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EIGENVECTOR ASSIGNMENT FOR DAMAGE LOCALIZATION WITH INVARIANT EIGENVALUES

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Abstract. A recently proposed scheme infers the spatial location of structural damage by interrogating the damage-induced shifts in assigned closed-loop system eigenvalues. The concept is to render the assigned eigenvalues invariant to model-based damage pattern postulates using eigenstructure assignment with left eigenvectors. In an idealized setting, damage is then confined to the pattern for which the damage-induced shifts in the assigned eigenvalues are zero. In practice, invariance is unattainable due to disturbances and the inevitable discrepancy between the physical system and its model representation, thus damage is located when the induced eigenvalue shifts are minimized. This paper explores how the assignment of the associated left eigenvectors affects the methodological robustness and proposes three different eigenvector assignment procedures for promoting it. The first procedure, which constitutes an optimization problem, explicitly maximizes the sensitivity to damage outside the postulated patterns, while the second procedure seeks to maximize the noted sensitivity in a direct setting based on singular value decomposition (SVD). The third procedure, which is also cast as an optimization problem, seeks to promote robustness by minimizing the uncertainties on the assigned eigenvalues. The proposed procedures are tested in a numerical example, and their individual viability, in terms of damage resolution promotion and computational efficiency, is discussed.

Key words: Damage localization, Virtual output feedback, Invariant eigenvalue assignment

1 Introduction

A commonly encountered challenge in vibration-based damage localization is that the identifiable vibration features carry insufficient information on damage to provide the required level of robustness [1, 2]. In the attempt to resolve the robustness problem, different feedback strategies have been adopted from control engineering [3]. These strategies are typically based on eigenstructure assignment, which allows for designing closed-loop (CL) system eigenvalues and

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eigenvectors in accordance with some predefined objective [4, 5]. In the context of damage localization, and structural health monitoring (SHM) in general, a common assignment objective is to maximize the sensitivity to damage of (a part of) the eigenspectrum [6–10].

The practical feasibility of the feedback strategies has been hindered by the required availability of appropriate hardware [3]. This roadblock has, however, been mitigated by the introduction of a virtual implementation, where the feedback is realized by offline processing of open-loop (OL) input-output data [11]. This virtual approach has been utilized in a recently proposed damage localization method [12, 13], in which the assignment objective is to render a subset of eigenvalues invariant to postulated damage patterns. As such, damage is, in an idealized setting, located in the pattern associated with the feedback gain for which the assigned eigenvalues do not change due to damage. It is opportune to note that the method relies on the availability of OL input-output data in the reference and damaged configurations and on the assumption that the system behaves linearly during the data collection.

In the addressed damage localization method, the invariance property for each of the eigenvalues is attained by assigning the associated left eigenvector to the null space of an augmented matrix. If the null space is multi-dimensional, a decision must be made as to how the eigenvector is extracted. The present paper examines the effect of this extraction on the damage localization robustness and proposes three assignment procedures, which, in theory, all should promote the robustness. The first two procedures seek to maximize the sensitivity to damage outside the postulated damage pattern, while the third procedure seeks to minimize the uncertainties on the assigned eigenvalues. The procedures are described and subsequently tested—in terms of robustness promotion and computational efficiency—in the context of a numerical example.

The paper is organized as follows. Section 2 reviews the damage localization method, while Section 3 outlines the three eigenvector assignment procedures. Section 4 presents the numerical example, and the paper closes with some concluding remarks in Section 5.

2 Damage localization using perturbation-invariant eigenvalues

We consider a system with input $u(t) \in \mathbb{R}^r$ and output $y(t) \in \mathbb{R}^m$. It is assumed that the dynamics of the system can be described by the linear, time-invariant (LTI) model

$$\dot{z}(t) = Az(t) + Bu(t), \tag{1a}$$

$$y(t) = Cz(t), \tag{1b}$$

in which $z(t) \in \mathbb{R}^n$ is the state vector with $n \in 2\mathbb{N}$, while $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, and $C \in \mathbb{R}^{m\times n}$ are the state, input, and output matrices. It is assumed that the direct transmission in system (1) is zero or subtracted from the output in (1b). It follows that

$$A \triangleq \begin{bmatrix} 0 & I \\ -\mathcal{M}^{-1}\mathcal{K} & -\mathcal{M}^{-1}\mathcal{D} \end{bmatrix}, \tag{2a}$$

