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ULTRASONIC PROPAGATION IN FRACTAL POROUS MATERIAL HAVING RIGID FRAME

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Abstract. This paper discusses the impact of the fractal structure of porous materials on wave behavior. In addition to commonly used parameters such as porosity, tortuosity, viscous and thermal characteristic lengths, a fractal dimension (α) can be introduced to represent the self-similarity of the material. The Helmholtz equation for wave propagation in a porous medium can then be modified to depend on non-integer dimensions. The fractal structure affects the wave's speed, attenuation, and phase shifts, with supersonic wave speeds possible for for certain values of the fractal dimension. The equivalent thickness of the material also varies with the fractal dimension becoming larger for low values of the fractal dimension. This provides us with valuable insights into the potential for creating novel metamaterials with exceptional acoustic properties by adjusting the fractal dimension value, enabling the attainment of supersonic velocities and substantial equivalent thicknesses. Understanding the effect of the fractal dimension on wave behavior has important implications for developing effective acoustic materials and can inform research in noise reduction, metamaterials, medicine, seismology, geophysics, and petroleum engineering.

Key words: Porous Materials, Fractals, Ultrasound, Attenuation, Fractal Dimension, supersonic waves

1 INTRODUCTION

Fractals are complex geometric structures that exhibit self-similarity at different scales or magnifications, and have been widely applied in various scientific and technological fields, including physics, biology, finance, mathematics, computer science, acoustics, and non-destructive

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testing [1, 2, 3, 4]. In the field of materials mechanics, fractals have been used to model the structure and properties of fractal materials such as polymers, metals, composites, and porous materials, and to predict their mechanical properties, including fracture toughness and hardness. In acoustics, fractals have been used to model the propagation of sound in complex and heterogeneous media such as porous materials [5, 6] and granular media, and to analyze complex acoustic signals such as cardiac and vocal signals. In non-destructive testing, fractals have been used to detect defects in materials such as cracks, inclusions, and porosities, and to measure surface properties of materials, such as roughness [7] and texture, using fractal-based image processing techniques.

In recent years, there has been growing interest in the study of sound propagation in porous media with fractal structure. Porous media with self-similar structure, referred to as fractal porous media, are characterized by complex geometries and exhibit unique acoustic properties that are different from those of traditional porous media. Several studies [5, 6, 8, 9] have investigated the acoustic behavior of fractal porous media, including the effects of fractal geometry on acoustic absorption, transmission, and scattering.

According to the power-law relationship proposed by Mandelbrot [1], the mass M of a spherical portion of a fractal material is related to its radius R through the expression: $M \propto R^D$,, where D represents the mass fractal dimension. This mathematical relationship implies that the rate at which the mass of a fractal object increases is determined by its fractal dimension D. As the size of the object increases, its mass grows at a specific rate in proportion to the power of its fractal dimension, as stated by Crownover [10]. If a porous material demonstrates a high degree of self-similarity, it can be considered as a fractal [11, 12, 13]. This self-similarity is observed in the repetition of the material's structure at different scales, particularly in the pore structure, where smaller pores share similar shapes and arrangements with larger ones [14].

The study of differential operators in non-integer dimensional spaces is a crucial area of research in fractal geometry [15, 16]. Traditional differential calculus deals with smooth, continuous spaces such as Euclidean space or the real line. However, many natural phenomena exhibit non-smooth or discontinuous behavior, which is better modeled using fractals or other non-integer dimensional spaces [17].

1.1 NON-INTERGER DIMENSIONAL SPACE MODEL

An approach proposed by Wilson [18], Stillinger [19], and Collins [20] involves integration and differentiation on non-integer spaces. This approach employs integration on such spaces for dimensional regularization in physics domains such as quantum theory [20, 21, 22] and physical kinetics [23, 24]. The Laplace scalar operators for non-integer dimensional spaces introduced by Stillinger [19], Palmer, and Stavrinou [15] have proven to be effective in various areas of physics, including quantum mechanics [19, 15, 25], electrodynamics [26, 27], scattering processes [28], and general relativity [29]. However, the product measure approach [19, 15] fails to consider the generalizations of gradient, divergence, curl, and Laplace vector operators for fractional and non-integer spaces, which are necessary to describe anisotropic fractal media. Tarasov's work provides these generalizations [17, 16]. Balankin *et al.* [30, 31] have developed a differential vector calculus based on the concept of metric derivative and propose two possible definitions of vector differential operators in fractional space.

