Available online at www.eccomasproceedia.org
Eccomas Proceedia SMART (2023) 640-650

ECCOMAS Proceedia

X ECCOMAS Thematic Conference on Smart Structures and Materials
SMART 2023
D.A. Saravanos, A. Benjeddou, N. Chrysochoidis and T. Theodosiou (Eds)

ADAPTIVE SEGMENTATION APPROACH TO HANDLE MATERIAL AND LOADING DISCONTINUITIES IN THE ANALYTICAL SOLUTIONS OF STRUCTURAL PROBLEMS

AGYAPAL SINGH[*] AND NIKOLAOS KARATHANASOPOULOS[*†]

[*]Department of Engineering
New York University (NYU)
Abu Dhabi Campus, United Arab Emirates
e-mail: agyapal.singh@nyu.edu

[†] Department of Mechanical and Aerospace Engineering, Tandon School of Engineering
New York University (NYU)
Brooklyn, NY, 11201, USA
e-mail: n.karathanasopoulos@nyu.edu

Abstract. A novel segment-based adaptive approach in conjunction with the extended Kantorovich method is proposed to handle material and loading discontinuities efficiently in elasticity-based analytical solutions for classical mechanics problems of beams and plates. This approach enables the consideration of all displacement, in-plane stresses, and transverse stresses as the primary variables and ensures the pointwise/exact satisfaction of interlayer and intersegment continuity conditions along with arbitrary edge boundary conditions. First, the present proposed approach is utilized for an accurate stress analysis of arbitrary-supported composite beams and plates subjected to discontinuously distributed loads in the form of patches. Later this approach can also be utilized for edge-bonded beams of different materials subjected to discontinuous loads. The correctness and efficacy of the present mathematical model are established by comparing the present numerical result with whatever is available in the open literature and with 2D/3D finite elements results of Abaqus for other cases. The study shows that localized distributed load in the form of patches leads to various localized stress concentrations, which could further lead to localized failures like cracks and delamination, consequently needing to be predicted accurately for efficient designs. The present approach has great potential to work as a benchmark tool for such predictions and design analysis. Further, the present analytical results will be beneficial to assess the accuracy and efficacy of different beam theories and the convergence of numerical approaches developing in the future for the analysis of beam and plate structures with loading or material discontinuities.

Key words: Composite Structures, Material and loading discontinuities, Elasticity-based analytical solution, Extended Kantorovich method

Peer-review under responsibility of the organizing committee of SMART 2023.
doi: 10.7712/150123.9818.445569

1 INTRODUCTION

Composite beams play a crucial role in the field of engineering and are commonly used in various applications. To accurately analyze the structural response of these beams, specialized mathematical models have been developed in recent years for investigating their behavior under different conditions such as static, vibration, and buckling [1, 2, 3]. In stress analysis of laminated beams, Airy-stress polynomial functions were introduced by Lekhnitskii [4] for precise elasticity analysis of beams. Building upon this approach, exact solutions for laminated beams were developed by Silverman [5], Hasin [6], Gerstner [7], Rao and Ghosh [8] and Cheng et al. [9]. Additionally, Esendemir et al.[10] developed an analytical solution based on elasticity for bending analysis of beam structures. Analyzing laminated systems within the framework of elasticity poses a significant challenge in developing an analytical solution that accounts for arbitrary boundary conditions due to the complexity of these systems and computational difficulties. As a result, many researchers have explored numerical approaches for beams [2]. Chen et al.[11] developed an elasticity-based semi-analytical solution for analyzing flexural and natural frequencies of laminated beams using the differential quadrature method (DQM). Furthermore, Subramanian and Mulay [12] extended Pagano's theory to analyze the flexural behavior of laminated beams. Trinh et al.[13] utilized the inverse differential quadrature method with zigzag theory, and Doeva [14] employed a Variational Iteration Method (VIM) for static analysis of laminated composite beams.

The existing analytical and numerical solutions for analyzing laminated beams are limited to uniformly distributed continuous loads. For localized loading, Pagano [15]] developed an exact solution for simply-supported laminates under cylindrical bending and subjected to both concentrated and distributed loads. Kapuria et al. [16] evaluated the zigzag theory under plane-stress conditions for simply-supported composite beams subjected to static patch loads. However, to the best of the author's knowledge, no analytical solution has been reported for composite structures in the literature which can handle loading and material discontinuity with arbitrarily support condition a. This is a significant limitation since discontinuous loads and material are prevalent in practice. Therefore, the ability of mechanical analysis models to accurately and rigorously evaluate the internal stress state of such structural components under general types of loading and support conditions is of paramount importance.