$$B \triangleq \begin{bmatrix} 0 \\ \mathcal{M}^{-1}\mathcal{B}_2 \end{bmatrix}, \tag{2b}$$

$$C \triangleq \begin{bmatrix} \mathcal{C}_\text{d} - \mathcal{C}_\text{a}\mathcal{M}^{-1}\mathcal{K} & \mathcal{C}_\text{v} - \mathcal{C}_\text{a}\mathcal{M}^{-1}\mathcal{D} \end{bmatrix}, \tag{2c}$$

where $\mathcal{K}$, $\mathcal{D}$, $\mathcal{M} \in \mathbb{R}^{\frac{n}{2}\times\frac{n}{2}}$ are the stiffness, damping, and mass matrices, $\mathcal{B}_2 \in \mathbb{R}^{\frac{n}{2}\times r}$ is the input distribution matrix, and $\mathcal{C}_\text{d}$, $\mathcal{C}_\text{v}$, $\mathcal{C}_\text{a} \in \mathbb{R}^{m\times\frac{n}{2}}$ are selection matrices for displacement (d), velocity (v), and acceleration (a) output.

Assumption 1 *System* (1) *is controllable and observable with* $\text{rank}(B) = r$*,* $\text{rank}(C) = m$*,* $\max(r, m) < n$*, and* $\mathcal{M}$*,* $\mathcal{D}$*,* $\mathcal{K} \succ 0$.

2.1 Virtual output feedback

Virtual static output feedback in system (1) is attained by offline processing of the OL triplet (A, B, C) based on the feedback law

$$u(t) = -Gy(t) + v(t), \tag{3}$$

where $v(t) \in \mathbb{R}^r$ contains the OL inputs and $G \in \mathbb{C}^{r\times m}$ is the gain, which, due to the virtual implementation, can be complex-valued. Hereby, the virtual CL system becomes

$$\dot{z}(t) = \bar{A}z(t) + Bv(t), \tag{4a}$$

$$y(t) = Cz(t), \tag{4b}$$

with the CL state matrix

$$\bar{A} \triangleq A - BGC. \tag{5}$$

It is opportune to note that CL system (4) is controllable and observable if OL system (1) complies with Assumption 1 [12].

2.2 Perturbation-invariant eigenpair assignment

Let (λ_j, ϕ_j) be the j^th left eigenpair of $\bar{A}$, with eigenvalue $\lambda_j \in \mathbb{C}$ and left eigenvector $\phi_j \in \mathbb{C}^{1\times n}$. It then follows that

$$\phi_j(A - BGC - \lambda_j I) = 0, \tag{6}$$

or, in a partitioned form,

$$\begin{bmatrix} A^\top - \lambda_j I & -C^\top \end{bmatrix} \begin{bmatrix} \phi_j^\top \\ \gamma_j \end{bmatrix} = 0 \tag{7}$$

with $\gamma_j \triangleq G^\top B^\top \phi_j^\top$. Eq. (7) offers a standard eigenstructure assignment formulation without explicit consideration of the assignment objective.

We will now address the specific assignment objective of imposing perturbation-invariance in the assigned left eigenpairs. Let $\delta\bar{A}$ be the damage-induced perturbation of $\bar{A}$, then the further developments confine $\delta\bar{A}$ to comply with Assumption 2.

Assumption 2 *Let system* (4) *have the spatial domain* $\Omega = \Omega_1 \cup \Omega_2$*, with* $\Omega_1 \neq \varnothing$*,* $\Omega_2 \neq \varnothing$*, and* $\Omega_1 \cap \Omega_2 = \varnothing$*. Then, any non-zero part in* $\delta\bar{A}$ *is confined to* Ω_2*, with* $\text{rank}(\delta\bar{A}) < n$.