Our research focuses on Tarasov's method [17, 16] to analyze wave propagation in fractal porous media. While a previous study investigated this phenomenon in the time domain [8], our approach solves the problem in the frequency domain to gain deeper insights into the interplay between the medium's structural properties and wave propagation.

Tarasov's method adapts the product measure technique, which was originally proposed by Stillinger [19] and Svozil [32]. The key to Tarasov's method is defining the nabla operator [17] as follows:

$$\nabla_\alpha = \sum_{i=1}^{3} \frac{\mathbf{e}i}{W(\alpha_i, x_i)} \cdot \nabla x_i \tag{1}$$

The term α_i denotes the non-integer fractal dimension of a line in a given direction, ranging from 0 (for a perfectly fractal material) to 1 (for a non-fractal medium). The density of states, denoted by the function $W(\alpha_i, x_i)$, is defined by the measure for integration in non-integer dimensional space and can be expressed as follows:

$$W(\alpha_i, x_i) = \frac{\pi^{\alpha_i/2}}{\Gamma(\alpha_i/2)} \left|x_i\right|^{\alpha_i - 1}, \tag{2}$$

our main objective is to apply Tarasov's method to reframe the fundamental equations of acoustics in a rigid, porous medium. Our ultimate aim is to solve the propagation equation and obtain the reflection and transmission operators in the frequency domain.

2 ACOUSTIC OF POROUS MEDIA

The motion of a fluid, including the propagation of waves, can be described by the Euler Equations, a subset of the Navier-Stokes equations that accounts for the absence of viscosity [33, 34, 35]. When a wave interacts with a fluid, it causes minor fluctuations in the fluid's velocity, pressure, and density. The Euler equations can be expressed as:

$$\frac{\partial}{\partial t}(\rho_0 \mathbf{v}) + \nabla p \;= \mathbf{0}, \quad \frac{\partial p}{\partial t} + K_a \nabla \cdot \mathbf{v} \;= 0. \tag{3}$$

where $\mathbf{v}$ is the fluid velocity, p the pressure, ρ_0 the density, and K_a is the bulk modulus of the fluid. Using Equations (3), we can easily obtain the following wave equation:

$$\Box p(\mathbf{r}, t) = 0, \tag{4}$$

where $\Box$ is the d'Alembert operator defined as $\Box = \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2$, $c_0 = \sqrt{K_a/\rho_0}$ is the velocity of sound in air. The presence of solid particles or other porous structures in a medium can influence the fluid motion when a wave propagates through it. To describe fluid motion in such a medium, we can extend the Euler equations to account for the porous structures. These equations are known as the Biot equations [36]. In the case of a wave propagating through a homogeneous and isotropic porous material saturated with air, such as porous foams, we can simplify the Biot model significantly. In this simplified scenario, the behavior of waves in a porous medium can be approximated using the equivalent fluid model [37]. This model assumes that the solid structures in the porous medium are rigid and do not vibrate when interacting with the wave

$$i\omega\rho(\omega)\mathbf{V}(\omega) = -\nabla P(\omega), \qquad i\omega P(\omega) = -K(\omega)\nabla\mathbf{V}(\omega), \tag{5}$$

here, $P(\omega)$, and $\mathbf{V}(\omega)$, respectively, represent the Fourier transform of $p(t)$, and $\mathbf{v}(t)$. In this case we have the equivalent density $\rho(\omega) = \rho_0\epsilon(\omega)$, instead of just the density ρ_0, taking into account the viscous interactions between the fluid and the solid structure that are described by the dynamic tortuosity $\epsilon(\omega)$. Moreover, we have $\frac{1}{K(\omega)} = \frac{\beta(\omega)}{K_a}$ instead of K, taking into account the thermal exchanges that are described by the thermal compressibility $\beta(\omega)$. The dynamic tortuosity [38] and the thermal compressibility [39] can be written as:

$$\epsilon(\omega) = \epsilon_\infty + \frac{2\epsilon_\infty}{\sqrt{\Lambda}}\left(\frac{\eta}{i\omega\rho}\right)^{\frac{1}{2}}, \quad \beta(\omega) = 1 + \frac{2(\gamma-1)}{\sqrt{\Lambda'}}\left(\frac{\eta}{i\omega\rho P_r}\right)^{\frac{1}{2}}, \tag{6}$$

where ϵ_∞ is the tortuosity, Λ the viscous characteristic length, η the dynamic viscosity, γ the adiabatic constant, Λ' the thermal characteristic length, and P_r Prandtl number. Using Equations (5), we can easily obtain the Helmholtz equation in the frequency domain:

$$\nabla^2 P(\mathbf{r},\omega) + k(\omega)^2 P(\mathbf{r},\omega) = 0, \tag{7}$$

where $k(\omega) = w/c(\omega)$ is the characteristic wave number, and $c(\omega) = \overline{\frac{K_a}{\rho\epsilon(\omega)\beta(\omega)}}$, is the phase velocity.

3 FRACTAL POROUS MATERIAL

A fractal porous material is a type of porous material that displays a complex, self-similar geometric structure, as described by Mandelbrot [1] and Adler [12] using fractals, which are mathematical objects exhibiting repeating patterns at different scales. Fractal geometry in porous materials is a result of the way the material is constructed, such as particle packing or crystal growth, and has significant effects on its physical and mechanical properties, including permeability, porosity, and mechanical strength [12].

The pore space in a fractal porous medium has a complex and self-similar structure characterized by a non-integer fractal dimension, making it unsuitable to describe using standard Euclidean geometry based on integer dimensions. By using the previously defined fractal gradient operator (1), the Euler equations (5), and the Helmholtz equation (7) can be expressed as follows:

$$i\omega\rho(\omega)\mathbf{V}(\mathbf{r},\omega) = -\nabla_\alpha P(\mathbf{r},\omega), \quad \frac{1}{K(\omega)} i\omega P(\mathbf{r},\omega) = -\nabla_\alpha \mathbf{V}(\mathbf{r},\omega) \quad , \tag{8}$$

$$\triangle_\alpha P(\mathbf{r},\omega) + k(\omega)^2 P(\mathbf{r},\omega) = 0, \tag{9}$$

when examining the propagation of a wave in a fractal porous material along the x-axis, the aforementioned equations can be expressed as:

$$V(x,\omega) = -\frac{1}{i\omega\rho(\omega)W(x,\alpha)}\frac{\partial}{\partial x}\left(\frac{1}{W(x,\alpha)}\frac{\partial P(x,\omega)}{\partial x}\right) \tag{10}$$

$$P(x,\omega) = -\frac{i\omega K(\omega)}{W(x,\alpha)}\frac{\partial}{\partial x}\left(\frac{1}{W(x,\alpha)}\frac{\partial V(x,\omega)}{\partial x}\right) \tag{11}$$

$$\frac{1}{W(x,\alpha)}\frac{\partial}{\partial x}\left(\frac{1}{W(x,\alpha)}\frac{\partial P(x,\omega)}{\partial x}\right) + k(\omega)^2 P(x,\omega) = 0, \tag{12}$$

where $k(\omega) = \omega\sqrt{\frac{\rho(\omega)}{K(\omega)}}$. The equation identified as (12) can be restated as:

$$\nabla^2\left(\frac{P(x,\omega)}{x^{\alpha-1}}\right) + k_\alpha(\omega)^2 P(x,\omega) = 0 \tag{13}$$

where $k_\alpha(\omega) = k(\omega)\frac{\pi^{\alpha_i/2}}{\Gamma(\alpha_i/2)}$ is an equivalent wave number that is proportional to the original wave number, but is also modified by the fractal dimension of the medium. In a porous medium, the wave number can be a complex quantity due to the presence of attenuation and dispersion effects [37, 40]. The real part of the wave number represents the spatial variation of the wave, while the imaginary part represents the attenuation of the wave due to the dissipative properties of the medium. The wave number $k(\omega)$ in a porous medium can be expressed as follows [37]:

$$k(\omega) = \frac{\omega}{c_p(\omega)} + ia(\omega), \tag{14}$$

where $c_p(\omega)$ is the phase velocity, and $a(\omega)$ is the attenuation, In the case of a fractal porous medium, we obtain:

$$k_\alpha(\omega) = \frac{\omega}{c_p^\alpha(\omega)} + ia_\alpha(\omega), \tag{15}$$

where

$$c_p^\alpha(\omega) = c_p(\omega)\frac{\pi^{\alpha_i/2}}{\Gamma(\alpha_i/2)}, \qquad a_\alpha(\omega) = a(\omega)\frac{\pi^{\alpha_i/2}}{\Gamma(\alpha_i/2)}, \tag{16}$$

are , respectively, the equivalent phase velocity, and the equivalent attenuation.

