$\mathrm{L}_{\mathbf{v}}\left(q^{i_1\cdots i_r}{}_{j_1\cdots j_s}\right)_{n+1} = \frac{1}{\Delta t}[\left(q^{i_1\cdots i_r}{}_{j_1\cdots j_s}\right)_{n+1} - \frac{\partial x^{i_1}_{n+1}}{\partial x^{k_1}_{n}}\cdots\frac{\partial^{i_r}_{n+1}}{\partial x^{k_r}_{n}}\frac{\partial x^{l_1}_{n}}{\partial x^{j_1}_{n+1}}\cdots\frac{\partial x^{l_s}_{n}}{\partial x^{j_s}_{n+1}}\left(q^{k_1\cdots k_r}{}_{l_1\cdots l_s}\right)_{n}], \tag{25}$$

where the tensor with components

$$(f^{i}{}_{j})_{n+1} = \frac{\partial x^{i}_{n+1}}{\partial x^{j}_{n}} = \frac{\partial x^{i}_{n+1}}{\partial X^{I}}\frac{\partial X^{I}}{\partial x^{j}_{n}} = (F^{i}{}_{I})_{n+1}((F^{-1})^{I}_{j})_{n}, \tag{26}$$

is defined as *the relative deformation gradient* with respect to the configuration $b_{n+1}$ (e.g., Simo and Hughes, [42, p. 279]). By means of Equation (25) an approximation for the covariant $\binom{0}{2}$ metric **g** may be derived as

$$\mathrm{L}_{\mathbf{v}}\left(g_{ij}\right)_{n+1} = \frac{1}{\Delta t}[\left(g_{ij}\right)_{n+1} - \frac{\partial x^{i}_{n}}{\partial x^{k}_{n+1}}\frac{\partial x^{j}_{n}}{\partial x^{l}_{n+1}}\left(g_{kl}\right)_{n}], \tag{27}$$

or equivalently

$$\mathrm{L}_{\mathbf{v}}\mathbf{g}_{n+1} = \frac{1}{\Delta t}(\mathbf{g}_{n+1} - \mathbf{f}^{T-1}_{n+1}\mathbf{g}_{n}\mathbf{f}^{-1}_{n+1}). \tag{28}$$

In a similar manner the approximation for the contravariant $\binom{2}{0}$ (body) metric **b** can be found to be

$$\mathrm{L}_{\mathbf{v}}\left(b^{ij}\right)_{n+1} = \frac{1}{\Delta t}[\left(b^{ij}\right)_{n+1} - \frac{\partial x^i_{n+1}}{\partial x^k_n}\frac{\partial x^j_{n+1}}{\partial x^l_n}\left(b^{kl}\right)_n], \tag{29}$$

or equivalently

$$\mathrm{L}_{\mathbf{v}}\mathbf{b}_{n+1} = \frac{1}{\Delta t}(\mathbf{b}_{n+1} - \mathbf{f}_{n+1}\mathbf{b}_n\mathbf{f}^T_{n+1}). \tag{30}$$

Accordingly, the time discrete counterparts of Equations (15) and (20) may be determined as

$$\boldsymbol{\tau}_{n+1} = \gamma \ln\sqrt{I_{3,n+1}}\mathbf{g}^{-1}_{n+1} + \mu I^{-\frac{1}{3}}_{3,\mathrm{n}+1}[\mathbf{b}_{n+1} - \mathbf{g}^{-1}_{n+1}], \tag{31}$$