Proposition 1 *Let $\lambda(\bar{A})$ be the eigenspectrum of $\bar{A}$ and let $(\tilde{\lambda}_j, \tilde{\phi}_j)$ be an eigenpair of $\bar{A} + \delta\bar{A}$. Then, $\tilde{\lambda}_j \in \lambda(\bar{A})$ if $\tilde{\phi}_j^\top \in$ null$(\delta\bar{A}^\top)$, where* null *denotes (right) null space.*

Let $\mathcal{Z}_j = \begin{bmatrix}\mathcal{X}_j^\top & \mathcal{Y}_j^\top\end{bmatrix}^\top$ be a basis for the null space of the matrix in (7), with $\mathcal{X}_j \in \mathbb{C}^{n\times m}$ and $\mathcal{Y}_j \in \mathbb{C}^{m\times m}$. Then, (7) provides

$$\phi_j^\top = \mathcal{X}_j\alpha_j, \tag{8}$$

where $\alpha_j \in \mathbb{C}^m$. According to Proposition 1, eigenvalue perturbation-invariance requires $\phi_j^\top \in$ null$(\delta\bar{A}^\top)$. It thus follows from (8) that

$$\delta\bar{A}^\top\mathcal{X}_j\alpha_j = 0, \tag{9}$$

for which rank$(\delta\bar{A}) < m$ provides a sufficient condition for consistency when $\alpha_j \neq 0$. Combining (7) and the invariance relation of Proposition 1 yields the augmentation

$$\begin{bmatrix} A^T - \lambda_j I & -C^\top \\ \delta\bar{A}^\top & 0 \end{bmatrix}\begin{bmatrix}\phi_j^\top \\ \gamma_j\end{bmatrix} = 0, \tag{10}$$

which constitutes a perturbation-invariant eigenstructure assignment formulation.

Let $\mathcal{P}_j = \begin{bmatrix}\mathcal{Q}_j^\top & \mathcal{R}_j^\top\end{bmatrix}^\top$ be a null space basis for the matrix in (10), where $\mathcal{Q}_j \in \mathbb{C}^{n\times l}$ and $\mathcal{R}_j \in \mathbb{C}^{m\times l}$, with l denoting the nullity of the considered matrix. Hereby,

$$\phi_j^\top = \mathcal{Q}_j\beta_j, \tag{11a}$$

$$\gamma_j = \mathcal{R}_j\beta_j, \tag{11b}$$

where $\beta_j \in \mathbb{C}^l$ is arbitrary. Now, let $\Gamma \triangleq \begin{bmatrix}\gamma_1 & \gamma_2 & \dots & \gamma_p\end{bmatrix} \in \mathbb{C}^{m\times p}$, $\tilde{\Phi} \triangleq \begin{bmatrix}\phi_1^\top & \phi_2^\top & \dots & \phi_p^\top\end{bmatrix} \in \mathbb{C}^{n\times p}$, and $\Theta \triangleq B^\top\tilde{\Phi}$, then it follows that

$$G^\top\Theta = \Gamma. \tag{12}$$

The gain G, which assigns the p perturbation-invariant eigenvalues, can be computed by inversion when $p = r$ and Θ has full rank. In the overdetermined case with $p > r$, (12) is generally inconsistent and the assignment will thus not be exact. Consequently, we confine the discussion to $p = r$, which, under Assumption 2, implies that r eigenvalues can be assigned invariant to $\delta\bar{A}$ for rank$(\delta\bar{A}) < m$.

2.3 Damage localization formulation

By implementation of the perturbation-invariant eigenstructure assignment procedure outlined in Subsection 2.2, damage localization can be accomplished by interrogating virtual CL system (4). A physics-based model of the system in its reference configuration is used to postulate different damage patterns. Each damage pattern represents a configuration where damage is present in a specific subdomain; with the subdomain confined such that the resulting state matrix perturbation is rank $m-1$ or less. For each postulated damage pattern, a gain, G_i, is designed to render r left eigenvalues invariant to damage in the i^{th} damage pattern.

The damage localization is based on a metric, $\mathcal{E}_i$, which expresses the deviation between the estimates of the assigned eigenvalues in the reference and damaged configurations for the i^{th} damage pattern. In particular,

$$\mathcal{E}_i \triangleq \sum_{j=1}^{r} \frac{|\Im(\delta\lambda_j^{(i)})|}{\Im(\lambda_j^{(i)})}, \tag{13}$$

where $\Im(\lambda_j^{(i)})$ is the imaginary part of the j^{th} assigned eigenvalue in the reference configuration, while $\Im(\delta\lambda_j^{(i)})$ is the shift in $\Im(\lambda_j^{(i)})$. Hereby, damage is located in the pattern for which (13) is minimized.