This article aims to present novel segment-based adaptive approach in conjunction with the extended Kantorovich method to handle material and loading discontinuities efficiently in elasticity-based analytical solutions for beam and plate structures. This proposed solution is the first of its kind and builds upon the Kantorovich method by extending it to the discrete loading and material analysis space

2 ELASTICITY BASED FORMULATION FOR 2D BEAM

A single-layer composite beam ($x = (0, a)$, $z = -h/2,\ h/2$), as presented in Figure 1, is considered for the present study. Here. $\zeta = (z - h/2)/t$ is a non-dimensionalized thickness parameter and t denotes the thickness of the beam. Similarly, the beam span is divided into

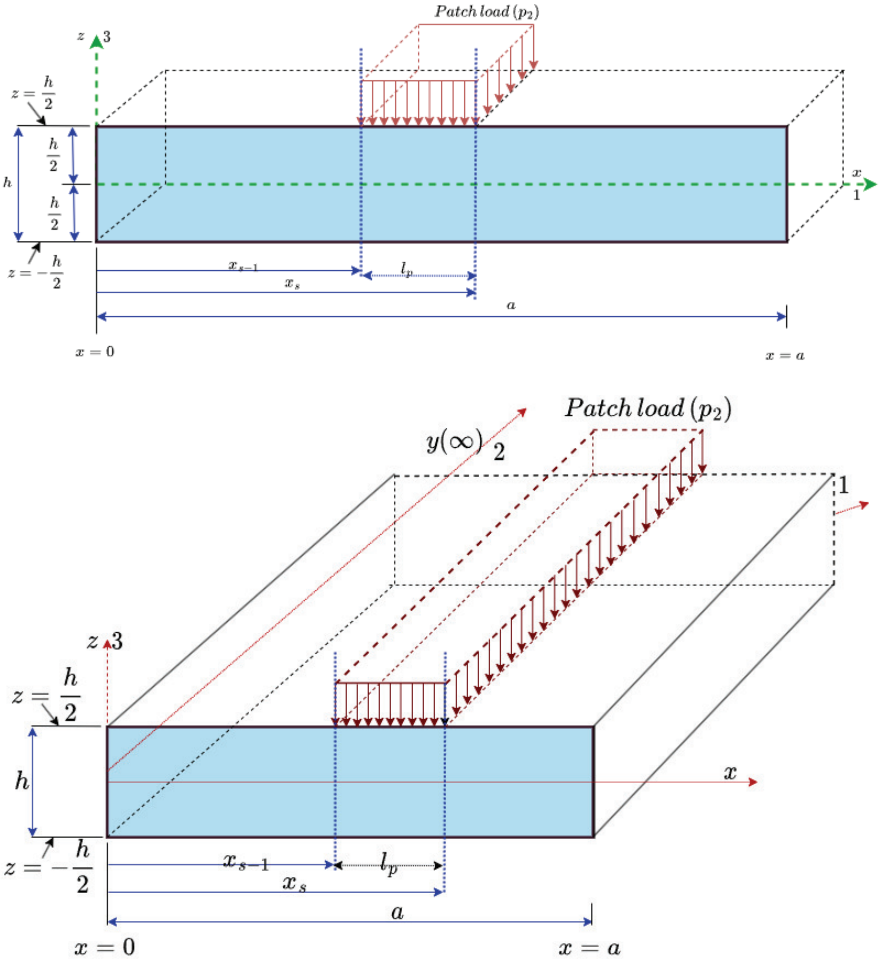

Figure 1: Geometry of the beam and plate considered for present study.

segments with and without loading, and $\xi^{(s)} = (x_s - x_{s-1})/l^{(s)}$ is a non-dimensional axial parameter defined for each segment, with $l^{(s)}$ denoting the length of that segment. The superscript 's' will be omitted in the further mathematical expressions, unless specifically used for clarity. The parameter $\xi_1 = x/a$ is a global in-plane parameter.

The linear strain-displacement relations for the 2D straight beam can be written as, $\varepsilon_x = u_{,x}$; $\gamma_{zx} = w_{,x} + u_{,z}$ and $\varepsilon_z = w_{,z}$. The elasticity based constitutive equations for the composite beam can be expressed as, $\varepsilon_x = s_{11}\sigma_x + s_{13}\sigma_z$; $\varepsilon_z = s_{13}\sigma_x + s_{33}\sigma_z$; $\gamma_{zx} = s_{55}\tau_{zx}$.