Consider a porous material plate of thickness L, traversed by an incident acoustic wave and producing a reflected wave and a transmitted wave, taking into account the continuity conditions of pressure p and flow ϕv, we obtain the following expressions for the reflection and transmission coefficients

$$R = \frac{x - y\tanh(\gamma_\alpha/2)}{x + y\tanh(\gamma_\alpha/2)}, \tag{17}$$

$$T = \frac{2x}{x + y\tanh(\gamma_\alpha/2)}. \tag{18}$$

where $x = \frac{1}{2}(1 + Z^2)$, $y = Z\sqrt{x^2 - 1}$, $Z = \phi\frac{Z_f}{Z_m}$, $Z_f = \sqrt{\rho_0 K_a}$, $Z_m = \sqrt{\rho(\omega)K(\omega)}$, $\gamma_\alpha = k_\alpha L_\alpha$. $L_\alpha = L^\alpha/\alpha$ is a measure of the effective path length or size of the medium that considers its intricate, self-similar structure.

4 Discussion

Porous materials are commonly characterized by parameters such as porosity (ϕ), tortuosity (ϵ_∞), and the viscous and thermal characteristic lengths (Λ and Λ') [37]. However, by considering the fractal nature of the material, an additional parameter, α, can be introduced to represent the fractal dimension of the material. This dimension takes non-integer values and provides insight into the level of self-similarity exhibited by the porous medium in a specific direction, such as the direction of wave propagation. As the fractal dimension of the porous material approaches zero, the material demonstrates a high degree of self-similarity at different scales. When $\alpha = 1$, the material is considered to be non-fractal. Incorporating the self-similar structure of the material can result in a more comprehensive understanding of its properties and increased accuracy of the parameters ϕ, α_∞, Λ, and Λ'.

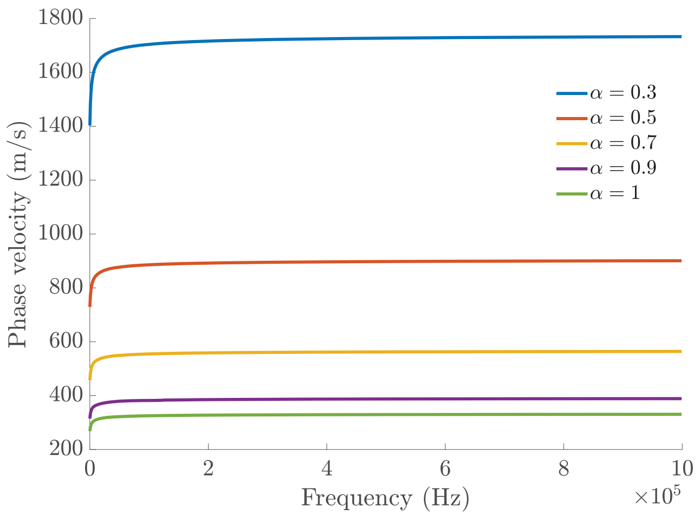

Figure 1: Plot of the equivalent phase velocity defined by Eq.16 with respect to the frequency for different values of the fractal dimension α

The introduction of the fractal dimension in the Helmholtz equation governing wave propagation in a porous medium leads to the dependence of the resulting wave number in Equation (15), phase velocity and attenuation in (16) on non-integer dimensions. This observation emphasizes the impact of the porous medium's fractal structure on the wave's behavior and its propagation characteristics. In other words, the medium's self-similarity is a key factor in shaping the wave's movement through it.