3 Eigenvector assignment procedures

We recall that any l-dimensional vector can be used as β_j to compute ϕ_j after (11a). If $l > 1$, then the selection of β_j may, as exemplified in Section 4, influence the damage localization robustness substantially. Therefore, we now formulate three procedures for designing β_j, with the aim of promoting the robustness offered by damage metric (13).

The three procedures are all, to more or less extent, related to the eigenvalue sensitivity, which writes

$$\frac{\partial\lambda_j}{\partial\theta} = \frac{1}{\phi_j\psi_j}\phi_j\frac{\partial\bar{A}}{\partial\theta}\psi_j, \tag{14}$$

where $\theta \in \mathbb{R}$ is a damage-related parameter and $\psi_j \in \mathbb{C}^n$ is the right eigenvector. It follows

$$\left|\frac{\partial\lambda_j}{\partial\theta}\right| = \frac{1}{|\phi_j\psi_j|}\left\|\phi_j\frac{\partial\bar{A}}{\partial\theta}\psi_j\right\|_2 \leq \frac{1}{|\phi_j\psi_j|}\|\phi_j\|_2\left\|\frac{\partial\bar{A}}{\partial\theta}\right\|_2\|\psi_j\|_2\,, \tag{15}$$

which constitutes the basis for the three procedures.

3.1 Procedure 1: *explicit maximization of damage sensitivity*

With reference to (15), this procedure aims at promoting robustness by maximizing the sensitivity of λ_j to damage outside the associated damage pattern through maximization of $\|\phi_j\|_2$. The procedure is cast as an optimization problem, which yields an optimal β_j as

$$\hat{\beta}_j = \underset{\beta_j}{\arg\max} \quad \|\mathcal{Q}_j\beta_j\|_2. \tag{16}$$

3.2 Procedure 2: *implicit maximization of damage sensitivity*

We now formulate a computationally relaxed alternative to procedure 1. In particular, (16) is cast as a direct problem by introducing the constraint $\|\beta_j\|_2 = 1$. Let the SVD of $\mathcal{Q}_j$ be

$$\mathcal{Q}_j = U_j\Sigma_jV_j^{\mathsf{H}}, \tag{17}$$

where $U_j \in \mathbb{C}^{n\times n}$ and $V_j \in \mathbb{C}^{l\times l}$ contain, respectively, the left- and right-singular vectors, and $\Sigma_j \in \mathbb{R}^{n\times l}$ is a diagonal matrix containing the singular values. Let $V_j = \begin{bmatrix} v_j^{(1)} & v_j^{(2)} & \ldots & v_j^{(l)} \end{bmatrix}$ and take $v_j^{(1)}$ as the right-singular vector associated with the largest singular value of $\mathcal{Q}_j$, then

$$v_j^{(1)} \triangleq \underset{\|x\|_2=1}{\arg\max} \quad \|\mathcal{Q}_j x\|_2. \tag{18}$$

Thus, by setting $\hat{\beta}_j = v_j^{(1)}$, we attain the configuration that maximizes the ℓ_2-norm of the product of $\mathcal{Q}_j$ and any ℓ_2-unit-norm vector.

3.3 Procedure 3: *minimization of eigenvalue uncertainty*

Consider the simple eigenvalue λ_j of $\bar{A}$ and assume that $\|\phi_j\|_2 = 1$ and $\|\psi_j\|_2 = 1$. The conditioning of λ_j is quantified by the condition number

$$\kappa(\lambda_j) = \frac{1}{|\phi_j \psi_j|}, \tag{19}$$

so, with reference to (15), it follows that an increase in $\kappa(\lambda_j)$ will increase the sensitivity of λ_j to perturbations in $\bar{A}$ *not* related to the postulated damage. The current procedure minimizes the eigenvalue condition numbers in order to promote robustness to model errors and other non-damage-related sources of perturbations in $\bar{A}$. In particular, let $c_\lambda = \begin{bmatrix} \kappa(\lambda_1) & \kappa(\lambda_2) & \ldots & \kappa(\lambda_r) \end{bmatrix}^\top$, then the optimal β_j is attained as