Using strain-displacement and constitutive equations, Reissner-type mixed variational principle for a beam without body force can be written as [17, 18, 19],

$$\int_V [\delta\sigma_x\{s_{11}\sigma_x + s_{13}\sigma_z - u_{,x}\} + \delta\sigma_z\{s_{13}\sigma_x + s_{33}\sigma_z - w_{,z}\} \\ + \delta\tau_{zx}\{s_{55}\tau_{zx} - u_{,z} - w_{,x}\}] + \delta u(\sigma_{x,x} + \tau_{xz,z}) + \delta w(\tau_{zx,x} + \sigma_{z,z})dV = 0 \quad (1)$$

It is assumed that the top surface and bottom surface of the beam are shear tractions free (τ_{zx}=0) and the beam is subject to a uniform distributed patch load p_2 at the top surface. The displacements (u ,w) and transverse stresses (σ_z, τ_{zx}) at the segment interfaces need to satisfy the following condition at the inter-segment interface

$$[(u,\, w,\, \sigma_x,\, \tau_{zx})|_{\xi^s=1}]^{(s)} = [(u,\, w,\, \sigma_x,\, \tau_{zx})|_{\xi^s=0}]^{(s+1)} \tag{2}$$

Along x-axis, the composite beam can have any type of support such as, Simply-supported: $\sigma_x = w = 0$; Clamped: $u = w = 0$; and Free: $\sigma_x = \tau_{xz} = 0$

3 EKM ANALYTICAL SOLUTION APPROACH

In the present mathematical model, both displacements (u, w) and stresses (σ_x, σ_z, τ_{zx}) are considered as primary variables and solved using the multi-term extended Kantorovich method [20, 21, 23**?**]. The field variables for a segment can be expressed as follows:

$$[\, u \quad w \quad \sigma_x \quad \sigma_z \quad \tau_{zx}\,]^T = \sum_{i=1}^{n} [\, f_1^i g_1^i \quad f_2^i g_2^i \quad f_3^i g_3^i \quad f_4^i g_4^i + [p_a^s + z p_d^s] \quad f_5^i g_5^i \,]^T \tag{3}$$

where f_l and g_l are unknown functions of ξ and ζ for the x and z-direction, respectively, and $p_a^s = -(p_2^s)/2$ and $p_d^s = -(p_2^s)/h$. The functions $f_l^i(\xi)$ depend on the s_{th} segment while the $g_l^i(\xi)$ functions are valid for all segments.

3.1 FIRST ITERATIVE STEP - SOLVING FUNCTIONS $g_l^i(\zeta)$

In this iteration step, the thickness function $g_l^i(\zeta)$ will be solved and obtained in closed-form. Hence, variation can be expressed as follows:

$$[\, \delta u \quad \delta w \quad \delta\sigma_x \quad \delta\sigma_z \quad \delta\tau_{zx}\,]^T = \sum_{i=1}^{n} [\, f_1^i \delta g_1^i \quad f_2^i \delta g_2^i \quad f_3^i \delta g_3^i \quad f_4^i \delta g_4^i \quad f_5^i \delta g_5^i \,]^T \tag{4}$$

The functions $g_l^i(\zeta)$ can be divided into two parts as $\bar{\mathbf{G}}$ and $\hat{\mathbf{G}}$. Here $\bar{\mathbf{G}}$ is a column vector of dimension $4n$ and contains independent variables that appear in the boundary conditions of the top and bottom surfaces of the beam, while $\hat{\mathbf{G}}$ is a column vector of size $1n$ and contains the remaining dependent variables. Considering the solution Equation 3, and its variational part Equation 4 and substituting into Equation 2, integration by parts along the x-axis can be performed. Since the variation is arbitrary, the coefficient of δg^i must vanish, which yields the following set of governing equations for $\bar{\mathbf{G}}$ and $\hat{\mathbf{G}}$, as follows:

$$\bar{\mathbf{G}}_{,\zeta} = \mathbf{M^{-1}}[\bar{\mathbf{A}}\bar{\mathbf{G}} + \hat{\mathbf{A}}\hat{\mathbf{G}} + \bar{\mathbf{Q}}_{\mathbf{p}}] \tag{5}$$
$$\hat{\mathbf{G}} = [\mathbf{K^m}]^{-1}[\tilde{\mathbf{A}}\bar{\mathbf{G}} + \tilde{\mathbf{Q}}_{\mathbf{p}}] \tag{6}$$

Here, M, $\bar{\mathbf{A}}$, $\hat{\mathbf{A}}$, $\mathbf{K}$, and $\tilde{\mathbf{A}}$ are coefficient matrices. The substitution of $\hat{\mathbf{G}}$ from Equation 7 into Equation 5 transforms equation 5 as follows:

$$\bar{\mathbf{G}}_{,\zeta} = \mathbf{A}\bar{\mathbf{G}} + \mathbf{Q}_{\mathbf{p}} \tag{7}$$

where Equation 7 is a set of $4n$ first-order coupled ODEs with constant coefficients and this system of ODEs can be solved analytically by following solution approach suggested by [24].