and

$$\begin{cases}
\text{If } \mathrm{g}_{\mathrm{n}+1} \le 0: \text{ Elastic domain,} \\
\quad dev[\frac{1}{\Delta t}(\mathbf{b}_{n+1} - \mathbf{f}_{n+1}\mathbf{b}_n\mathbf{f}^T_{n+1})] = \mathbf{0},\ \alpha_{n+1} = \alpha_{\mathrm{n}} \\
\text{If } \mathrm{g}_{\mathrm{n}+1} > 0 \text{ and } (f_{n+1} \le 0 \text{ or } \boldsymbol{\nu}_{n+1} : \frac{1}{\Delta t}(\mathbf{g}_{n+1} - \mathbf{f}^{T-1}_{n+1}\mathbf{g}_n\mathbf{f}^{-1}_{n+1}) \le 0): \text{ Viscous process,} \\
\quad \mu \mathrm{J}^{-\frac{2}{3}}_{\mathrm{n}+1} dev[\frac{1}{\Delta t}(\mathbf{b}_{n+1} - \mathbf{f}_{n+1}\mathbf{b}_n\mathbf{f}^T_{n+1})] = -2\bar{\mu}_{n+1}\frac{{g_{n+1}}^x}{\eta}\mathbf{n}_{n+1}, \\
\quad \frac{\alpha_{n+1} - \alpha_n}{\Delta t} = \sqrt{\frac{2}{3}}\frac{{g_{n+1}}^y}{\eta} \\
\text{If } f_{n+1} > 0 \text{ and } \boldsymbol{\nu}_{n+1} : \frac{1}{\Delta t}(\mathbf{g}_{n+1} - \mathbf{f}^{T-1}_{n+1}\mathbf{g}_n\mathbf{f}^{-1}_{n+1}) > 0: \text{ Viscoplastic process,} \\
\quad \mu I^{-\frac{1}{3}}_{3,\mathrm{n}+1} dev[\frac{1}{\Delta t}(\mathbf{b}_{n+1} - \mathbf{f}_{n+1}\mathbf{b}_n\mathbf{f}^T_{n+1})] = -2\bar{\mu}_{n+1}\frac{{g_{n+1}}^x}{\eta}\mathbf{n}_{n+1} - \\
\quad -2\bar{\mu}_{n+1}\frac{1}{\beta}\mathbf{n}_{n+1}[\boldsymbol{\nu}_{n+1} : \frac{1}{\Delta t}(\mathbf{g}_{n+1} - \mathbf{f}^{T-1}_{n+1}\mathbf{g}_n\mathbf{f}^{-1}_{n+1})], \\
\quad \frac{\alpha_{n+1} - \alpha_n}{\Delta t} = \sqrt{\frac{2}{3}}\left(\frac{g^y}{\eta} + \frac{1}{\beta}[\boldsymbol{\nu}_{n+1} : \frac{1}{\Delta t}(\mathbf{g}_{n+1} - \mathbf{f}^{T-1}_{n+1}\mathbf{g}_n\mathbf{f}^{-1}_{n+1})]\right)
\end{cases} \tag{32}$$

where $\mathbf{g}_{n+1} = \mathbf{g}(\mathbf{x}_{n+1})$, $\mathrm{g}_{n+1}$, $f_{\mathrm{n}+1}$, $\mathbf{n}_{\mathrm{n}+1}$, $\bar{\mu}_{\mathrm{n}+1}$ are quantities which are expressible in terms of the basic variables. It is observed that Equation (31) and the rather complicated Equations (32) form a system of three independent equations in three unknowns ($\mathbf{b}_{n+1}$, $\alpha_{\mathrm{n}+1}$, $\boldsymbol{\tau}_{n+1}$). The solution of this system can be performed by a *three step predictor – corrector algorithm*, the steps of which are dictated by *the time discrete loading – unloading conditions* which are included in Equation (32) as well. Algorithmic details regarding the enforcement of the time discrete loading – unloading conditions and the solution of the system within the framework of a large

deformation rate – independent constitutive theory can be found in Panoskaltsis et al. [21, 24]).
*REMARK 4.13.* We note the absence of the consistency condition and accordingly of the consistency parameter from the time discrete equations. Due to this absence the resulting system is simpler than the classical elastic – plastic case and more computer power is preserved.

*REMARK 4.14.* By noting that *any objective derivative of a tensorial quantity* **q** *differs from its convected Lie derivative by terms depending on **q** and the Lie derivative of the spatial metric* **g** (e.g., Marsden and Hughes [28, p.100], Stumpf and Hoppe [29]), Equation (25) can be used as a basis for an objective approximation of other objective derivatives which may be used in place of the convected derivative, used herein.

The predictions of the model introduced will be illustrated by considering two problems of large scale viscoplastic flow, namely the extension of a material block and the combined extension and inflation of a thick – walled cylinder.