$$\hat{\beta}_j = \underset{\beta_j}{\arg\min} \quad \|c_\lambda\|_2. \tag{20}$$

4 Numerical example

The viability of the three procedures proposed for designing β_j is examined in the context of a numerical example with the shear system depicted in Figure 1. The system, which is modeled with six degrees of freedom (DOF), is subjected to a known white noise input, $u(t)$, at DOF 2, and displacement outputs, $y_i(t)$, are captured at DOF 1, 3, and 6. We shall distinguish between the *design model* and *simulation models*, where the former is the model used for the gain design, while the simulation models yield input-output data in the reference and damage configurations. In the design model, all inter-story stiffnesses and floor masses are, respectively, $1,000$ and 1 in some consistent set of units, while classical damping is assumed, with each mode having a damping ratio of 5%. We use the design model to postulate six different damage patterns, where the i^{th} pattern is realized by a unit reduction in the i^{th} inter-story stiffness.

The system is simulated in its reference configuration and in six different damage configurations, where the i^{th} configuration is attained by reducing the i^{th} inter-story stiffness by 15%. In these simulation models, the mass and damping properties are identical to those of the design model, while each inter-story stiffness is randomly perturbed $\pm 2.5\%$ with respect to the design model stiffness to emulate model errors. All seven configuration are simulated in a Monte Carlo

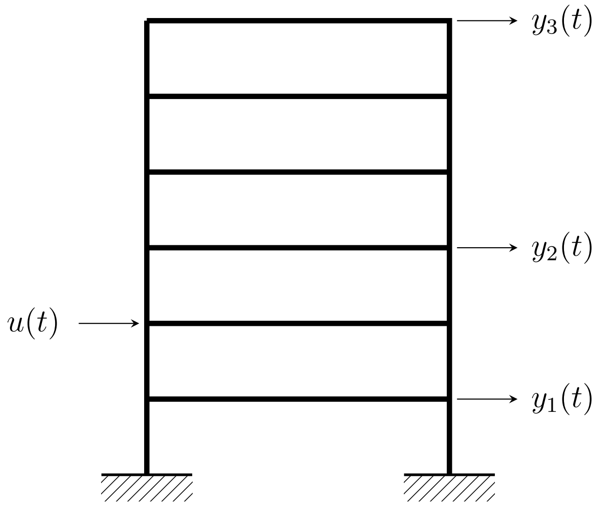


Figure 1: Shear system with deterministic input $u(t)$ and displacement outputs $y_i(t)$.

setting with 200 realizations. In each realization, the simulations are conducted with a sampling period of 0.02 seconds for $50,000$ samples and with the input temporally discretized using a zero-order hold. The attained input and output signals are corrupted by 5% additive white Gaussian noise and then used to estimate the system triplet (A, B, C) for each configuration via subspace identification [14] with a model order of 12.