3.2 SECOND ITERATIVE STEP - SOLVING FUNCTIONS $f_l^i(\xi_1)$

In first iterative step $g_l^i(\zeta)$ functions have been obtained in closed-form manners. Now, in this iteration step, the obtained $g_l^i(\zeta)$ functions are used to identify the f_l^i functions, which are assumed as unknown in this iteration. Therefore, variation is assumed in the f_l^i functions as follows:

$$[\,\delta u \quad \delta w \quad \delta\sigma_x \quad \delta\sigma_z \quad \delta\tau_{zx}\,]^T = \sum_{i=1}^{n} [\,g_1^i\delta f_1^i \quad g_2^i\delta f_2^i \quad g_3^i\delta f_3^i \quad g_4^i\delta f_4^i \quad g_5^i\delta f_5^i\,]^T \tag{8}$$

Similar to the first iteration, the in-plane functions $f_l^i(\xi)$ can also be split up into two-column vectors $\hat{\mathbf{F}}$ and $\bar{\mathbf{F}}$. Here, $\bar{\mathbf{F}}$ contains these specific independent variables that appear in the inter-segment continuity and edge conditions along the x-directions and the $\hat{\mathbf{F}}$ column vector contains the remaining dependent variables. Substituting Equation 8 in Equation 2 and equating the coefficient of δf_l^i to zero individually after performing integration along z-direction leads to a set of differential-algebraic equations as follows:

$$\bar{\mathbf{F}}_{,\xi} = \mathbf{N}^{-1}[\bar{\mathbf{B}}^{\mathbf{f}}\bar{\mathbf{F}} + \hat{\mathbf{B}}^{\mathbf{f}}\hat{\mathbf{F}} + \bar{\mathbf{P}}^{\mathbf{f}}_{\mathbf{m}}] \tag{9}$$

$$\hat{\mathbf{F}} = \mathbf{L}^{-1}[\tilde{\mathbf{B}}^{\mathbf{f}}\bar{\mathbf{F}} + \tilde{\mathbf{P}}^{\mathbf{f}}_{\mathbf{m}}] \tag{10}$$

However, $g_l^i(\zeta)$ are already known in closed-form manners from the previous iteration. Hence above set of equations can be solved similarly by applying the edge conditions and inter-segment continuity conditions along the x-direction on $\bar{\mathbf{F}}$. These two iterative steps can be continued to get the final converged solution up to the needed level of accuracy.

4 Numerical results and discussion

In this section, the present approach for analyzing the flexural response of beams with loading discontinuities is first elaborated. For this purpose, a single-layer aluminum beam (E=70 Gpa, ν=0.3) is being considered, as depicted in Figure 1. Numerical results are being presented for different boundary conditions and thickness ratios ($S = a/h$), and the obtained results being expressed in a non-dimensional form as:
$(\bar{u}, \bar{w}) = 100(u, w/S)E_0/p_0hS^3$; $(\bar{\sigma}_x,\, \bar{\tau}_{zx}) = (\sigma_x,\, S\tau_{zx})/p_0S^2$ with $E_0 = 70GPa$.

The longitudinal distribution of deflections and stresses under localized patch loads is a crucial aspect of beam mechanics that needs to be explored. The variation of deflections and stresses for a single-layer aluminum beam subject to a locally distributed patch load of length $0.1a$ at its center is depicted in Figure 2 for simply supported (S-S) and clamped–free supported (C-F) end conditions. To validate the numerical results, 2D finite element (FE) results are also included in Figure 2, along with the present analytical results. The 2D FE analysis was conducted using a plane stress element with a mesh size of 50×18 along the axial and transverse directions. Excellent agreement is observed between the two solutions, showing that the present approach can accurately predict the mechanical behavior of beams under loading discontinuity. It should be noted that the locally distributed load induces very high localized normal stress