## 4.2 Extension of a material block (plane strain)

The extension problem, in order to avoid necking phenomena within the specimen, is assumed to be that of a (laterally) constrained one, which is defined as

$$x^1 = (1+\lambda)X^1,\ x^2 = X^2,\ x^3 = X^3.$$

The basic material parameters are those considered in Simo and Hughes [42, p. 326] where the elastic – plastic upsetting of an axisymmetric billet is examined,

$$\gamma = 384.62,\ \mu = 833.33,\ \sigma_y = 1.00,\ K = 3.00.$$

The additional model parameters are set equal to

$$k = x = y = 1,\ \eta = 10{,}000,\ \beta = 1{,}000.$$

The predictions of the model for straining of the block up to 100%, for six different *displacement* rates ($\frac{d\lambda}{dt} = 10^{-5},\ 10^{-2,},\ 10^{-1},\ 3x10^{-1},\ 1,\ 10^{6}$) are shown in Figure 1. By referring to the results of this figure, we note that the predicted response is independent of the displacement rate, over a large dynamic range (from 1 to $10^6$). For those rates the response is *entirely governed by the rate – independent part* of the model. On the other hand there is *a strong rate – effect for lower rates* ($10^{-5} \le \frac{d\lambda}{dt} \le 10^{-1}$ sec$^{-1}$). In this sense the predicted response is identical to that described by Bodner [43] for a fully annealed FCC metal.

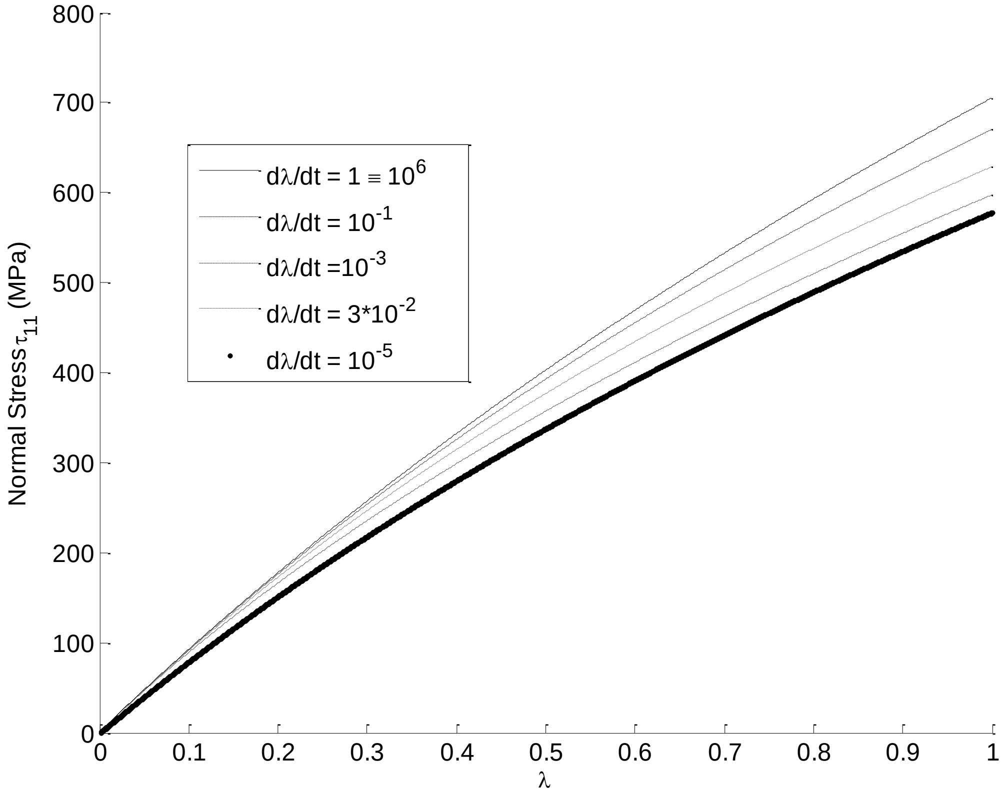


Figure 1: Restrained tension of a material block. Normal stress versus λ for different loading rates.