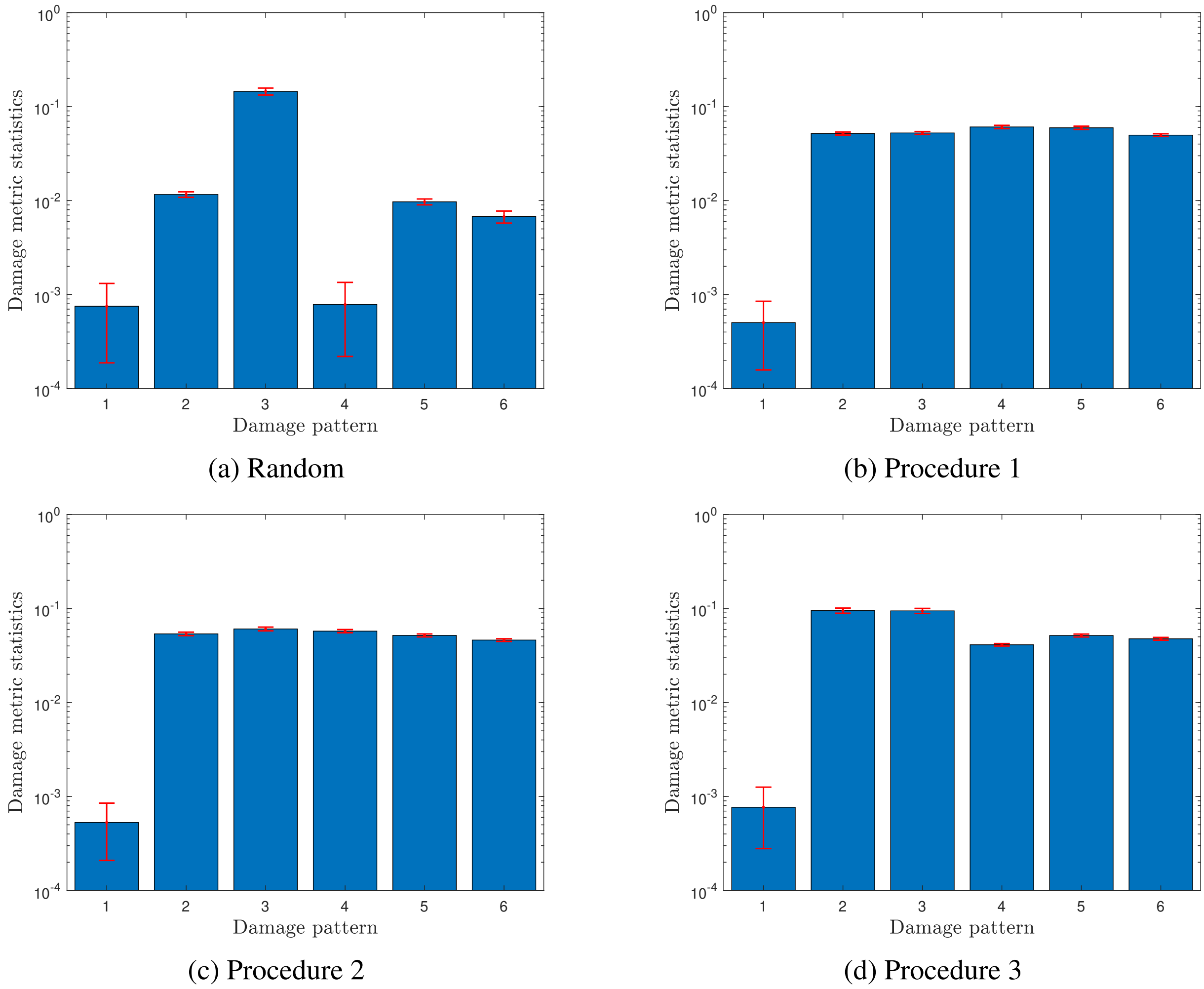
For each of the six postulated damage patterns, a gain is designed to assign a CL eigenvalue at $0.9\eta_1$, where η_1 is the first OL eigenvalue. The associated left CL eigenvector is realized in four different ways; namely, by a randomly selected β_j and by means of the three procedures presented in Section 3. The resulting damage localization rates (based on the 200 realizations) are presented in Table 1, where *successful localization* implies that the lowest damage metric, $\mathcal{E}_i$, is attained for the damage pattern coincident with the examined damage configuration. It is

Table 1: Successful damage localization rates obtained in the Monte Carlo experiment with a randomly chosen β_j and the three procedures outlined in Section 3.

Damage configuration	Localization rate (%)			
	Random β_j	Procedure 1	Procedure 2	Procedure 3
1	55.0	100	100	100
2	100	91.0	94.0	98.5
3	40.5	88.5	86.0	94.0
4	99.5	100	99.5	100
5	99.5	99.5	100	100
6	90.5	95.5	94.0	85.0
Mean localization rate (%)	80.9	95.8	95.6	96.2

opportune to recall that the i^{th} damage configuration refers to a reduction in the i^{th} inter-story stiffness. Evidently, the proposed eigenvector assignment procedures all improve the damage localization compared to that obtained with the randomly chosen β_j. In this regard, we note that five additional random realizations of β_j have been explored, and the results have been mean localization rates of 79.1%, 80.5%, 82.8%, 84.2%, and 87.2%.

To elaborate on the results provided in Table 1, Figures 2 and 3 present the mean value and standard deviation of the damage metric for, respectively, damage configuration 1 and 2 when using the different procedures for computing β_j. From inspection of the results for damage configuration 1, see Figure 2, it is evident that the three procedures all enhance the damage localization robustness compared to the randomly chosen β_j. In fact, the performances of the three procedures are comparable. For damage configuration 2, for which the results are seen