Figure (1) illustrates the variation of the phase velocity with respect to frequency for different

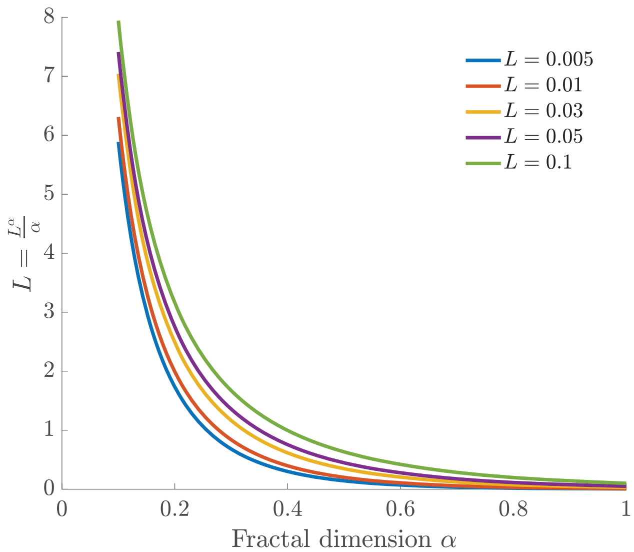

Figure 2: Plot of the equivalent thickness $L_\alpha = L^\alpha/\alpha$ with respect to the fractal dimension α for different values of the porous material thickness $L = [0.005, 0.01, 0.03, 0.05, 0.1]m$

values of the fractal dimension α. It shows some interesting results: for instance, for values of $\alpha < 1$, supersonic speeds of the wave are observed. Supersonic wave speeds refer to the propagation of waves that travel faster than the speed of sound in the medium, which is not typically observed in most materials and is generally associated with high-energy phenomena such as shock waves or explosions [41]. However, the fractal structure of a porous material can introduce irregularities in the medium that allow for the possibility of supersonic wave speeds under certain conditions. When a wave propagates through such a medium, it encounters a complex network of pores and voids that can lead to interference and scattering, causing the wave to travel faster than it would in a regular medium.

Incorporating the fractal approach reveals an intriguing outcome: the equivalent thickness of the material. This parameter provides valuable insights into wave behavior within a fractal porous medium (Figure 2). As the fractal dimension α approaches 0, L_α increases, while approaching 1 leads to L_α converging to L. When waves propagate through a highly self-similar fractal porous material, they encounter repetitive structures that can cause reflection or refraction. This phenomenon induces constructive or destructive interference, amplifying or attenuating the waves. Moreover, these repeated patterns increase the path length, enhancing scattering and attenuation. Conversely, as the fractal dimension approaches 1, the porous material exhibits a more random internal structure, reducing the chances of encountering similar structures across scales. Consequently, waves follow more direct paths, resulting in reduced scattering and attenuation. The concept of equivalent thickness reveals the direct influence of fractal self-similarity on the effective thickness of classical porous materials.

To further illustrate the influence of a material's self-similarity on wave behavior, two distinct

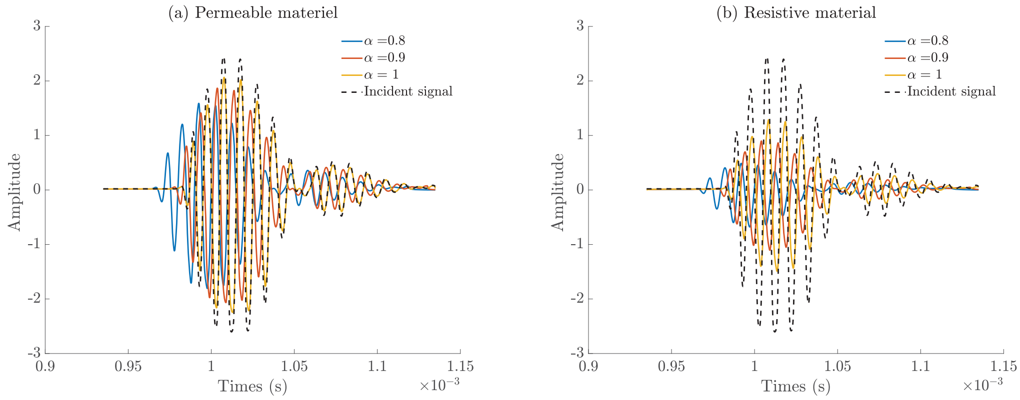

Figure 3: Transmitted waves for different values of the α fractal dimension, compared with an experimental incident wave (a) Highly permeable material, (b) Lowly permeable material