In order to check the model predictions in the case of an instantaneous change of rate in the course of deformation, we consider two additional straining histories. The first one involves an increasing change of rate and is given by

$$\frac{d\lambda}{dt}=\begin{cases}10^{-3} & for\ 0\le\lambda\le 0.1,\\ 3\times 10^{-2} & for\ 0.1\le\lambda\le 0.5\ ,\\ 10^{-1} & for\ 0.5\le\lambda\le 1.\end{cases}$$

The corresponding stress – deformation curve is shown in Figure 2, together with the original stress – deformation curves in the cases where the displacement rate remains constant throughout the test. By referring to the results of Figure 2, we conclude that after the first rate increase (from $\frac{d\lambda}{dt}=10^{-3}$ to $\frac{d\lambda}{dt}=3\times 10^{-2}$), the material point "leaves" the original stress – deformation curve (for $\frac{d\lambda}{dt}=10^{-3}$) and reaches asymptotically the original stress – deformation which corresponds to the higher rate $\frac{d\lambda}{dt}=3\times10^{-2}$. Upon a further rate increase (from $\frac{d\lambda}{dt}=3\times10^{-2}$ to $\frac{d\lambda}{dt}=10^{-1}$), the material point "crosses" the original stress – deformation curve corresponding to the lower rate ($\frac{d\lambda}{dt}=10^{-1}$) and reaches asymptotically the corresponding original stress – deformation curve which corresponds to the new higher rate ($\frac{d\lambda}{dt}=10^{-1}$).

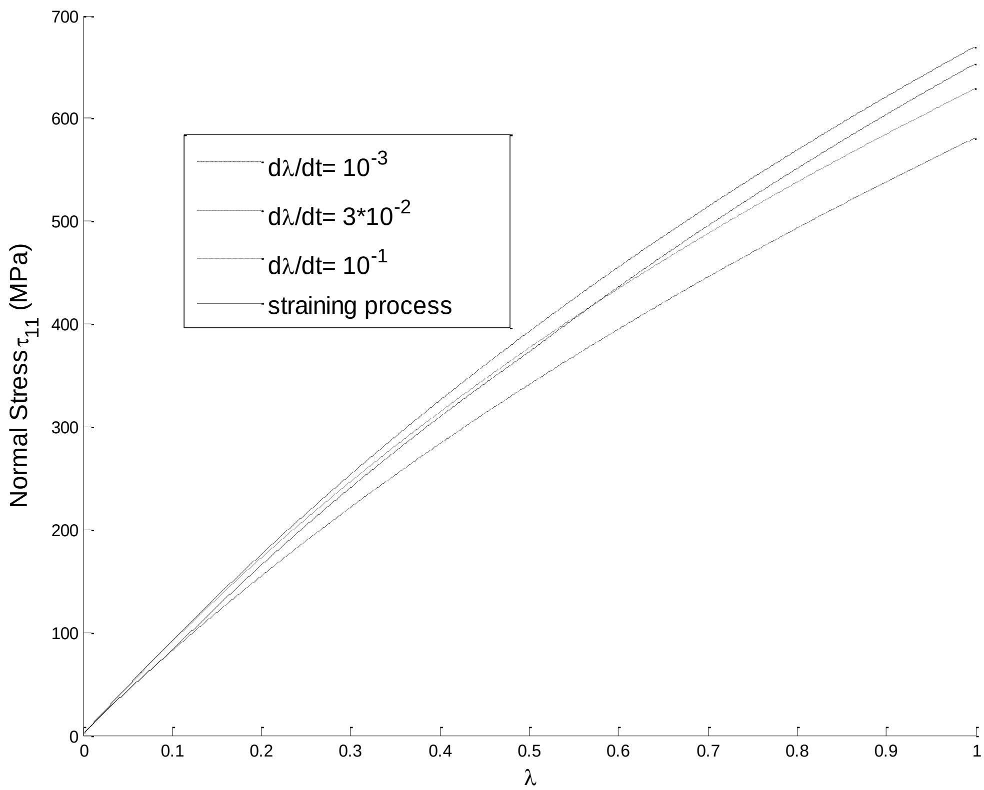


Figure 2: Restrained tension of a material block. Normal stress versus λ for increasing changes in displacement rate.