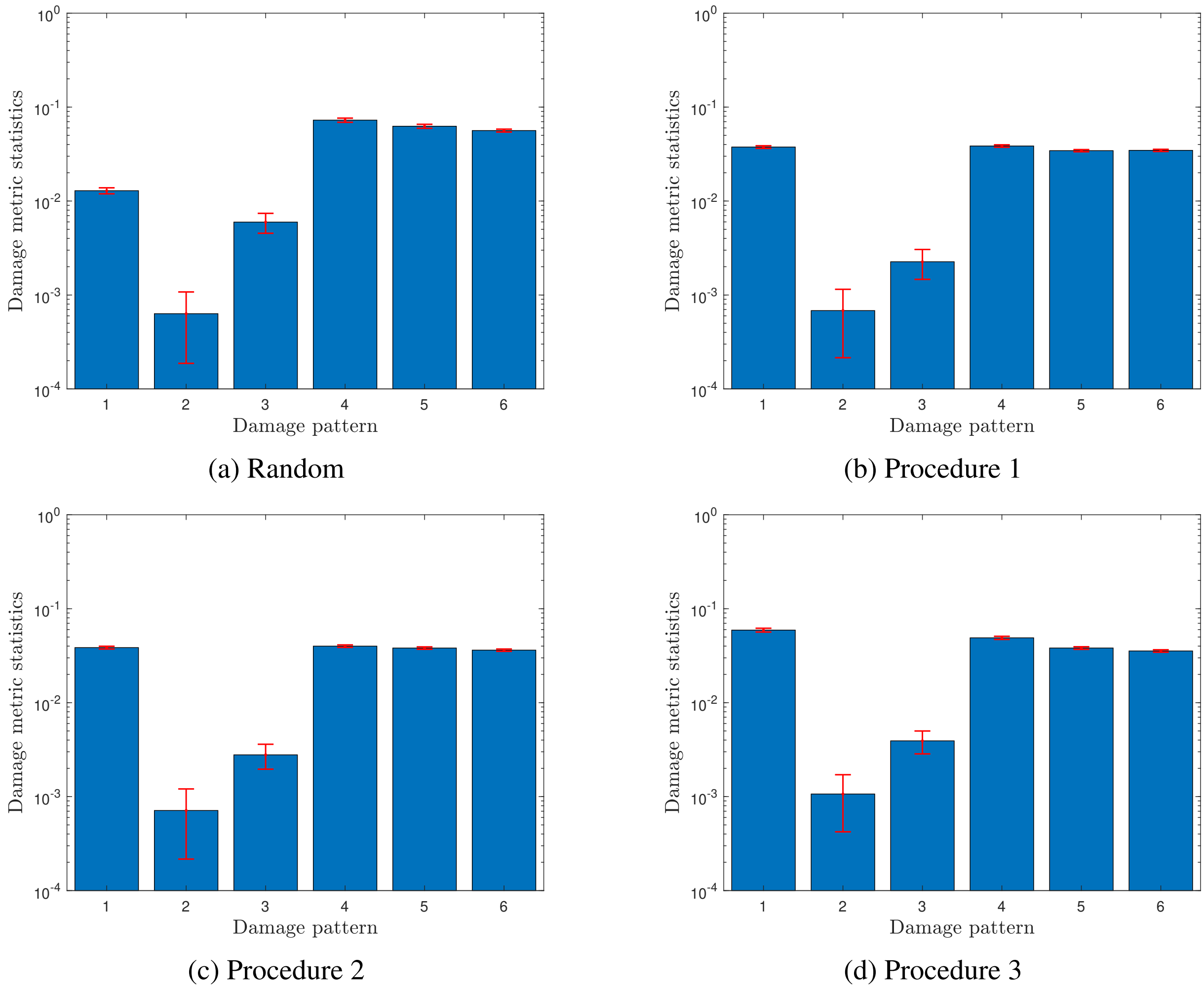
(a) Random

(b) Procedure 1

(c) Procedure 2

(d) Procedure 3

Figure 2: Mean value and standard deviation of the damage metric for damage configuration 1 attained with different eigenvector assignment procedures.



(a) Random (b) Procedure 1 (c) Procedure 2 (d) Procedure 3

Figure 3: Mean value and standard deviation of the damage metric for damage configuration 2 attained with different eigenvector assignment procedures.

in Figure 3, the randomly chosen β_j slightly outperforms the proposed procedures. The results attained with the three procedures are, however, still acceptable.