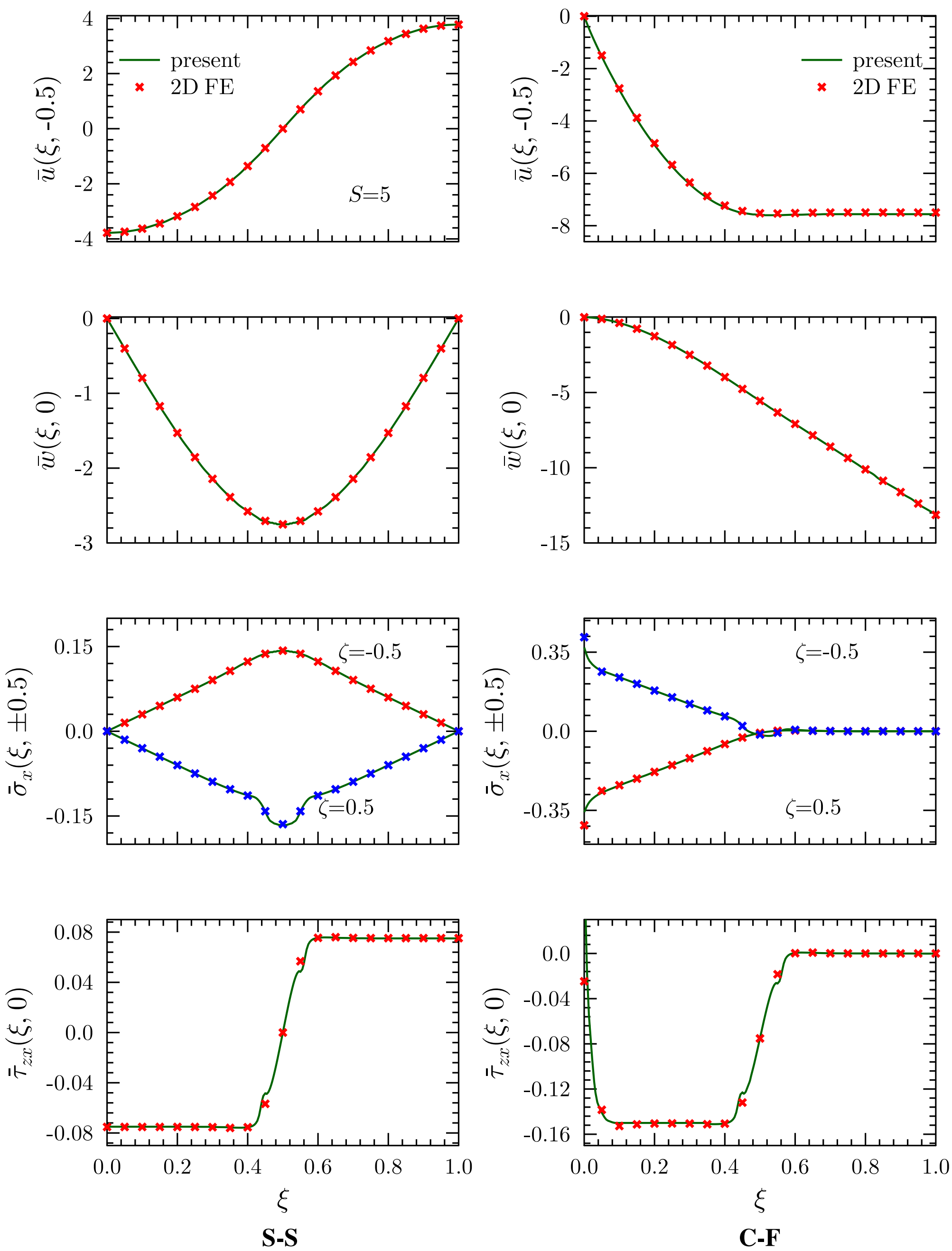

Figure 2: Longitudinal variations of displacements and stresses under S-S and C-F boundary conditions for thick (S=5) single-layered beam subjected to the locally distributed patch load of length $0.1a$ at center.

concentrations in the vicinity of the applied load, while the stress concentration at the bottom of the beam is significantly lower than at the top. Additionally, a sharp change in transverse stress ($\bar{\tau}_{zx}$) from negative to positive is observed in the vicinity of the localized distributed load. The sharp variations in axial and transverse stresses in the vicinity of the localized load can be accurately and efficiently predicted by our analytical solution. Finally, it is noted that the variation of deflections is smooth along the length of the beam as compared to stresses, further emphasizing the importance of our study.