The second loading history comprises a decreasing change of loading rate and is given by

$$\frac{d\lambda}{dt} = \begin{cases} 10^{-1} & for\ 0 \leq \lambda \leq 0.3, \\ 3\times 10^{-2} & for\ 0.3 \leq \lambda \leq 0.7\ , \\ 10^{-3} & for\ 0.7 \leq \lambda \leq 1. \end{cases}$$

The results of this test are shown in Figure 3. By referring to these results, we conclude that after the first decrease of the displacement rate (from $\frac{d\lambda}{dt} = 10^{-1}$ to $\frac{d\lambda}{dt} = 3\times 10^{-2}$ ), the material point "leaves" the original stress – deformation curve (for $\frac{d\lambda}{dt} = 10^{-1}$) and after a while follows the original stress – deformation curve which corresponds to the lower rate ( $\frac{d\lambda}{dt} = 3\times 10^{-2}$ ). A similar response appears upon a further decrease of the rate (from $\frac{d\lambda}{dt} = 3\times 10^{-2}$ to $\frac{d\lambda}{dt} = 10^{-3}$) with the material point eventually following the original stress – deformation curve corresponding to the lowest displacement rate. In both cases the predicted response is identical with the experimentally observed behavior in metallic materials (see Metals Handbook [44]).

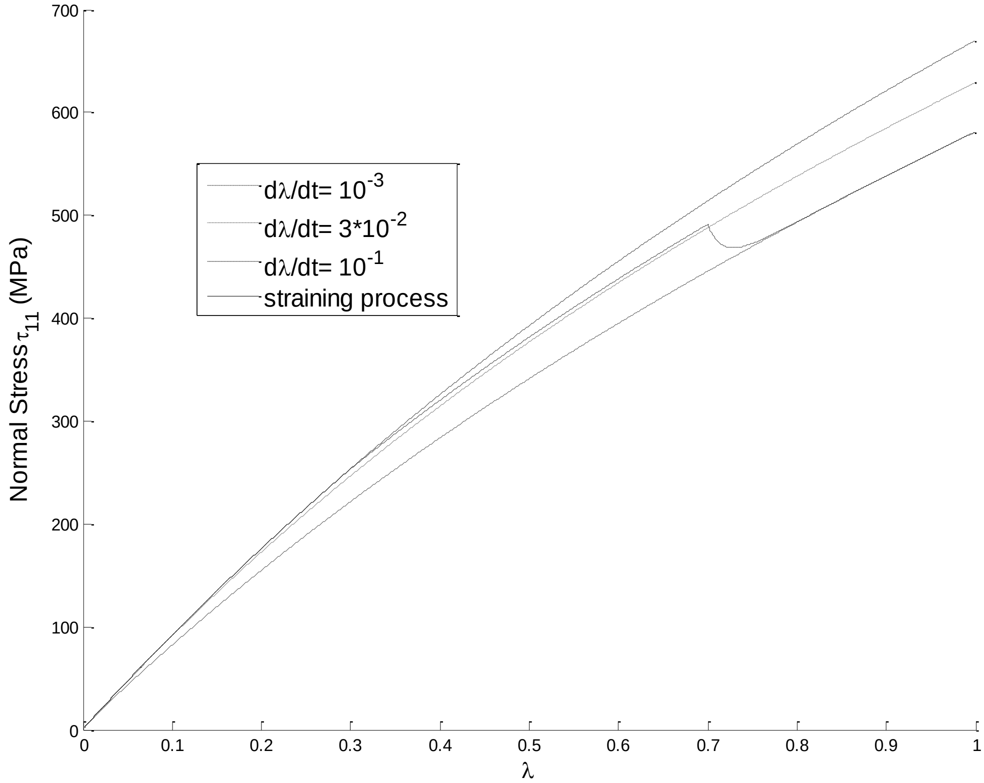


Figure 3: Restrained tension of a material block. Normal stress versus λ for decreasing changes in displacement rate.