A cursory examination of the results presented in Table 1 and Figures 2 and 3 reveals that the three assignment procedures are generally comparable in terms of damage localization performance. Some additional insight to the implementation of the procedures is provided in Table 2, which shows the computational time for each procedure when implemented in MATLAB 2022b [15] under different assignment conditions. Procedure 2 is implemented using MATLAB's built-in SVD algorithm, while procedures 1 and 3 are implemented by solving, respectively, (16) and (20) using MATLAB's genetic algorithm with a population size of 50, a maximum number of generations of 100, and a convergence criterion of an average change in the objective function value of at most 10^{-6} over the past 50 generations. As expected, the direct nature of

Table 2: Computational time for the eigenvector assignment procedures when assigning 1, 3, and 5 eigenpairs per gain.

Number of assigned eigenpairs	Computational time (seconds)		
	Procedure 1	Procedure 2	Procedure 3
1	1.22	0.03	5.93
3	2.48	0.03	16.1
5	4.02	0.04	26.0

procedure 2 promotes computational efficiency compared to the optimization-based procedures.

5 Concluding remarks

The invariance property of the assigned eigenvalues, which constitutes the basis of the considered damage localization scheme, is imposed by assigning each of the associated left eigenvectors to the null space of an augmented matrix. If the null space is multi-dimensional, then a decision must be made as to how each eigenvector is extracted. This paper proposes three different extraction procedures for promoting the damage localization robustness and examines their viability in a numerical example. It is found that the procedures all improve the localization results compared to that obtained with random extractions. In this context, it is observed that the localization performances of the three procedures are comparable, whereas the computational efficiency of procedure 2 is considerably better than that of the two other procedures.

REFERENCES

[1] S. Wernitz, "Damage localization in data-driven vibration-based structural health monitoring using linear quadratic estimation theory," Doctoral dissertation, Leibniz University Hannover, 2022.

[2] D. Bernal and M. D. Ulriksen, "Subspace exclusion zones for damage localization," *Mechanical Systems and Signal Processing*, vol. 114, pp. 120–127, 2019.

[3] S. Pedersen and M. D. Ulriksen, "Closed-loop experimental testing framework for structural control applications," *Structural Control and Health Monitoring*, vol. 28, p. e2765, 2021.

[4] V. Syrmos, C. Abdallah, P. Dorato, and K. Grigoriadis, "Static output feedback—a survey," *Automatica*, vol. 33, pp. 125–137, 1997.

[5] G. P. Liu and R. J. Patton, *Eigenstructure Assignment for Control System Design*, 1st ed. Wiley, 1998.

[6] L. R. Ray and L. Tian, "Damage detection in smart structures through sensitivity enhancing feedback control," *Journal of Sound and Vibration*, vol. 227, pp. 987–1002, 1999.

[7] J.-S. Lew and J.-N. Juang, “Structural damage detection using virtual passive controllers,” *Journal of Guidance, Control, and Dynamics*, vol. 25, pp. 419–424, 2002.

[8] L. J. Jiang and K. W. Wang, “An experiment-based frequency sensitivity enhancing control approach for structural damage detection,” *Smart Materials and Structures*, vol. 18, 2009.

[9] D. Bernal and M. D. Ulriksen, “Output feedback in the design of eigenstructures for enhanced sensitivity,” *Mechanical Systems and Signal Processing*, vol. 112, pp. 22–30, 2018.

[10] D. Bernal, E. Memarzadeh, and M. D. Ulriksen, “Limit cycle periods in damage detection,” *Mechanical Systems and Signal Processing*, vol. 162, p. 108037, 2022.

[11] D. Bernal, “Virtual output feedback in parameter estimation.” *Mechanical Systems and Signal Processing*, vol. 142, p. 106786, 2020.

[12] M. D. Ulriksen, S. Gres, and D. Bernal, “Invariant eigenvalue assignment and uncertainty quantification for fault localization,” *To appear*, 2023.

[13] M. D. Ulriksen, P. Z. Nieto, M. B. Dahl, S. Gres, and D. Bernal, “Statistical damage localization with perturbation-invariant eigenvalues,” *To appear*, 2023.

[14] P. van Overschee and B. De Moor, *Subspace identification for linear systems: Theory - Implementation - Applications*, 1st ed. Kluwer Academic Publishers, 1996.

[15] MATLAB, *Release 2022b*. Natick, Massachusetts: The MathWorks Inc., 2022.