To study the effect of thickness to span ration of beam on axial stress concentration under localized distributed load, the longitudinal variation of axial normal stress($\bar{\sigma}_x$) are plotted in Figure 3 for different aspect ratios *i.e.* S=5, 10 and 20. The results are plotted for thick aluminium beam $S = 5$ subjected to patch load of length $0.1a$ at center under S–S, C–S, C–C and C–F support conditions. It is observed that the localized load leads to very high localized

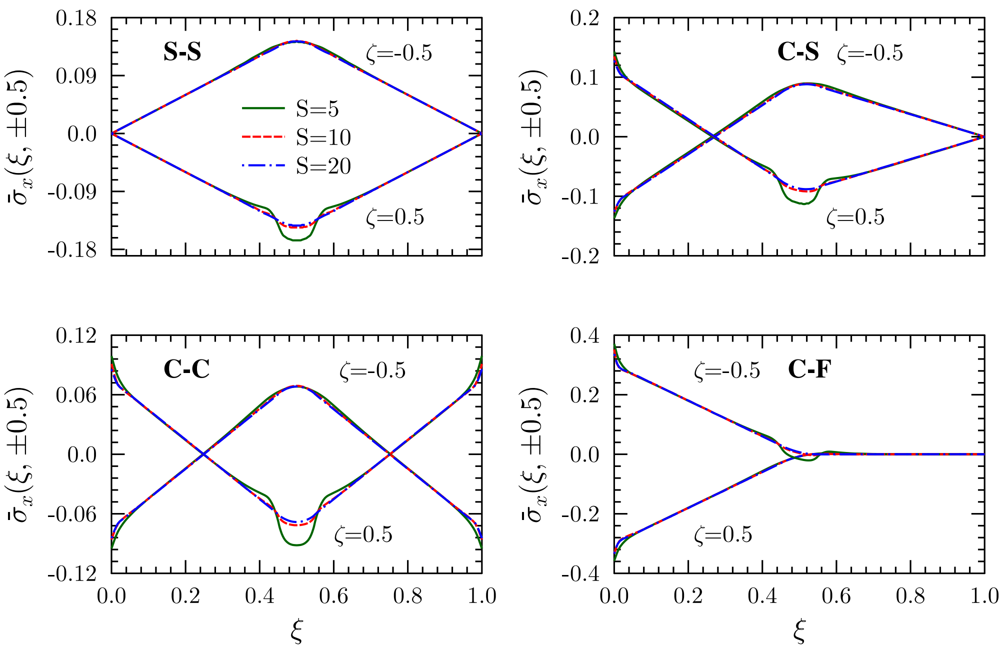

Figure 3: Variation of normal stress $\bar{\sigma}_x$ for S–S, C–S, C–C, and C–F beam pf different thickness ratios, $S = 5$, 10, and 20 subjected to a patch load of length $0.1l$ at center.

axial stress ($\bar{\sigma}_x$) in the vicinity of applied patch load in the thick beams. But as the thickness of beams decreases, the variation of normal axial stress becomes smooth and distributed. In the thick beams, the stress concentration in the vicinity of patch load is higher for the beam subjected to clamped-clamped (C-C) support conditions and lowest for the beam subjected to clamped-free (C-F) support conditions. The thick beams observe maximum stress concentration in the vicinity of the patch load under all the support conditions, where no stress concentration occurs in the thin beam.