### 4.3 Combined extension and inflation of a thick – walled cylinder

The combined extension and inflation of a thick - walled cylinder problem to be studied, is defined in a cylindrical polar coordinate system (see Ogden [45]) as

$$r^2 - a^2 = \frac{1}{\varpi}(R^2 - A^2),\ \theta=\Theta,\ \mathrm{z}=(1+\varpi)\mathrm{Z},$$

where $a$ and A are the inner and the outer radii of the cylinder and $\varpi$ is the axial displacement. The same problem has been also discussed within the context of the rate – independent theory by Panoskaltsis et al. [22]. The model parameters are those considered by Simo and Hughes [42, p. 324], where a similar plane strain problem is examined:

$$\gamma = 40{,}000,\ \mu=3{,}800,\ \sigma_{\mathrm{y}} = 0.5,\ K = 0.00 \text{ and k=x=y=1},\ \eta=\beta=10{,}000.$$

The cylinder is loaded under displacement control by controlling the displacement rates in both the azimuthal and the axial directions, simultaneously. Without loss of generality both rates are assumed equal. The specimen is loaded for three different rates ($\frac{d\varpi}{dt} = 5x10^{-2}$, $10^{-1}$ and 1) until the corresponding displacement reaches a prescribed constant value ($\varpi$ =18%), and then the displacement is kept constant leading to stress relaxation. The corresponding *time history of stress* in both radial and axial directions is depicted in Figures 4 and 5.

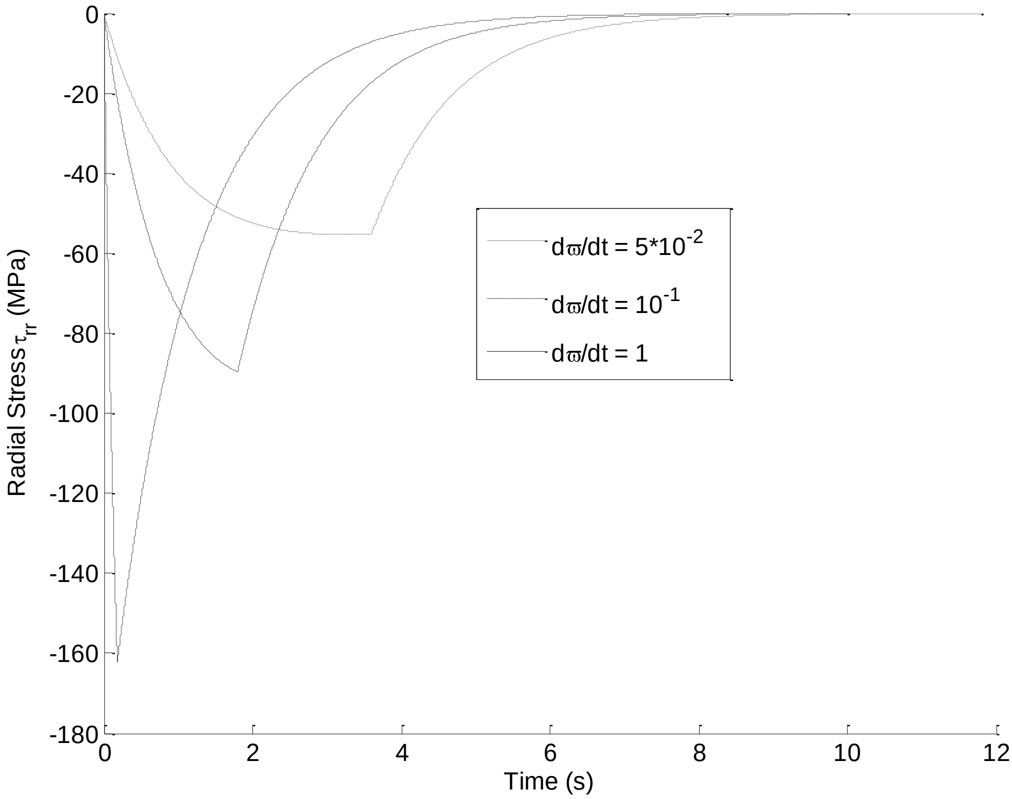


Figure 4: Combined Extension and inflation of a thick – walled cylinder. Radial stress versus time for different loading rates.