scenarios were analyzed. In the first scenario, depicted in Figure 3(a), a highly permeable material with a thickness of 0.5 cm, a porosity of 0.95, a tortuosity of 1.05, and a viscous characteristic length of 200 μm was used. The thermal length was set at $\Lambda' = 3\Lambda$. In contrast, the second scenario (Figure 3(b)) featured a very low permeability material with a thickness of 0.5 cm, a porosity of 0.7, a tortuosity of 1.35, and a viscous characteristic length of 65 μm. In both scenarios, the experimental incident wave, measured using airborne transducers with a frequency of 100 KHz in air, was compared to theoretical waves generated numerically using the transmission coefficient for different values of α. The comparison of these scenarios provides insight into how the self-similar structure of a porous medium affects the propagation of waves through it, as well as the resulting attenuation and phase shifts.

The figures obtained from the analysis reveal some interesting observations about the impact of fractal structure on wave behavior. Specifically, we observe that for values of $\alpha < 1$, the waves are generated ahead of the incident wave, indicating that their speed is higher than that of sound in air. This behavior has been discussed earlier. However, when we compare how fractal structure affects highly permeable versus low-permeability materials, we find significant differences.

One major difference pertains to the amplitude of the waves. We see that for $\alpha < 1$, the attenuation in the permeable material increases. This suggests that there exists a relationship between the fractal nature of the medium and the dissipation of energy in the wave. When a wave travels through a self-similar porous material, it encounters a complex and irregular structure that causes it to scatter and interact with the medium in a more intricate way. This leads to a greater degree of energy loss, or attenuation, as the wave propagates through the medium. Essentially, the more self-similar the porous material is, the more challenging it is for the wave to propagate through it without losing energy. In contrast, a low-permeability material exhibits even greater energy loss. This is because the tortuosity of such materials is relatively large, making it more difficult for the wave to propagate through the medium.

In summary, the figures obtained from the analysis illustrate how the fractal structure of a porous material can affect the behavior of waves propagating through it. By studying two distinct scenarios, we can see how the amplitude and attenuation of waves can vary depending on the degree of self-similarity and the material's properties.

The second contrast that can be observed between the permeable and low-permeability materials is related to the velocity of the wave, as shown in Figure **??**. Surprisingly, we find that the wave velocity increases even more in a highly permeable material, whereas in a low-permeability material, the wave encounters more resistance, resulting in a slower wave speed and greater attenuation. The low-permeability nature of the material impedes the propagation of the wave, leading to a lower speed and greater energy loss as the wave traverses the medium.

In conclusion, the study sheds new light on the complex interplay between a material's fractal structure and its acoustic properties. The findings suggest that low-permeability materials may have potential applications where wave propagation is critical. The observation that the fractal porous material has a higher wave speed than a non-fractal porous material implies that the fractal structure improves the material's ability to transmit energy, making it more effective for applications in which wave propagation is crucial, such as acoustics or telecommunications.

5 CONCLUSION

By using a fractal approach, it is possible to analyze the behavior of wave propagation within a fractal porous material. The fractal dimension of the material, which is denoted by the parameter "α", plays a crucial role in determining the wave behavior. As the fractal dimension "α" approaches 1, the material becomes less self-similar, and the wave propagation speed decreases. Conversely, as the fractal dimension "α" approaches 0, the material becomes more self-similar, and the wave propagation speed increases.

Furthermore, the degree of self-similarity, as represented by the fractal dimension "α", affects wave attenuation. As the degree of self-similarity increases, wave attenuation also increases, and vice versa. The behavior of wave propagation within a fractal porous material is dependent on the material's physical parameters. Generally, highly permeable materials (low tortuosity and high characteristic lengths) allow for relatively faster wave propagation and less attenuation, while materials with low permeability (high tortuosity and low characteristics lengths result in slower wave propagation and more attenuation.

The study of wave propagation within fractal porous materials has significant implications for various fields, such as materials science, geology, and acoustics. Understanding the wave behavior in these materials can aid in designing materials for numerous applications, such as sound insulation, building, biomechanics and oil recovery.

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