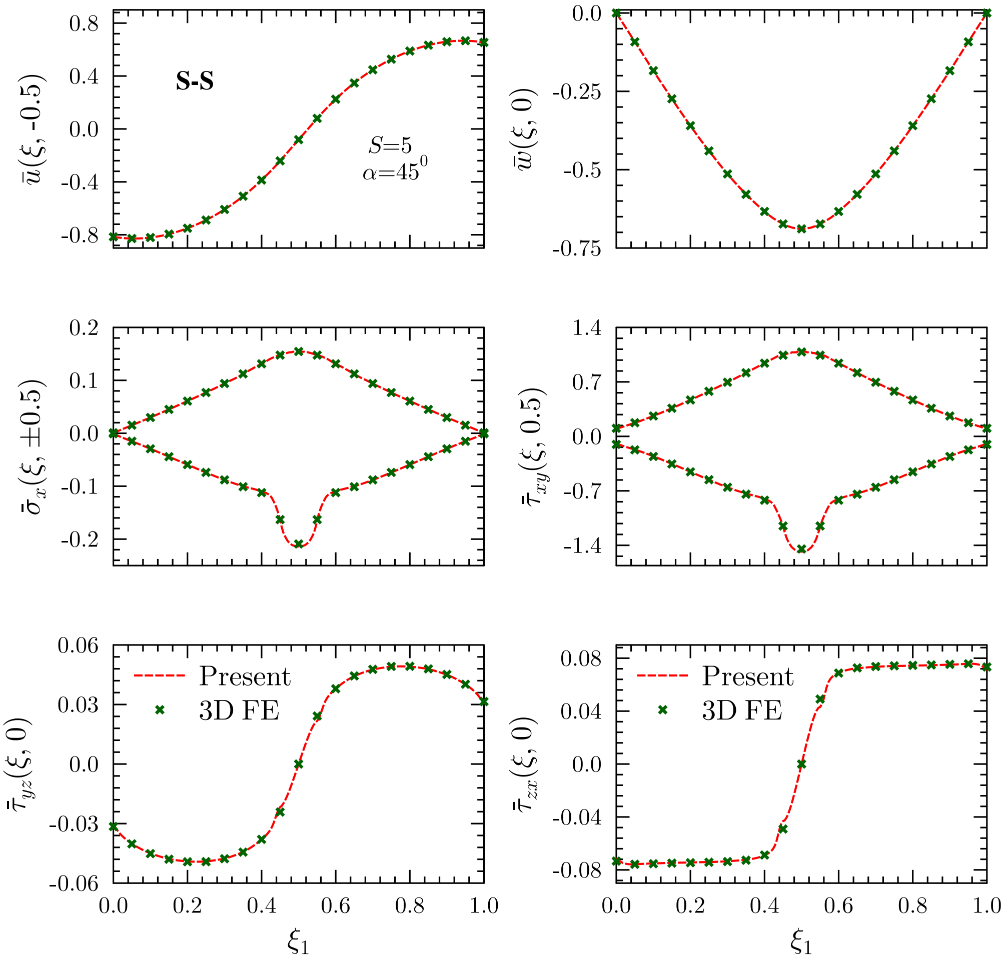

Figure 4: Longitudinal variations of displacements and stresses under S-S boundary conditions for thick (S=5) single-layered composite plate subjected to the locally distributed patch load of length $0.1a$ at center.

Further, the present approach has been extended to analyze an angle-ply plate subjected to a discontinuous load with material properties E_1=181 GPa, E_2=E_3=10.3 GPa, G_{23}=2.87 GPa, G_{13} = G_{12}=7.17 GPa, ν_{12}= ν_{13}= ν_{23}=0.33 . The variation of deflections and stresses for a single-layer angle-ply composite thick plate subject to a locally distributed patch load of length $0.1a$ at its center is depicted in Figure 4, for simply supported (S-S) end conditions. For numerical validation, 3D finite element (FE) results are also included in Figure 4, along with the present analytical results. A quadratic 20-node brick element with reduced integration (C3D20R) was used for the 3D FE analysis with a mesh size of $50 \times 50 \times 18$ along the in-plane and transverse directions, respectively, employing a plane stress element. The excellent agreement observed for the 3D case indicates the generality of the present approach. It is observed that the variation of normal axial ($\bar{\sigma}_x$) and shear ($\bar{\tau}_{xy}$) stresses is highly localized in the vicinity of the load. The high-stress concentration appears at the top surface in the vicinity of the localized load, and the distribution of normal and shear stresses is highly asymmetric across the thickness. These results demonstrate that the present approach can accurately predict the mechanical behavior of structures subjected to loading discontinuity.

5 Conclusions

An innovative approach has been developed for accurately mechanical analysis of the beams and plates subject to discontinue loading such as distributed patch loads, regardless of their support conditions. The approach employs a mixed variational principle-based formulation and the extended Kantorovich method to obtain an iterative analytical solution. Extensive numerical study is conducted to evaluate the flexural behavior of beams and composite plates under discontinue loading, which demonstrated that the model can efficiently and accurately capture the highly localized stress concentrations and their non-linear variation along the transverse direction. The analytical results provide can further be used for evaluating the static bending response of composite structures with material discontinuity in the axial direction. Additionally, the numerical results presented in this study serve as a benchmark for the general-case analytical evaluation of composite structures under discontinuous loads, surpassing the applicability limits of classical laminate theories.

REFERENCES

[1] Y. Ghugal and R. Shimpi, “A review of refined shear deformation theories for isotropic and anisotropic laminated beams,” *Journal of reinforced plastics and composites*, vol. 20, no. 3, pp. 255–272, 2001.

[2] K. Liew, Z. Pan, and L. Zhang, “An overview of layerwise theories for composite laminates and structures: Development, numerical implementation and application,” *Composite Structures*, vol. 216, pp. 240–259, 2019.

[3] A. S. Sayyad and Y. M. Ghugal, “Bending, buckling and free vibration of laminated composite and sandwich beams: A critical review of literature,” *Composite Structures*, vol. 171, pp. 486–504, 2017.