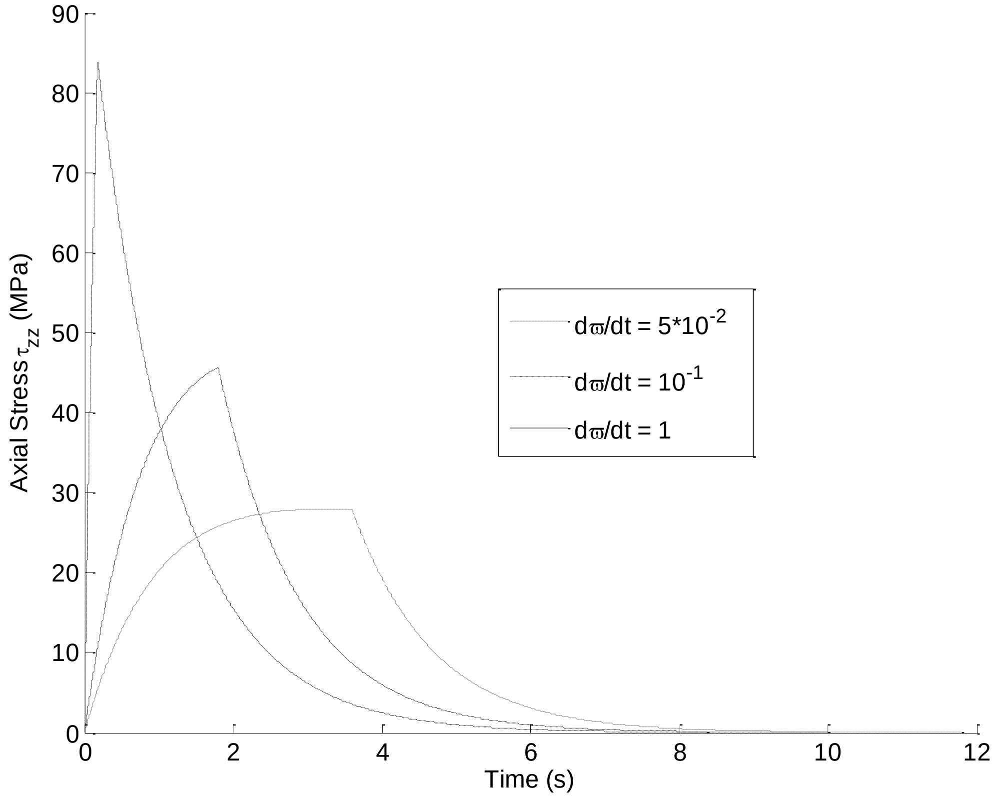


Figure 5: Combined extension and inflation of a thick – walled cylinder. Axial stress versus time for different loading rates.

## 5 CONCLUSIONS

The important problem of modeling solid materials with different characteristic times has been revisited. A new internal variable theory has been developed within finite deformations. The proposed theory relies crucially on the consistent combination of a general viscoplastic formulation and *a new version* of rate – independent generalized plasticity. In this new version of rate – independent generalized plasticity the concepts of viscous range and viscous process have been defined. The proposed formulation is presented in a covariant setting and in particular includes the following basic ingredients:

1. A manifold structure of the theory, which accounts not only for the body of interest and the ambient space, but for the state space as well. Accordingly, the state space is considered as a fiber over the material point, a procedure leading to a local vector bundle mapping, which is used for the identification of the dynamical process under examination.
2. Loading-unloading conditions, which have been derived in the material and in the spatial configurations.

Also, in the course of our development, important observations regarding the theories of plasticity and viscoplasticity, leading to a better understanding of those field theories, are made.

Furthermore, the concept of *physical metric* introduced by Valanis [30] and elaborated further by Valanis and Panoskaltsis [31] has been revisited within the context of a material model. The model is based on an extension of a $J_2$ flow theory to the finite deformation regime and considers the material metric as a basic internal variable describing both rate – dependent and rate – independent mechanisms within the body. For the numerical implementation of the model suitable approximations for the Lie derivatives of tensorial quantities are obtained. In addition, the loading-unloading conditions, which have been derived, are used. It is noted that the suitability of the model for large scale computations has been proved by:

(a) The derivation of a novel first order – accurate time integration algorithm.

(b) A series of numerical simulations, which show the ability of the model in predicting several patterns of the extremely complex response of metals under quasi – static and dynamic conditions.

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