[4] S. G. Lekhnitskii, “Anisotropic plates,” Foreign Technology Div Wright-Patterson Afb Oh, Tech. Rep., 1968.

[5] I. K. Silverman, “Orthotropic beams under polynomial loads,” *Journal of the Engineering Mechanics Division*, vol. 90, no. 5, pp. 293–319, 1964.

[6] Z. Hashin, “Plane anisotropic beams,” *Journal of Applied Mechanics*, vol. 34, no. 2, pp. 257–262, 1967.

[7] R. W. Gerstner, “Stresses in a composite cantilever,” *Journal of Composite Materials*, vol. 2, no. 4, pp. 498–501, 1968.

[8] K. Rao and B. Ghosh, "Exact analysis of unsymmetric laminated beam," *Journal of the Structural Division*, vol. 105, no. 11, pp. 2313–2325, 1979.

[9] S. Cheng, X. Wei, and T. Jiang, "Stress distribution and deformation of adhesive-bonded laminated composite beams," *Journal of engineering mechanics*, vol. 115, no. 6, pp. 1150–1162, 1989.

[10] Ü. Esendemir, M. R. Usal, and M. Usal, "The effects of shear on the deflection of simply supported composite beam loaded linearly," *Journal of reinforced plastics and composites*, vol. 25, no. 8, pp. 835–846, 2006.

[11] W. Chen, C. Lü, and Z. Bian, "A mixed method for bending and free vibration of beams resting on a pasternak elastic foundation," *Applied Mathematical Modelling*, vol. 28, no. 10, pp. 877–890, 2004.

[12] H. Subramanian and S. S. Mulay, "On the homogenization of a laminate beam under transverse loading: extension of pagano's theory," *Acta Mechanica*, vol. 232, no. 1, pp. 153–176, 2021.

[13] L. C. Trinh, S. O. Ojo, R. M. Groh, and P. M. Weaver, "A mixed inverse differential quadrature method for static analysis of constant-and variable-stiffness laminated beams based on hellinger-reissner mixed variational formulation," *International Journal of Solids and Structures*, vol. 210, pp. 66–87, 2021.

[14] O. Doeva, P. K. Masjedi, and P. M. Weaver, "A semi-analytical approach based on the variational iteration method for static analysis of composite beams," *Composite Structures*, vol. 257, p. 113110, 2021.

[15] N. Pagano and A. Wang, "Further study of composite laminates under cylindrical bending," *Journal of Composite Materials*, vol. 5, no. 4, pp. 521–528, 1971.

[16] S. Kapuria, P. Dumir, and N. Jain, "Assessment of zigzag theory for static loading, buckling, free and forced response of composite and sandwich beams," *Composite structures*, vol. 64, no. 3-4, pp. 317–327, 2004.

[17] P. Kumari and A. Singh, "Three-dimensional analytical solution for fgm plate with varying material properties in in-plane directions using extended kantorovich method," in *Recent Advances in Structural Engineering, Volume 1: Select Proceedings of SEC 2016*. Springer Singapore Singapore, 2018, pp. 611–621.

[18] A. Singh and P. Kumari, "Three-dimensional free vibration analysis of composite fgm rectangular plates with in-plane heterogeneity: An ekm solution," *International Journal of Mechanical Sciences*, vol. 180, p. 105711, 2020.

[19] A. Singh, S. Naskar, P. Kumari, and T. Mukhopadhyay, "Viscoelastic free vibration analysis of in-plane functionally graded orthotropic plates integrated with piezoelectric sensors: Time-dependent 3d analytical solutions," *Mechanical Systems and Signal Processing*, vol. 184, p. 109636, 2023.

[20] A. Singh and P. Kumari, "Two-dimensional free vibration analysis of axially functionally graded beams integrated with piezoelectric layers: An piezoelasticity approach," *International Journal of Applied Mechanics*, vol. 12, no. 04, p. 2050037, 2020.

[21] A. Singh and K. Poonam, "Two-dimensional elasticity solution for arbitrarily supported axially functionally graded beams," *Journal of Solid Mechanics*, vol. 10, no. 4, pp. 719–733, 2018.

[23] A. Singh and P. Kumari, "Analytical free vibration solution for angle-ply piezolaminated plate under cylindrical bending: A piezo-elasticity approach," *Advances in Computational Design*, vol. 5, no. 1, pp. 55–89, 2020.

[24] A. Singh and S. Kapuria, "Analytical elasticity solution for accurate prediction of localized stresses in laminated composites under patch loading," *European Journal of Mechanics-A/Solids*, vol. 95, p. 104624, 2022.